Sparse Fourier Transform Algorithms for Real-time Applications

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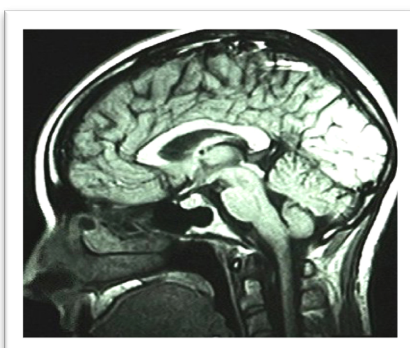
Fourier Transform Is Used Everywhere

Audio & Video Processing

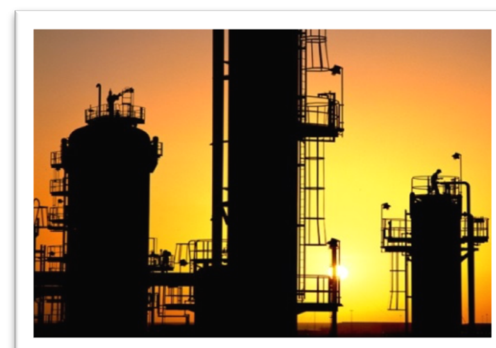
GPS

Radar



Medical Imaging

DNA Sequencing

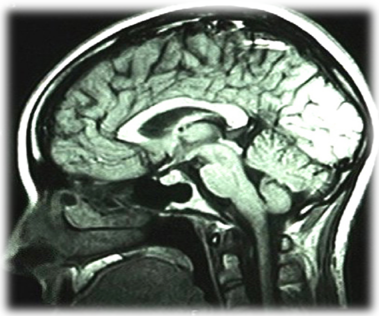


Oil Exploration

Computing the Fourier Transform

- In 1965, Cooley and Tukey introduced the FFT: $O(n \log n)$
- But … FFT is no longer fast enough

GPS



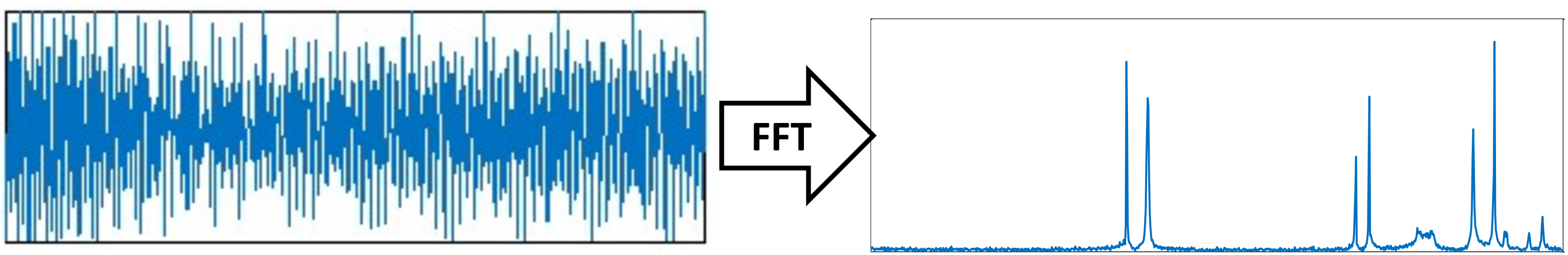
Medical Imaging

Astronomy

Can we compute the Fourier transform faster than FFT in sublinear time?

Key Idea: Leverage Sparsity

Often the Fourier Transform is dominated by a few peaks



Time

Frequency

We leverage sparsity to compute the Fourier transform faster than FFT

Sparsity appears in GPS, medical imaging, video, audio, astronomy data, seismic data, genomics

Theoretical Results

- Prior work:

  [Mansour '92, Gilbert, Muthukrishnan, Strauss '05, Akavia '10, Iwen'10]

  – For a signal of size n that is k-sparse i.e. has k large frequencies

  – Best run time: [Gilbert, Muthukrishnan, Strauss '05]

  – $O(k \log^4 n)$ → Improves over FFT for $k \ll n / \log^3 n$

- Sparse Fourier Transform [Hassanieh, Indyk, Katabi, Price 2012]

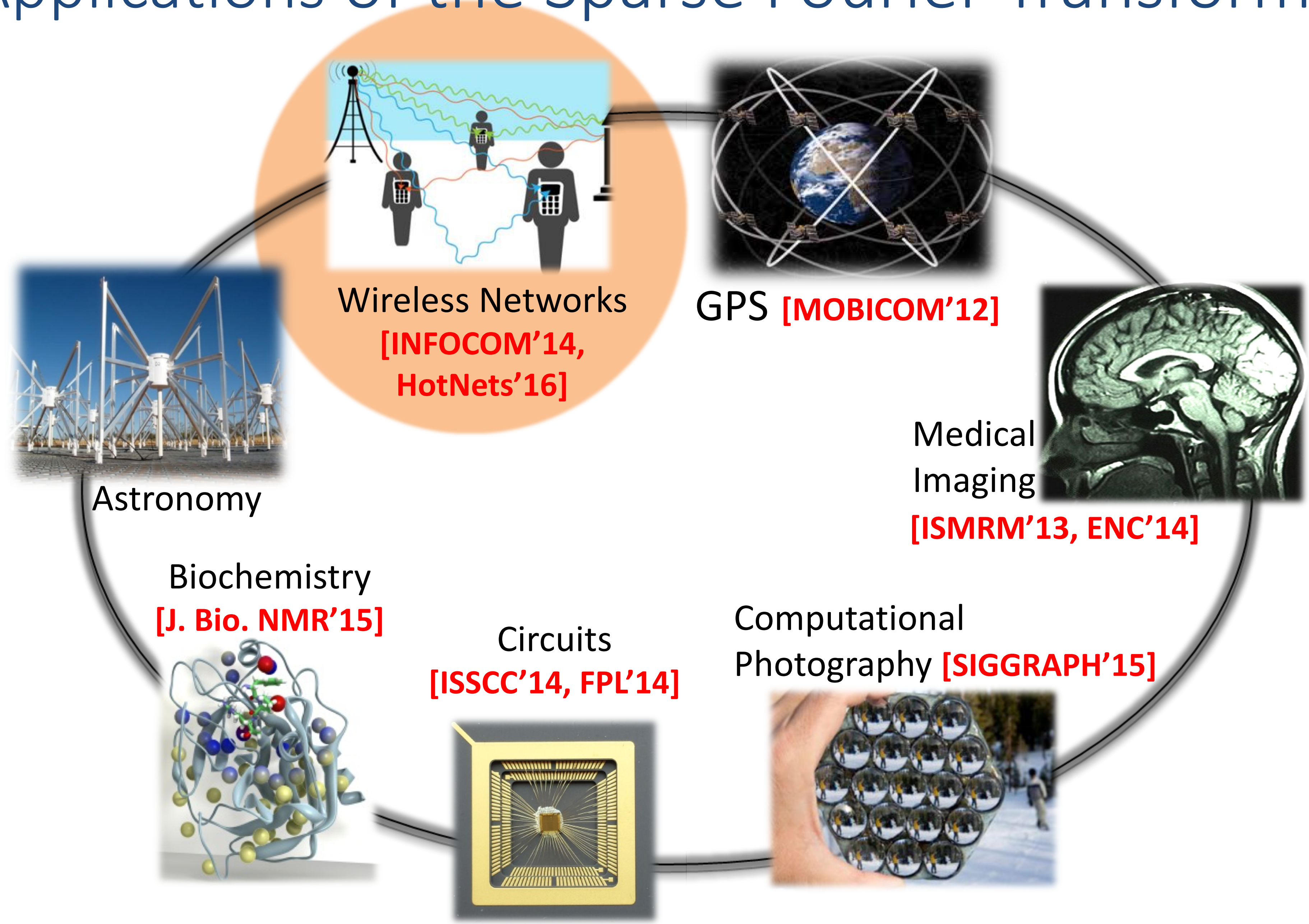
  – Exactly k-sparse case: $O(k \log n)$

    - Optimal if FFT is optimal

  – Approximately k-sparse case: $O(k \log(n) \log(n/k))$

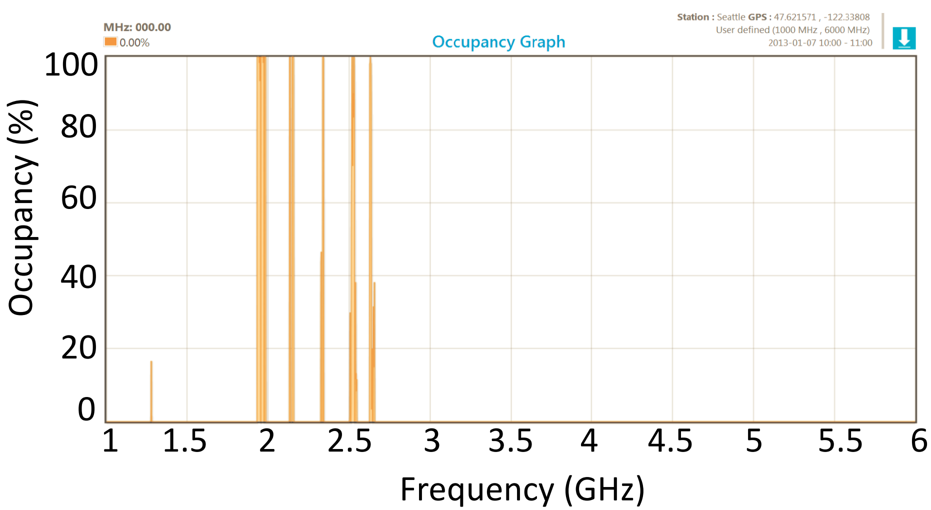
    - Improves over FFT for any $k = o(n)$

Applications of the Sparse Fourier Transform

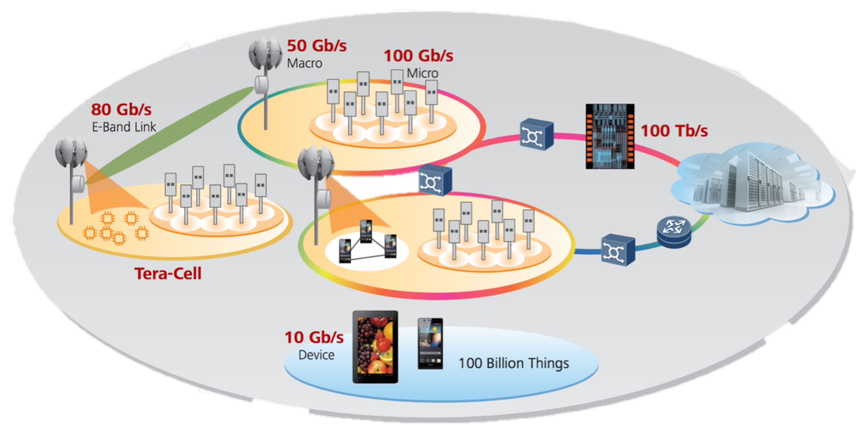


Real-time Applications to Wireless Networks

Spectrum Sensing & Acquisition

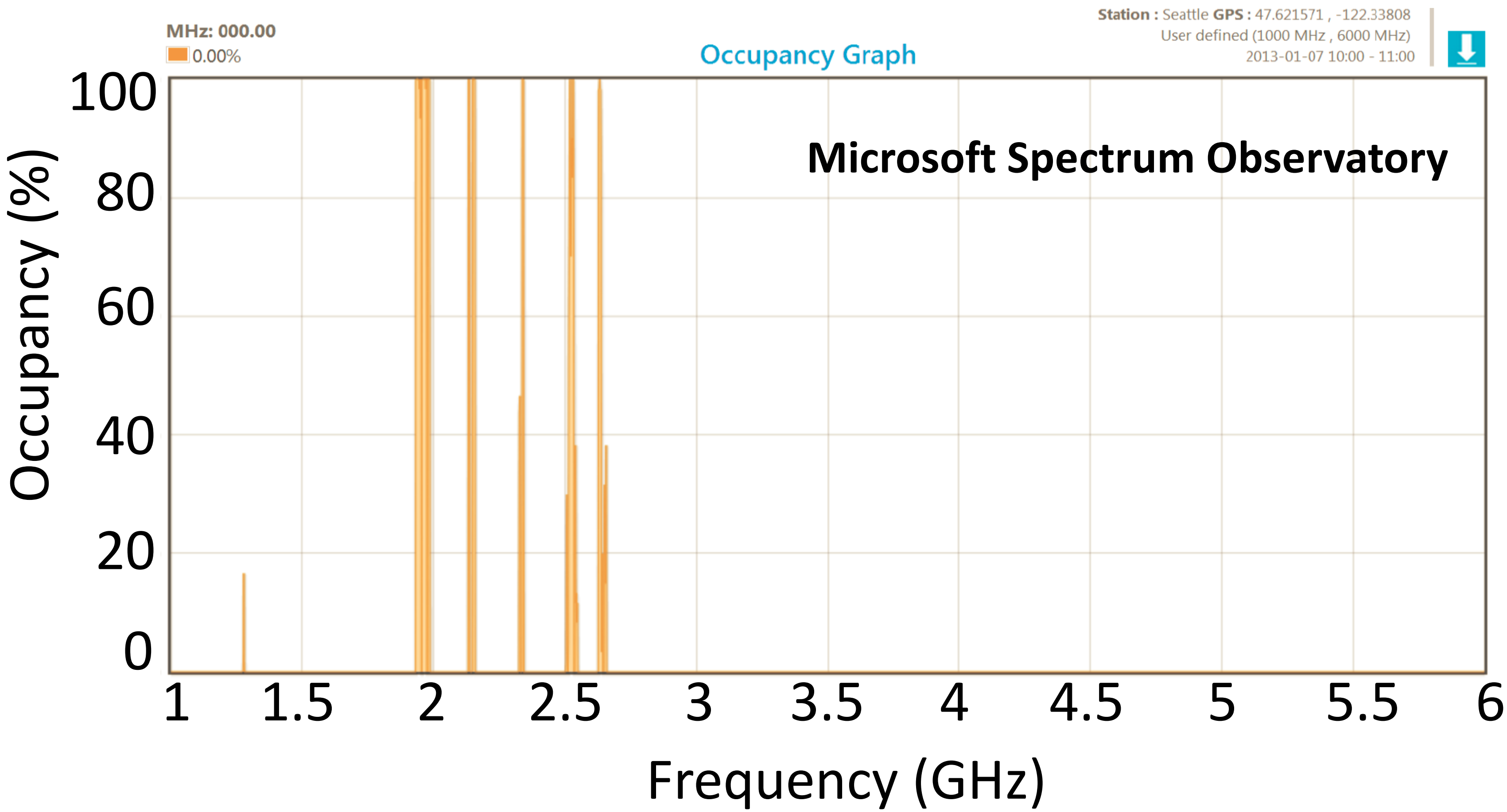


Millimeter Wave 5G Networks



Spectrum Scarcity

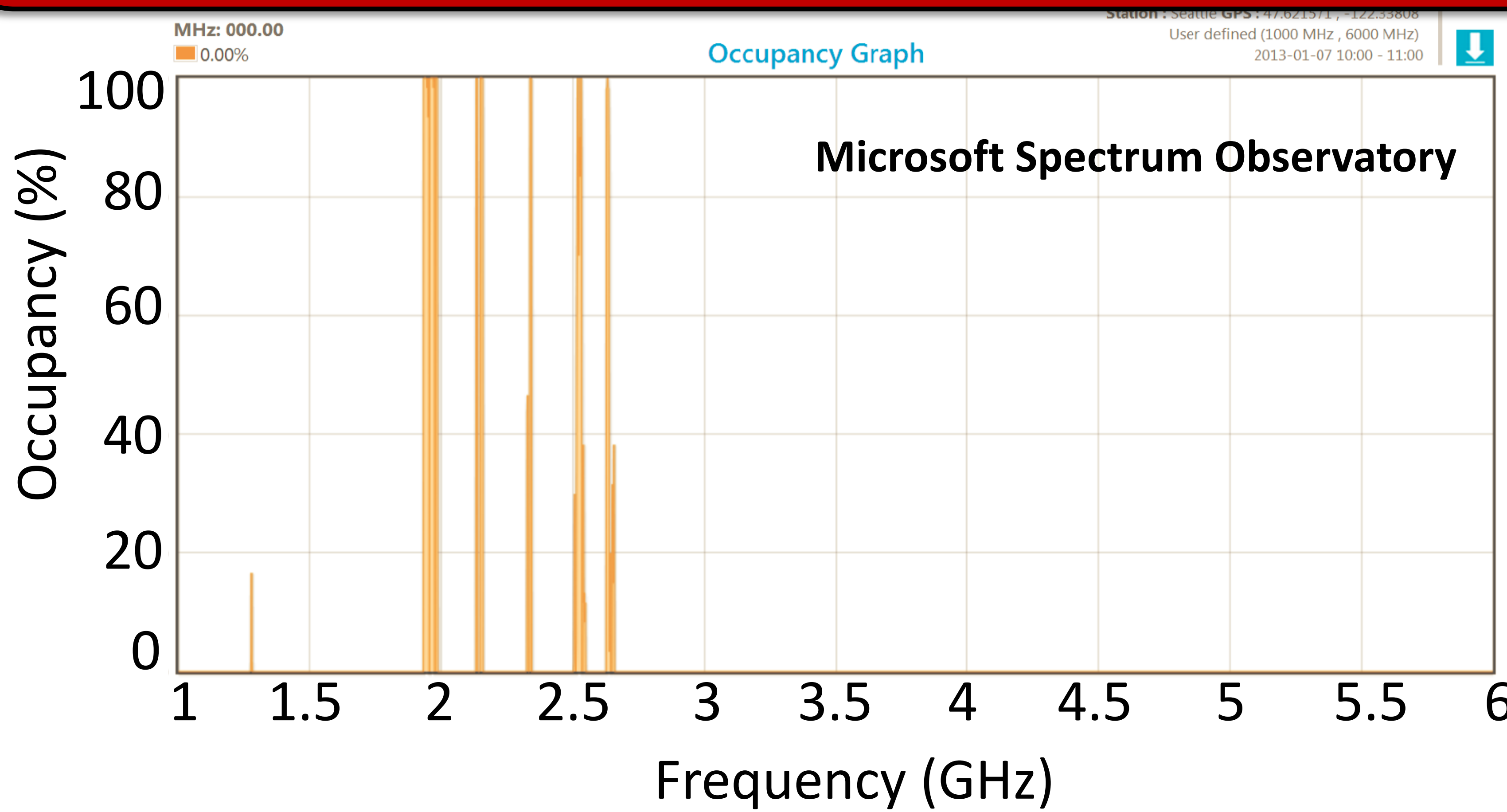
- The FCC: spectrum crunch started in 2013
- But at any time, most of the spectrum is unused



Dynamic Spectrum Access

Sense to find unused bands; Use them!

How do you capture GHz of spectrum?



Realtime GHz Spectrum Sensing is Difficult

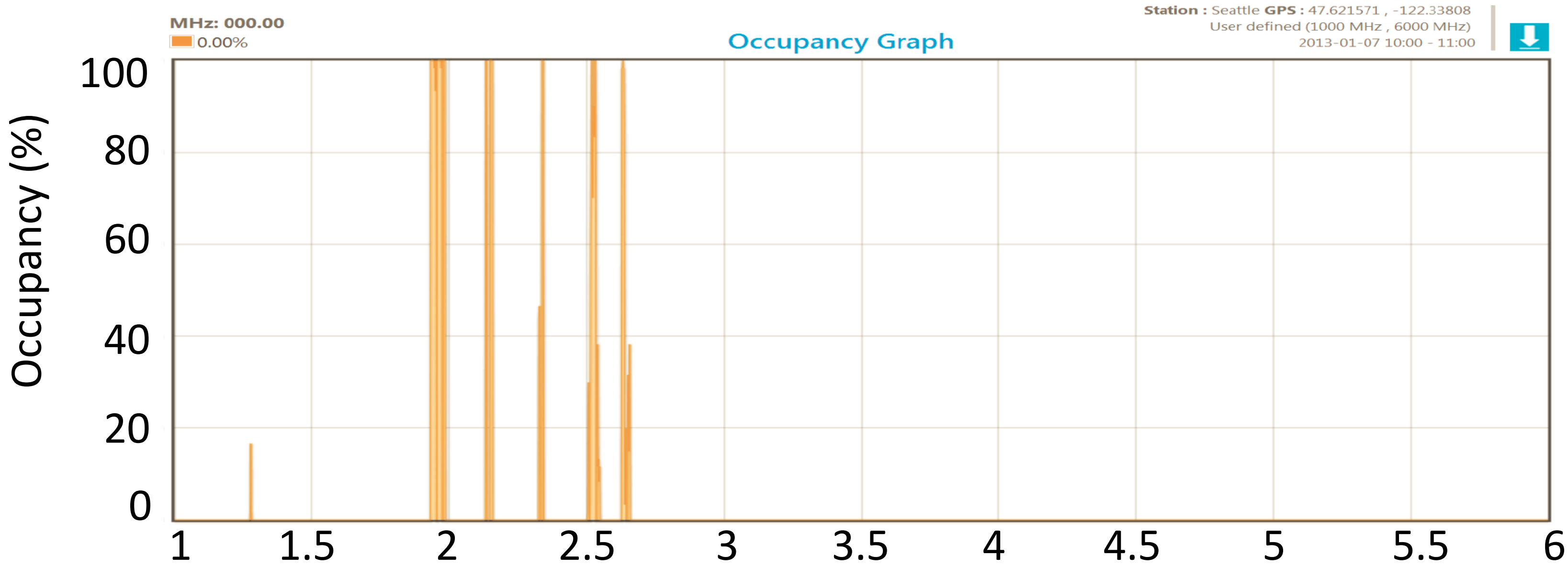
- Today, sequential scanning of tens of MHz
  ➔ Can easily miss fleeting signals

- Key Challenge: high-speed ADCs

Tens of MHz ADC
Low-power
High resolution
Cheap

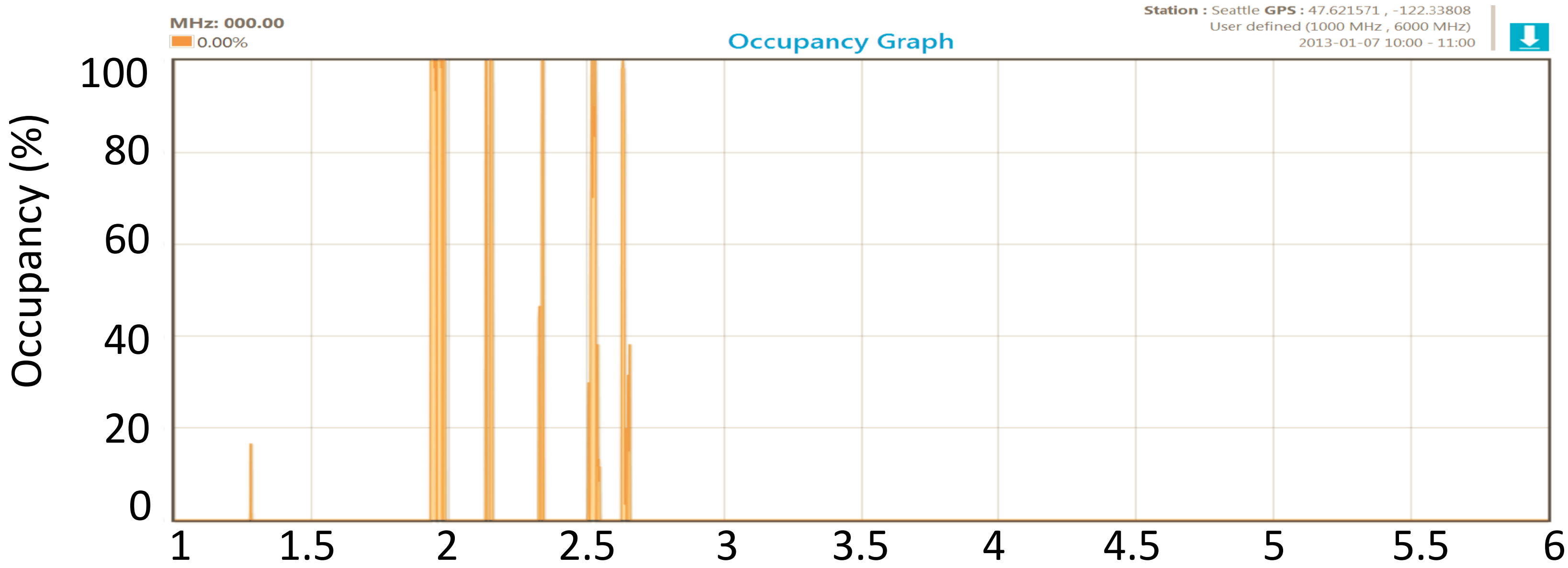
A Few GHz ADC
10x more power
Poor resolution
Expensive

Leverage Sparsity



- Sparse recovery tells us if the signal is sparse, we can capture it without sampling it very fast at Nyquist
- [Laska et al. ’11, Yoo et al. ‘12, Mishali et al. ’11] leverage sparsity using compressive sensing but
  - – Random Sampling which is difficult
  - – Computationally very expensive

Leverage Sparsity



Sparse Fourier Transform

No random sampling → Use a few low-speed ADCs

Sub-linear algorithm → Computes large FFT cheaply

How Does the Sparse Fourier Transform Work?

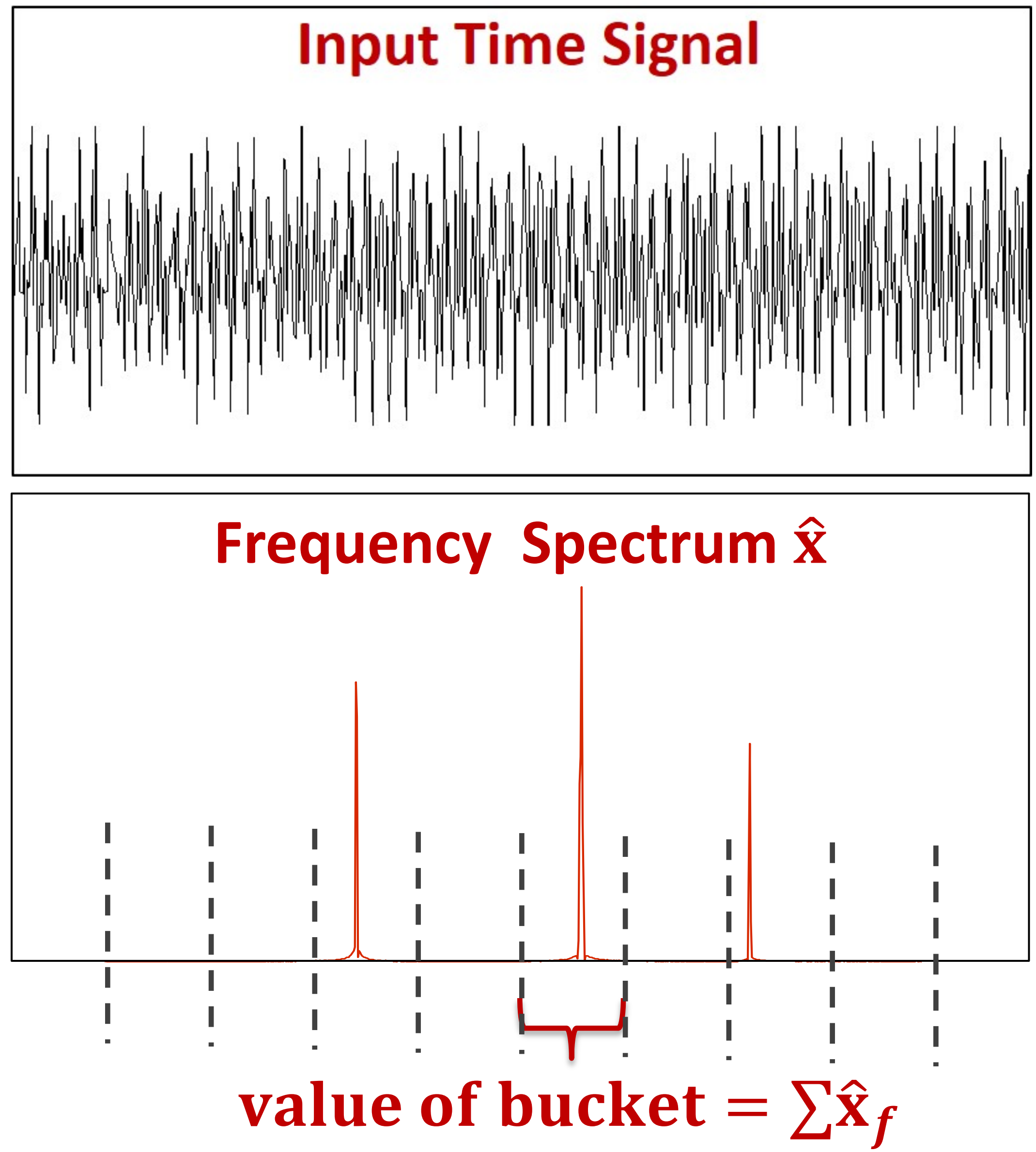
1- Bucketize

Divide spectrum into a few buckets

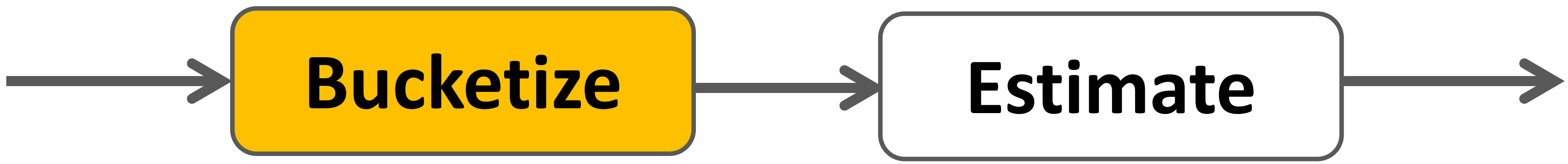
Most buckets are empty → Ignore

2- Estimate

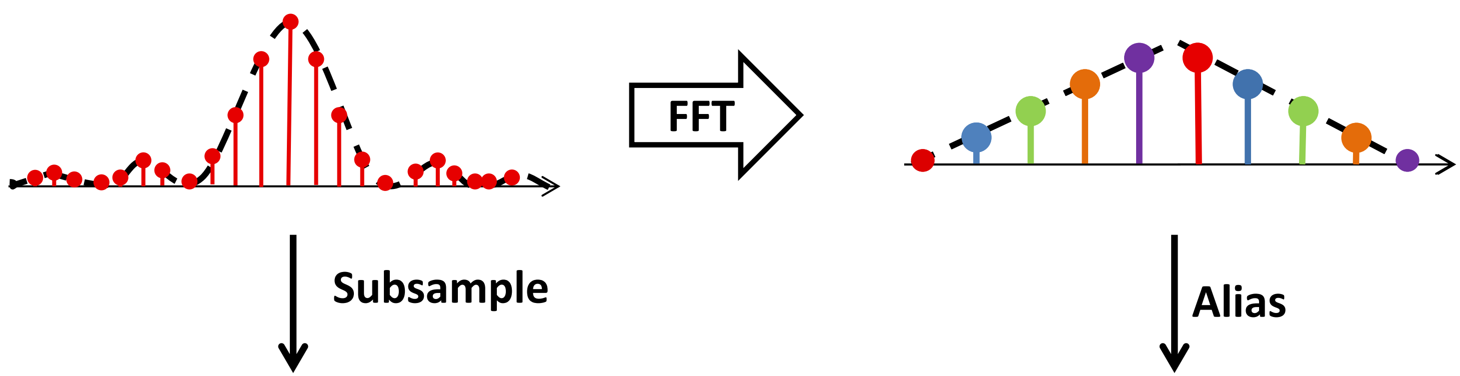
Estimate the large coefficients in non-empty bucket



How Does the Sparse Fourier Transform Work?

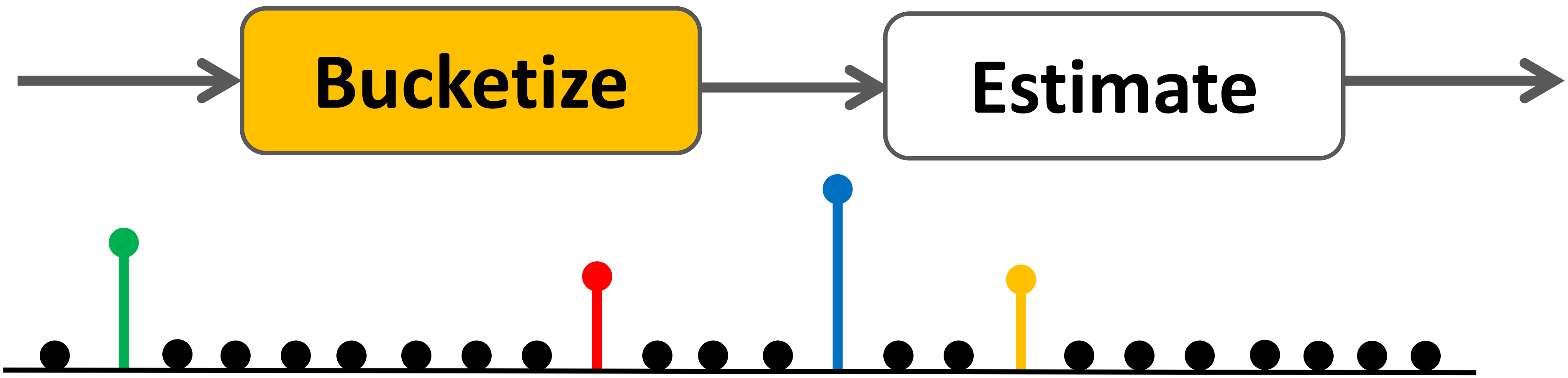


Sub-sampling in time FFT → Aliasing in frequency



Aliasing done by a small FFT → Cheap Bucketization

How Does the Sparse Fourier Transform Work?



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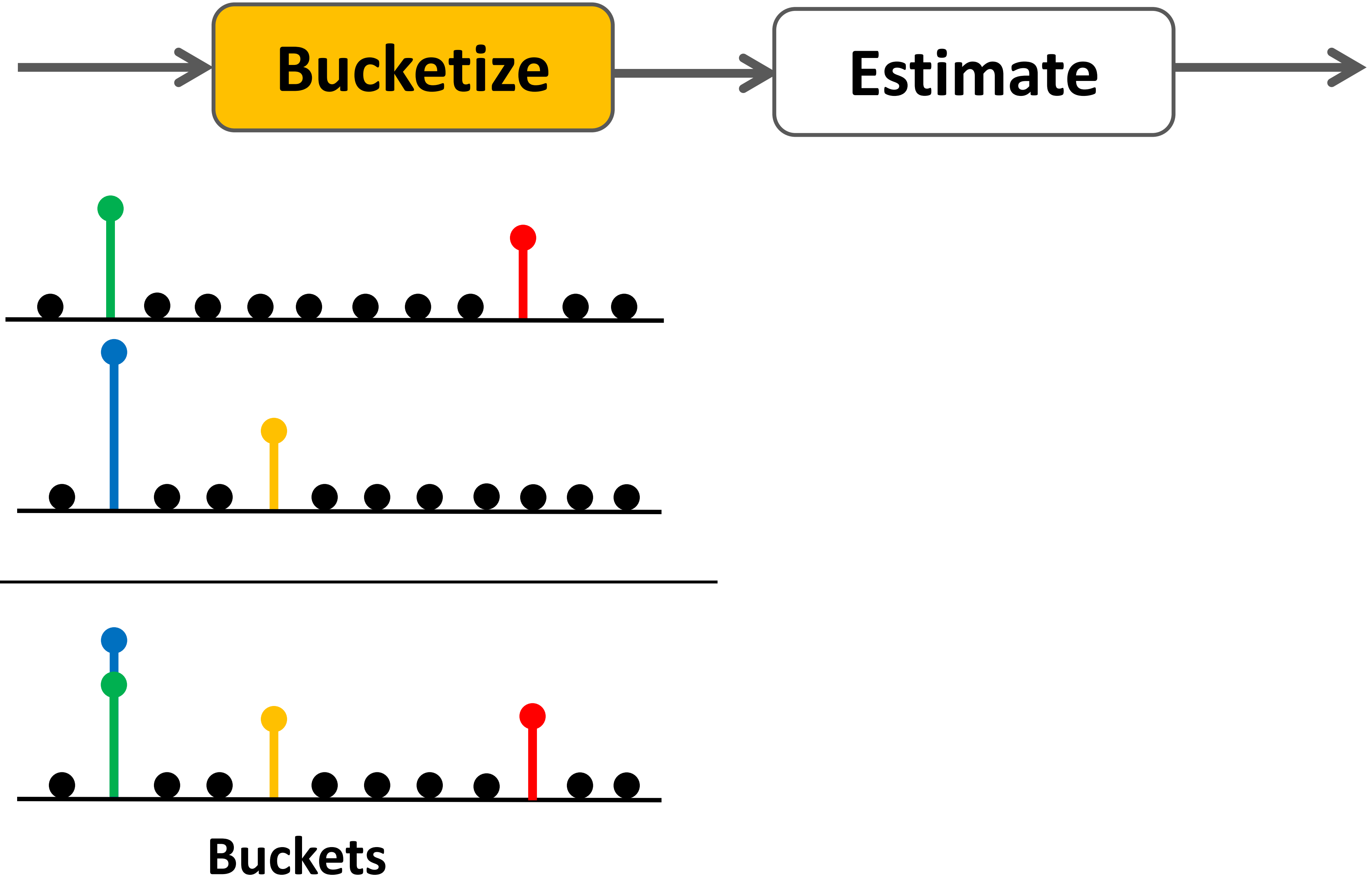

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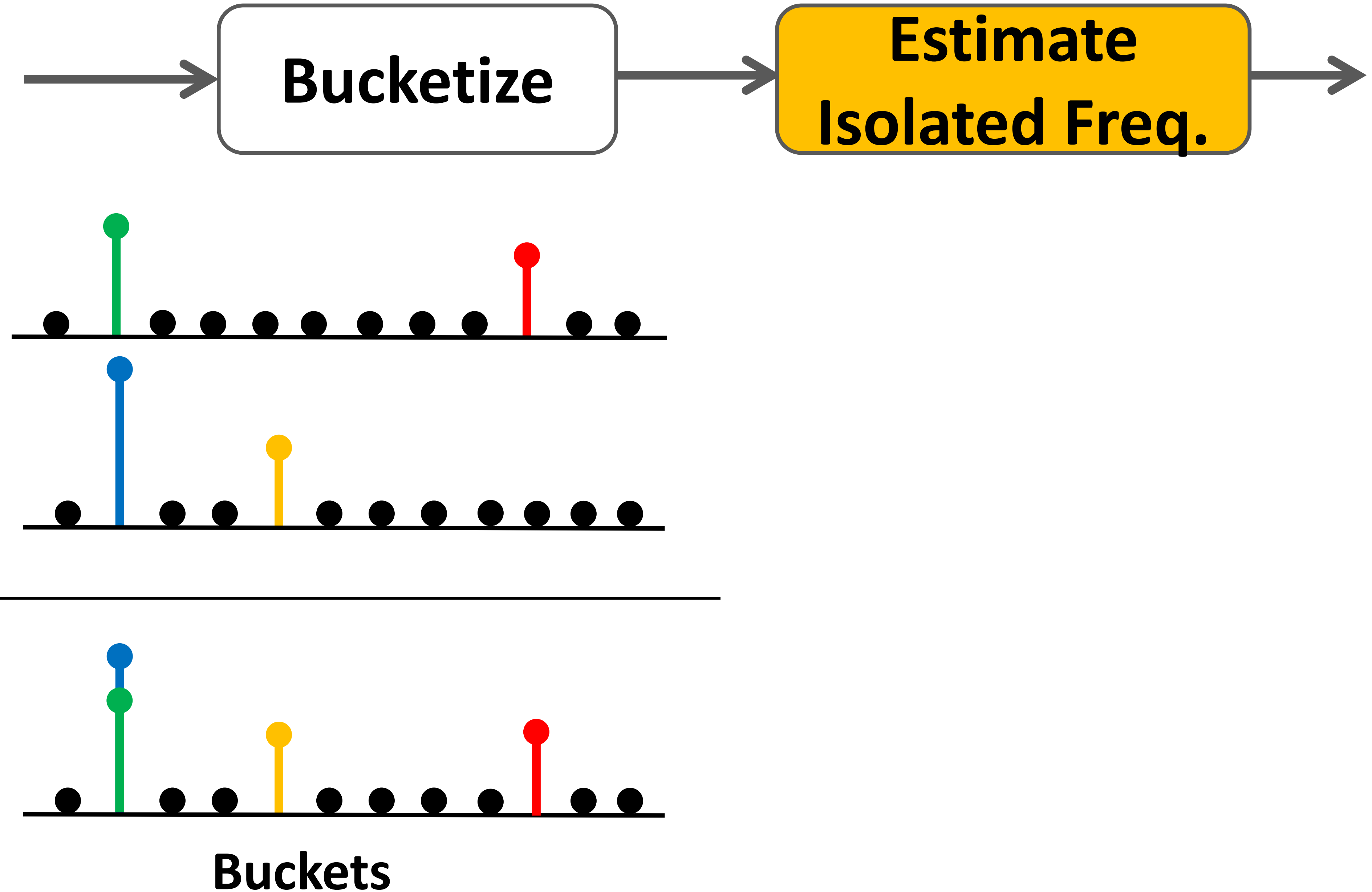

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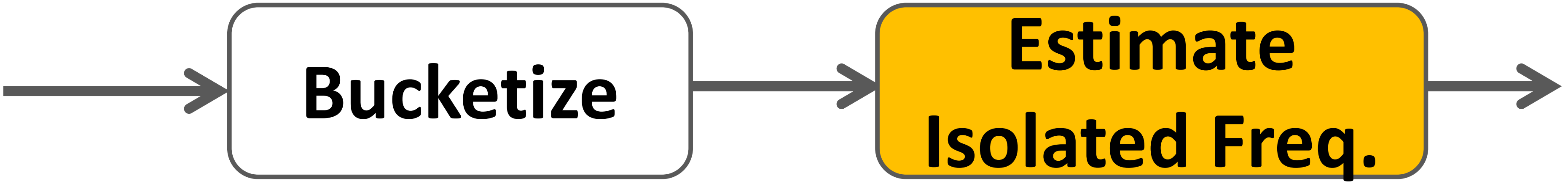
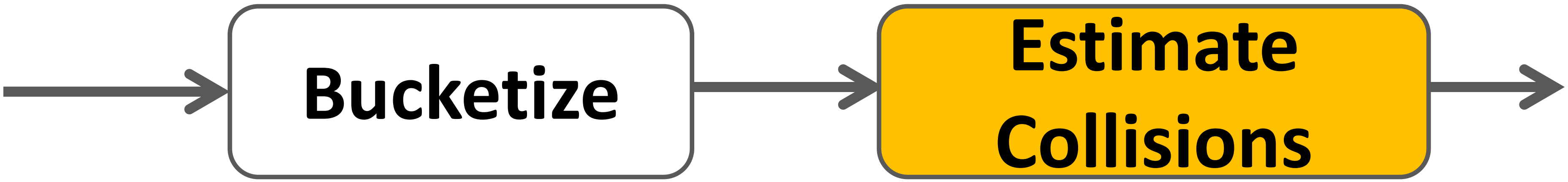


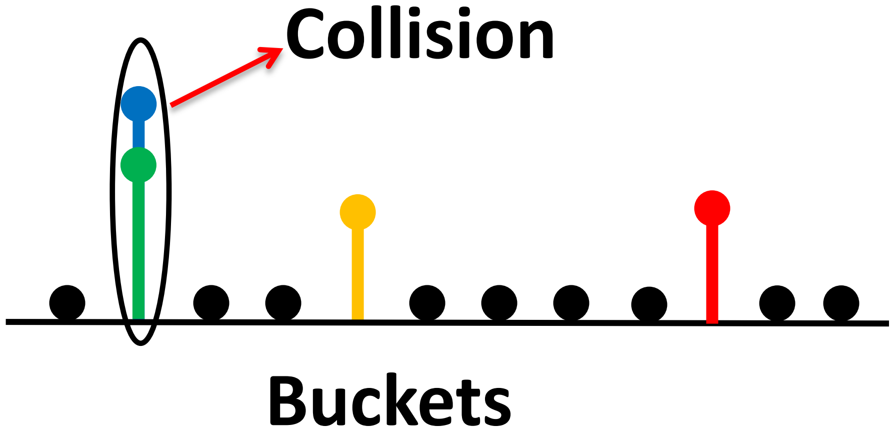
Shift in time ➔ Phase Rotation in frequency

Repeat bucketization with a **time shift τ**

Change in Phase = $2\pi f \tau$

How Does the Sparse Fourier Transform Work?





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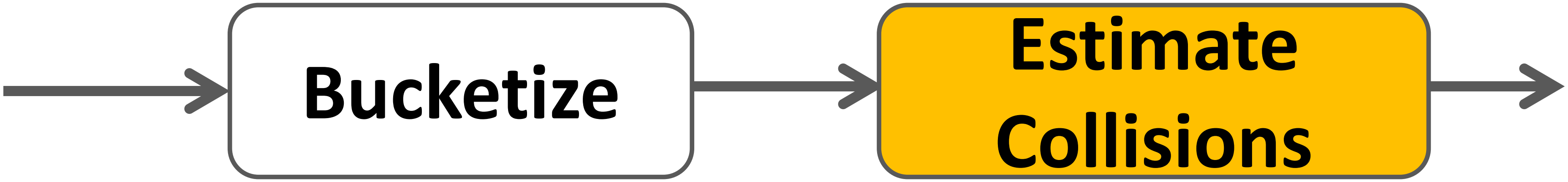
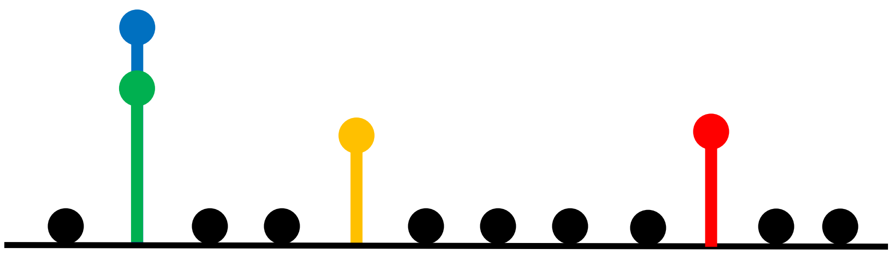
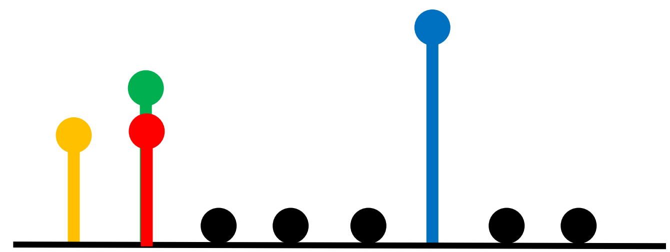


Bucketize multiple times using **co-prime sub-sampling**

Same frequencies don't collide in two bucketizations

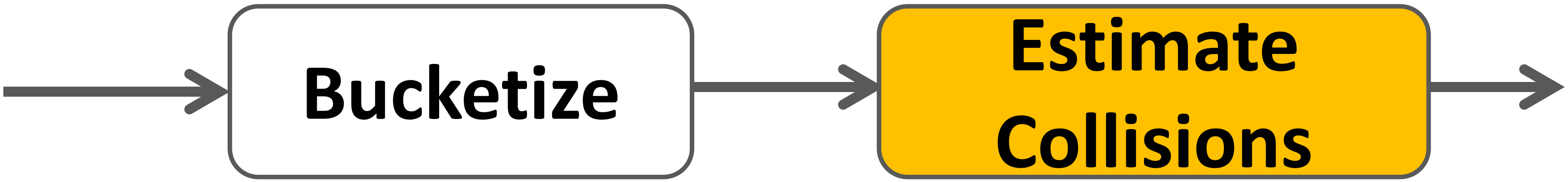


Buckets

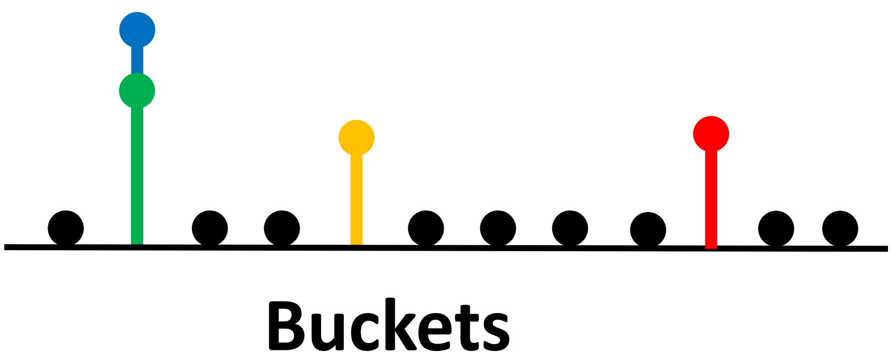


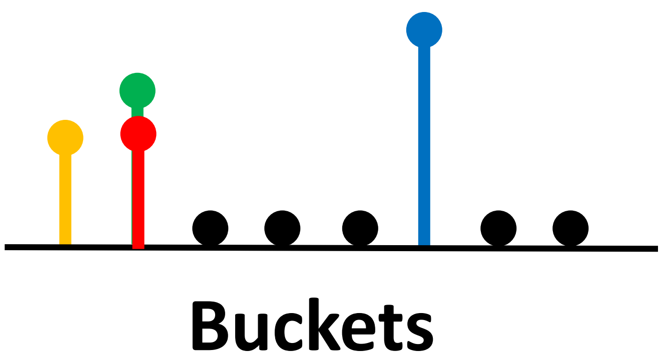
Buckets

How Does the Sparse Fourier Transform Work?



Estimate isolated freq. in one bucketization and subtract them from the other, and iterate …





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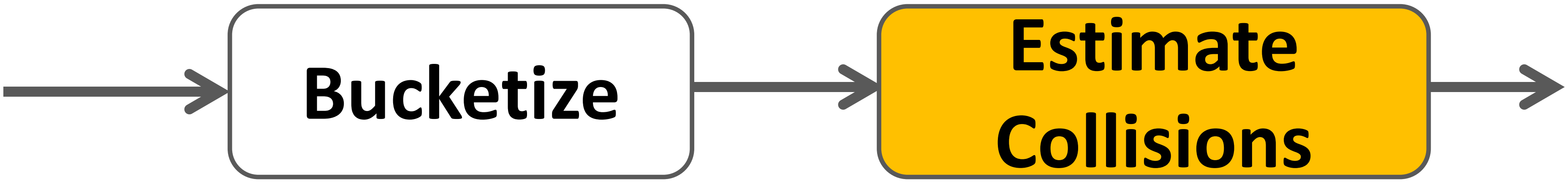

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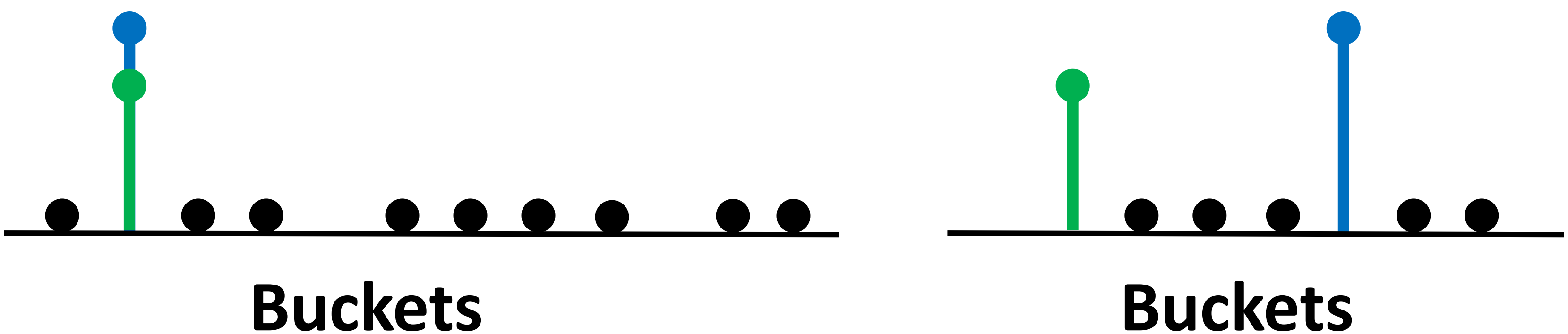


Theoretical Results

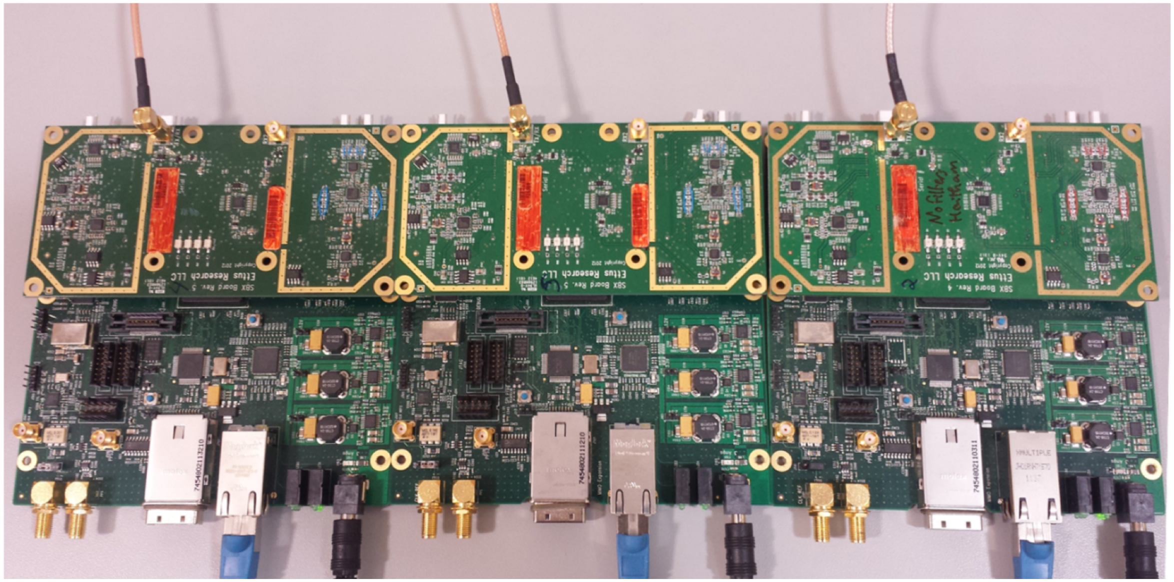
[Ghazi, Hassanieh, Indyk, Katabi, Price, Shi 2013]

- Exact k-sparse (randomly chosen):
  - $O(k)$ samples (Optimal)
  - $O(k \log n)$ computations
- Approximately k-sparse case:
  - $O(k \log(n))$ samples (Optimal)
  - $O(k \log^2 n)$ computations
- Similar result independently reached by [Pawar, Ramchandran 2013]

GHz Receiver for Sparse Signals

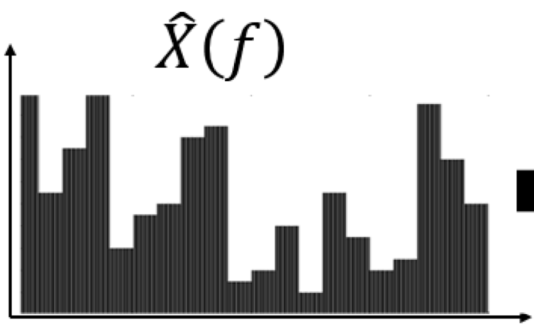
- Sub-sample the data → Can use low-speed ADCs
- Very fast algorithm → Lower-power consumption

Built a receiver that can capture GHz of spectrum using few tens of MHz ADCs with total digital bandwidth of 150 MHz

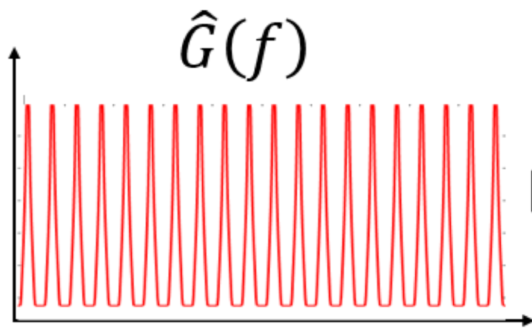


But, what if the spectrum is not sparse?!

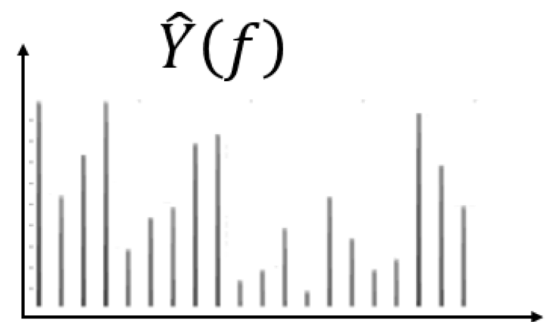
We make it sparse!



Non-Sparse Spectrum

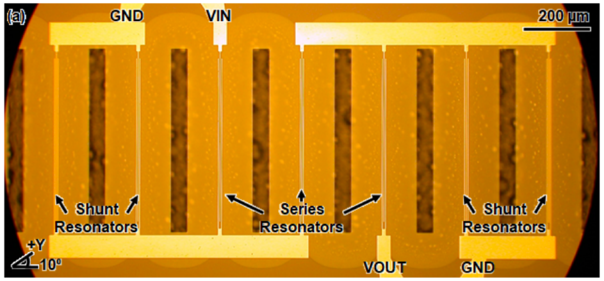


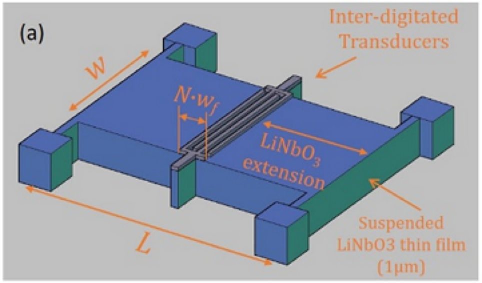
RF Hardware Comb Filter



Sparse Spectrum

Using MEMS acoustic resonators
→ built highly tunable RF Comb Filters



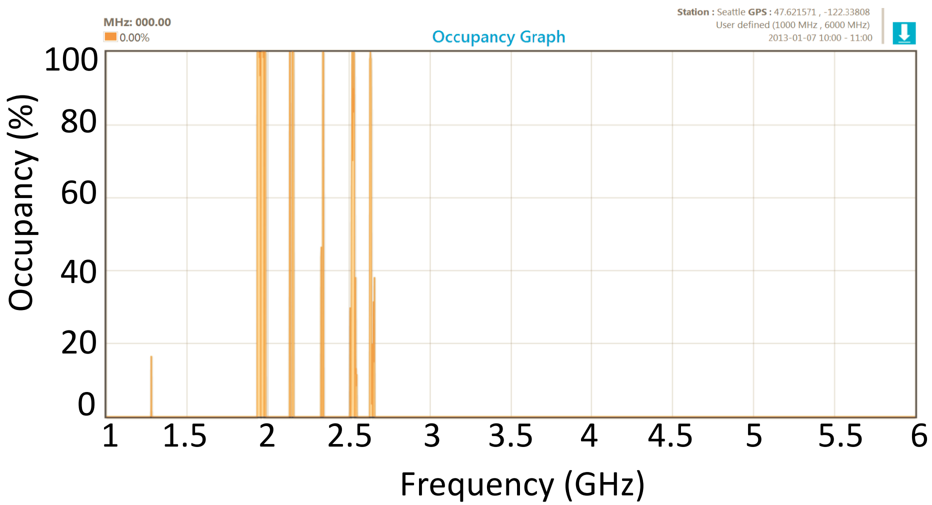


Use Sparse Fourier Transform

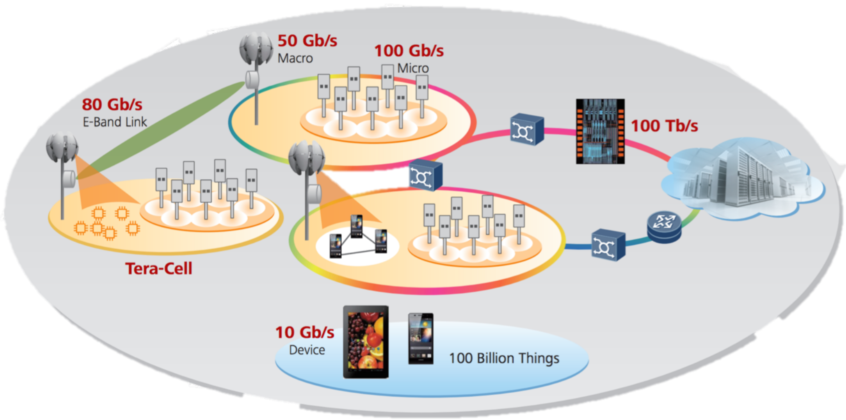
[Lu, Manzaneque, Yang, Zhou, Hassanieh, and Gong, 2018]

Real-time Applications to Wireless Networks

Spectrum Sensing & Acquisition



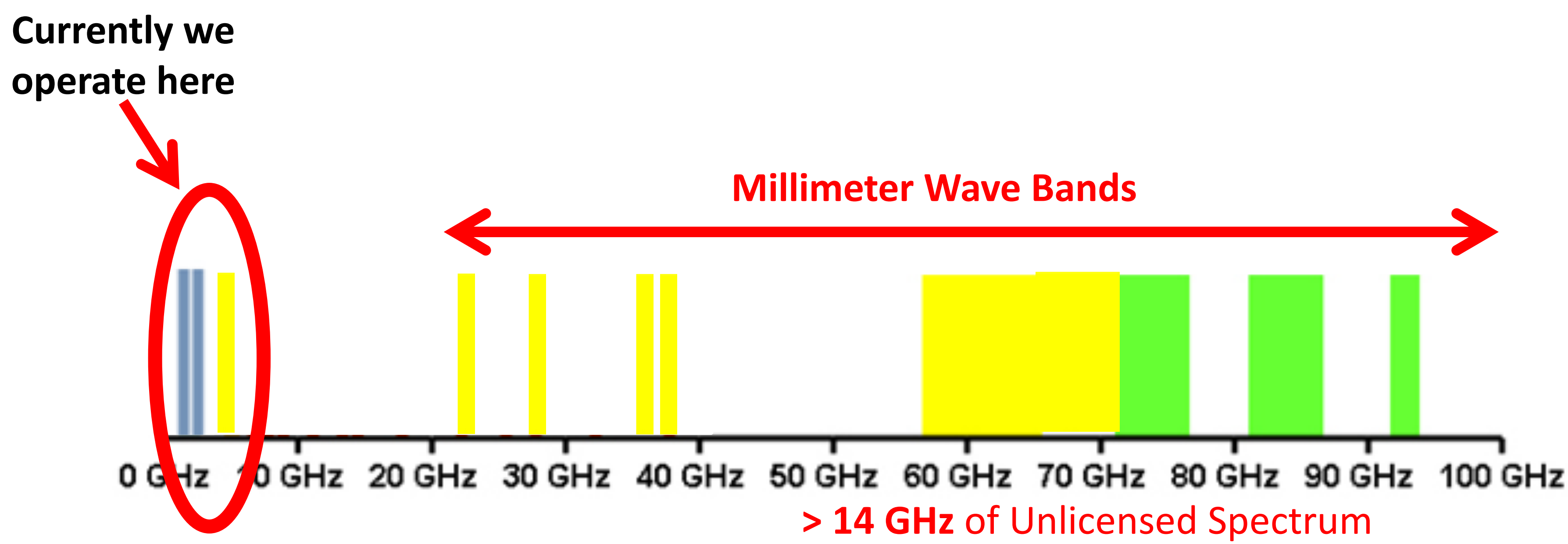
Millimeter Wave 5G Networks



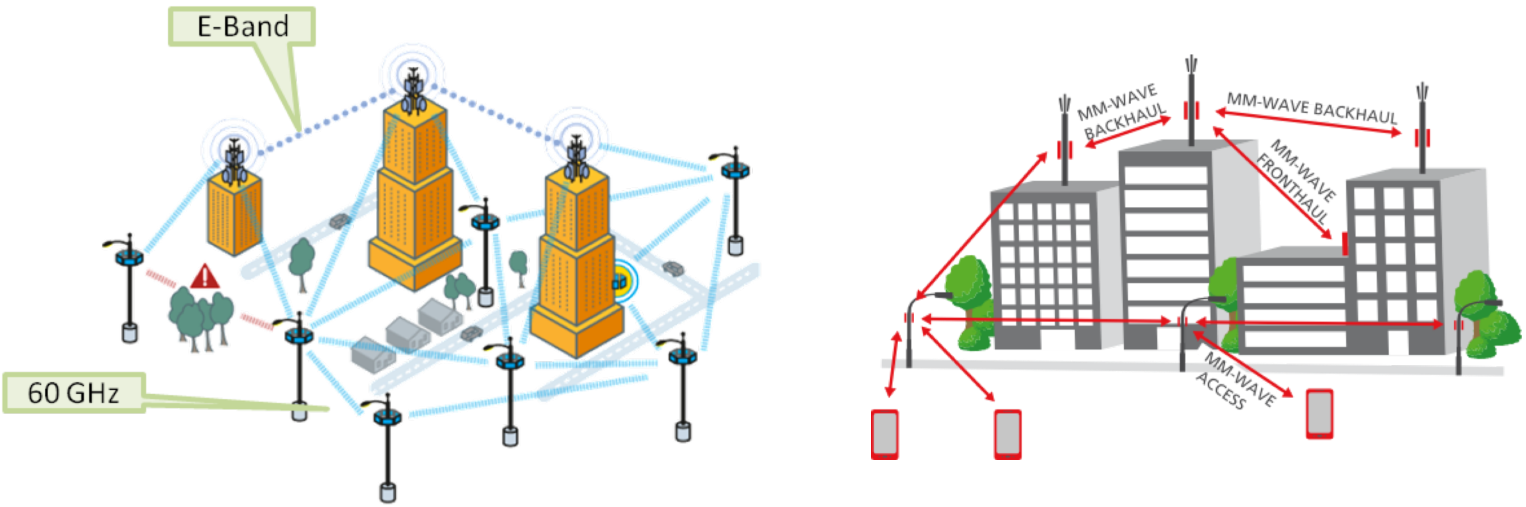
With Sparse Fourier Transform

Capture and sense GHz spectrum in real-time

Millimeter Wave for 5G



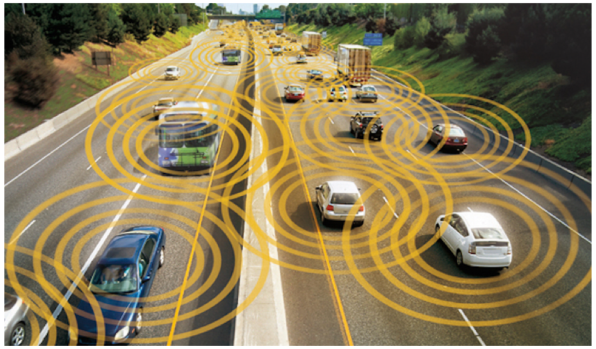
Millimeter Wave Wireless Applications



5G Wireless Backhaul & Access

Virtual Reality

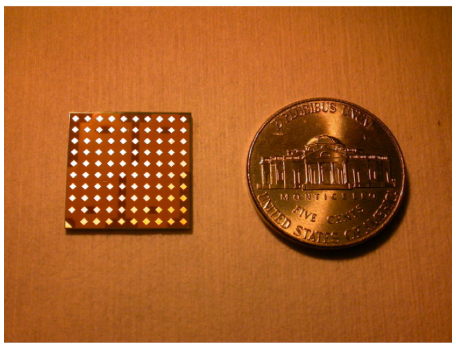
Wireless Data Centers

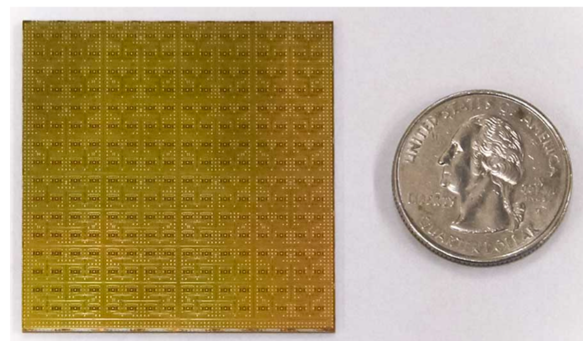


Connected Vehicles

Millimeter Waves Suffer from Large Attenuation

mmWave radios use phased antenna arrays to focus the power along one direction





Small Wavelength enables thousands of antennas to be packed into small space

→ Extremely narrow beams

mmWave changes how wireless systems operate

Today: Broadcast

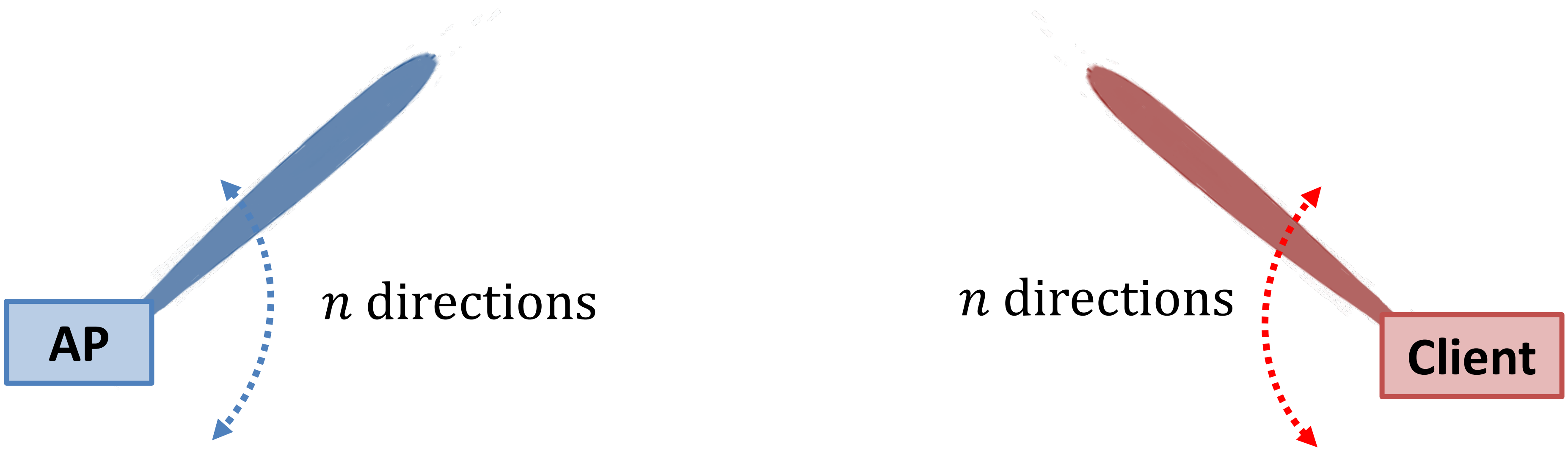
mmWave changes how wireless systems operate

mmWave: Pencil-beam Antennas

Communication is possible only when the beams are aligned

Naïve Approach: Exhaustive Scan

n : number of possible directions



$O(n^2)$ measurements → Too slow

802.11ad Scan

Stage 1: Client uses omni-directional; AP scans directions

802.11ad Scan

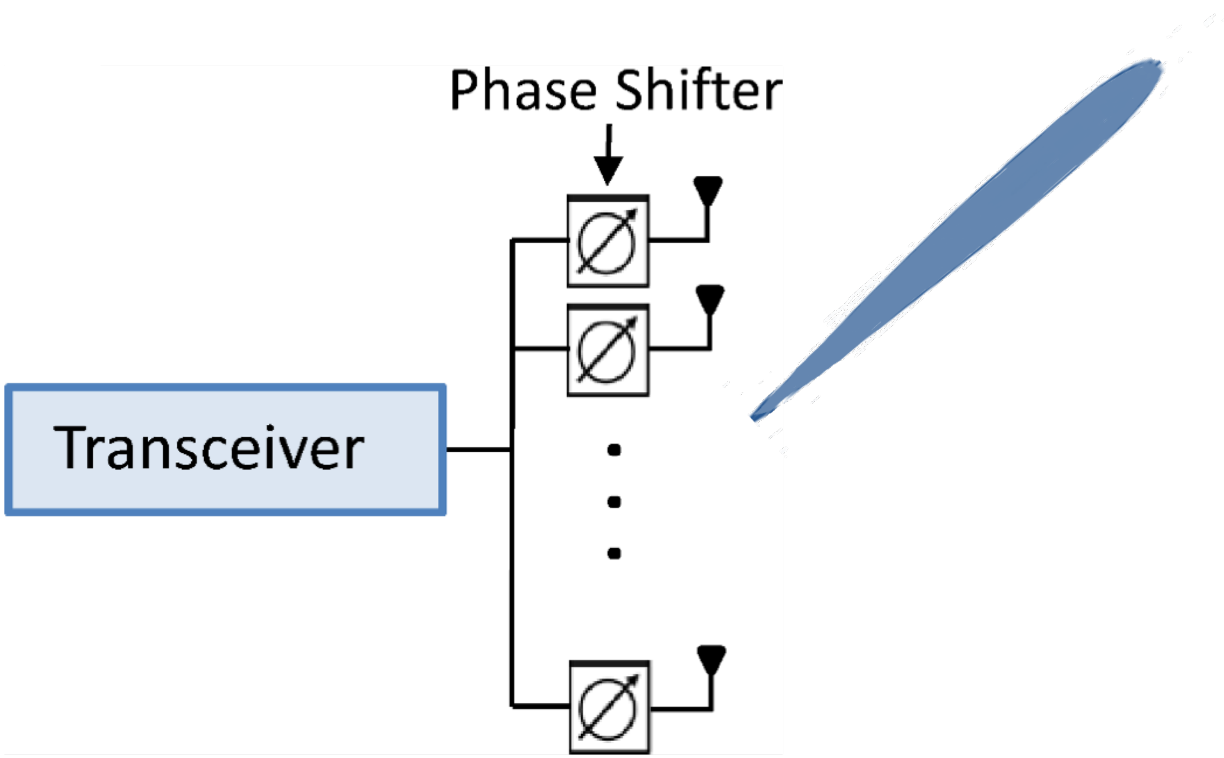
Stage 2: AP uses omni-directional; client scans directions

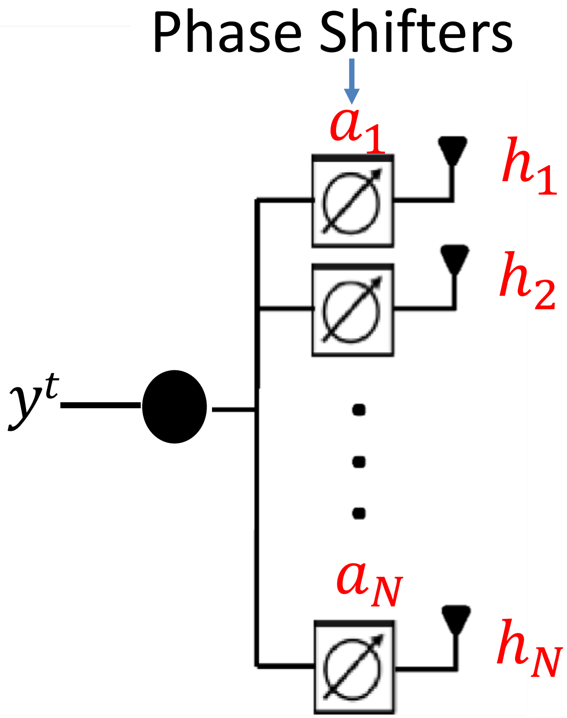
$O(n)$ measurements →Still Too Slow

[MOBICOM'14, SIGMETRICS'15, NSDI'16]

How can we find the right alignment in sublinear number of measurements without scanning all directions?

mmWave radios use phased arrays to create a beam





For an Antenna Array:

$\vec{h} = F'\vec{x}$, F' **is Inverse Fourier Matrix**

$$a_m = e^{j\phi_m}$$

$$y^t = \sum_{k}^{N} h_m e^{j\phi_{m,t}}$$

$$= \vec{a}^t \vec{h} = \vec{a}^t F' \vec{x}$$

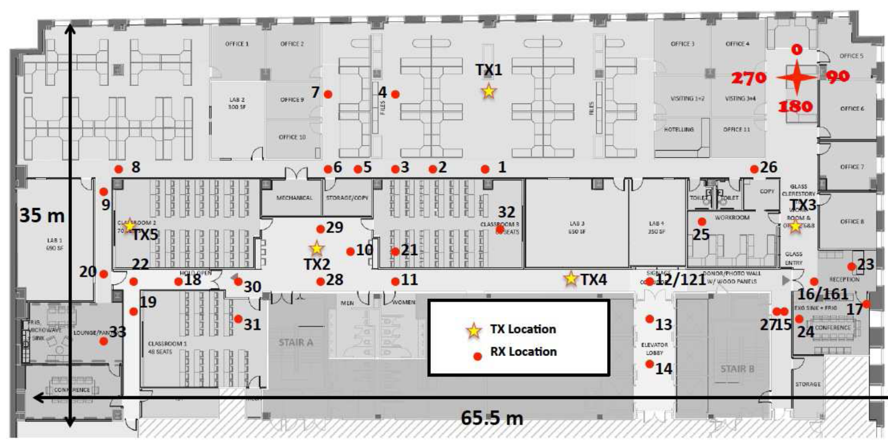
$\vdots$

$$\mathbf{Y} = \mathbf{A}\, F'\vec{x}$$

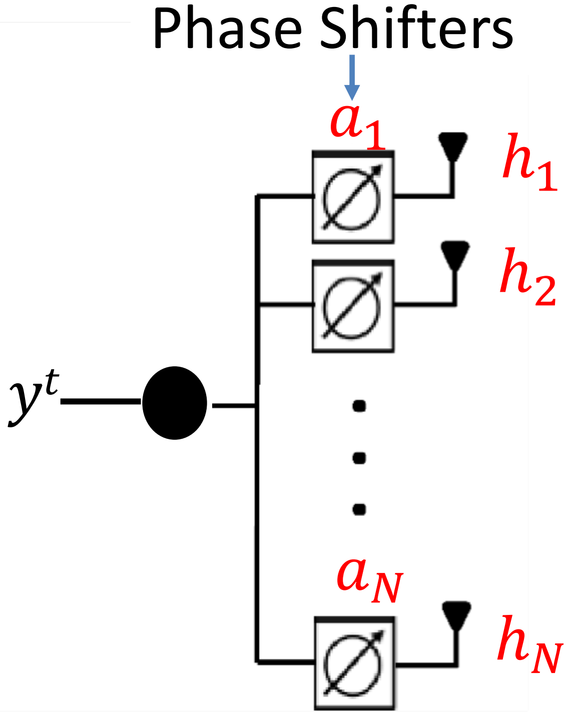
Leverage Channel Sparsity

- mmWave Wireless channels are very sparse: at most 3-4 paths exist between TX and RX [ICC'14, Proc. of IEEE'14, SIGMETRICS'15, NSDI'16…]

Outdoor



Indoor



For an Antenna Array:

$\vec{h} = F'\vec{x}$, F' **is Inverse Fourier Matrix**

$$a_m = e^{j\phi_m}$$

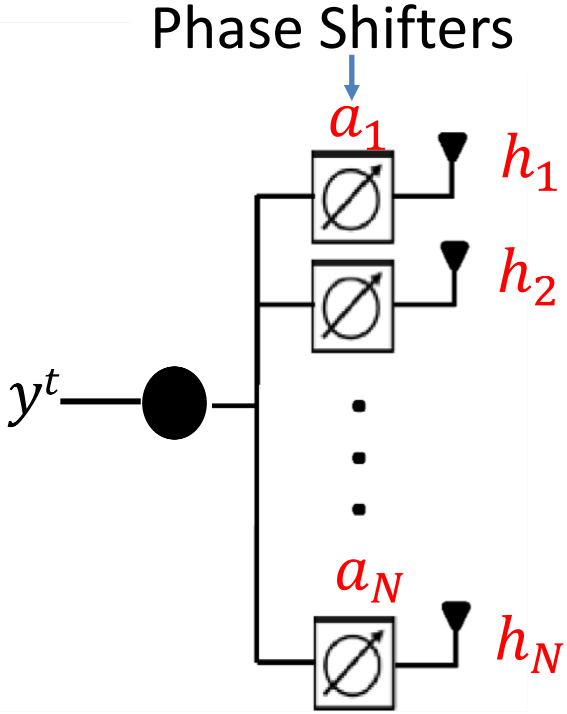
$$y^t = \sum_{k}^{N} h_m e^{j\phi_{m,t}} \times e^{j2\pi\Delta f_c t}$$

$$= \vec{a}^t \vec{h} = \vec{a}^t F' \vec{x}$$

⋮

$$Y = A\, F' \vec{x}$$

Carrier frequency offset (Δf_c) between TX and RX corrupts the phases of the measurements



For an Antenna Array:

$\vec{h} = F'\vec{x}$, F' **is Inverse Fourier Matrix**

$$a_m = e^{j\phi_m}$$

$$y^t = \left| \sum_{k}^{N} h_m e^{j\phi_{m,t}} \times e^{j2\pi\Delta f_c t} \right|$$

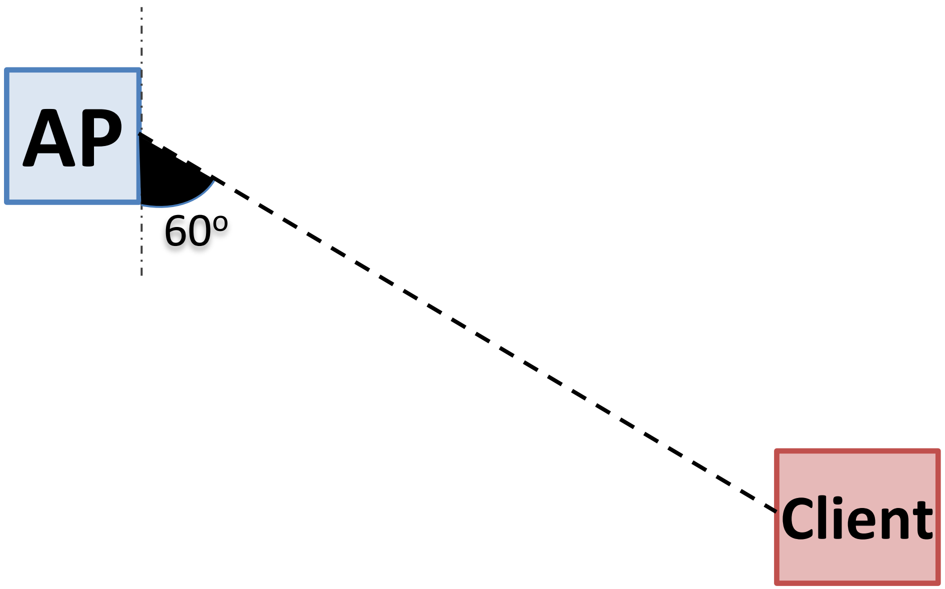
$$= |\vec{a}^t \vec{h}| = |\vec{a}^t F' \vec{x}|$$

⋮

$$\mathbf{Y} = |\mathbf{A}\, F'\vec{x}|$$

Sparse Phase Retrieval Problem

Solution Idea



Potential Direction of the Client:

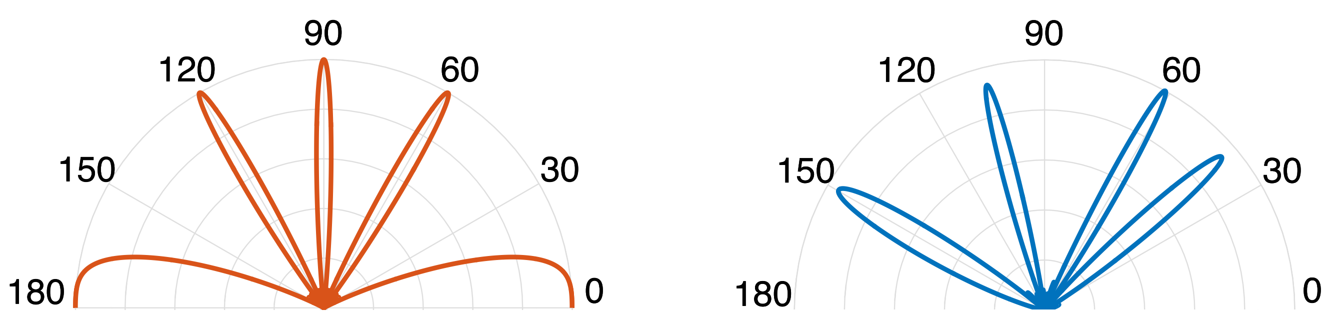
0°, 60°, 90° or 120°

40°, 60°, 100° or 150°

60° is direction of client

Construct a Multi-Armed Beam:

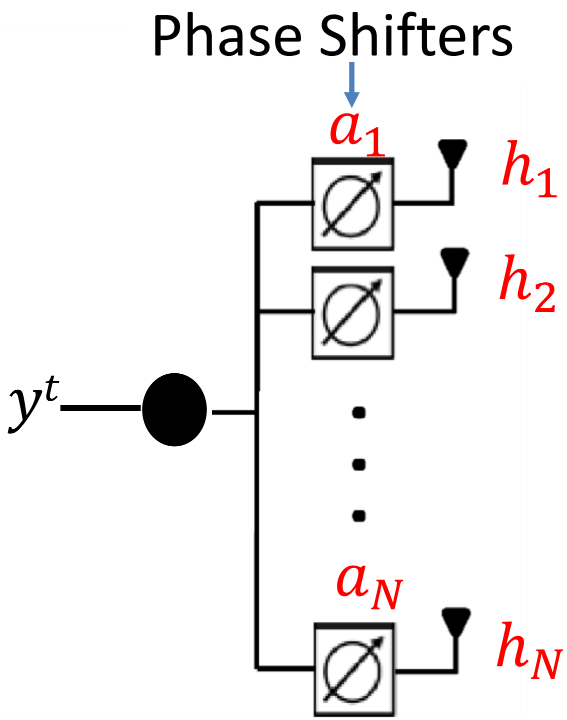
Simultaneously collects signals from multiple directions.



1. How can we generate multi-armed beams?

2. What is the best choice of multi-armed beams to quickly find the right direction?

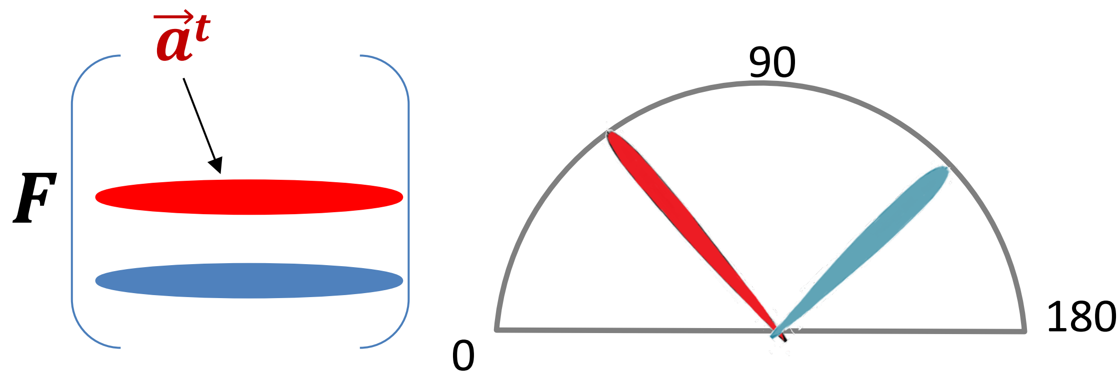
Creating Multi-Armed Beams



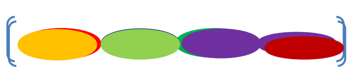
For an Antenna Array:

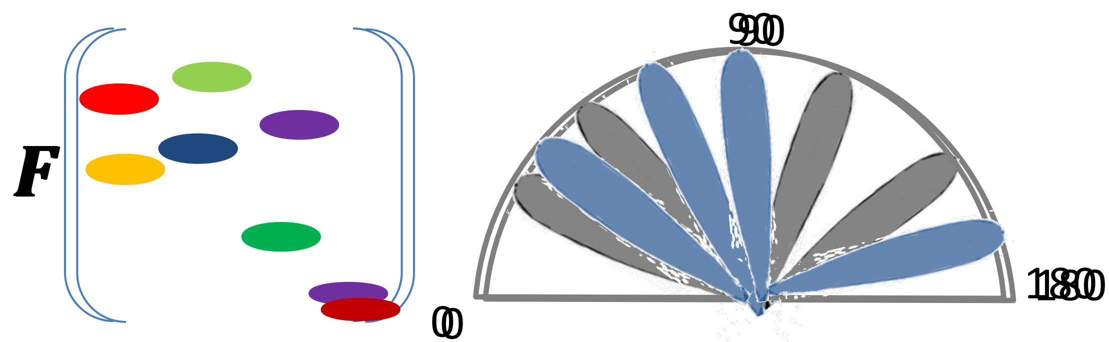
$\vec{h} = F'\vec{x}$, F' is Inverse Fourier Matrix

$y^t = |\vec{a}^t \vec{h}| = |\vec{a}^t F'\vec{x}|$



Divide $\vec{a}^t$ into segments

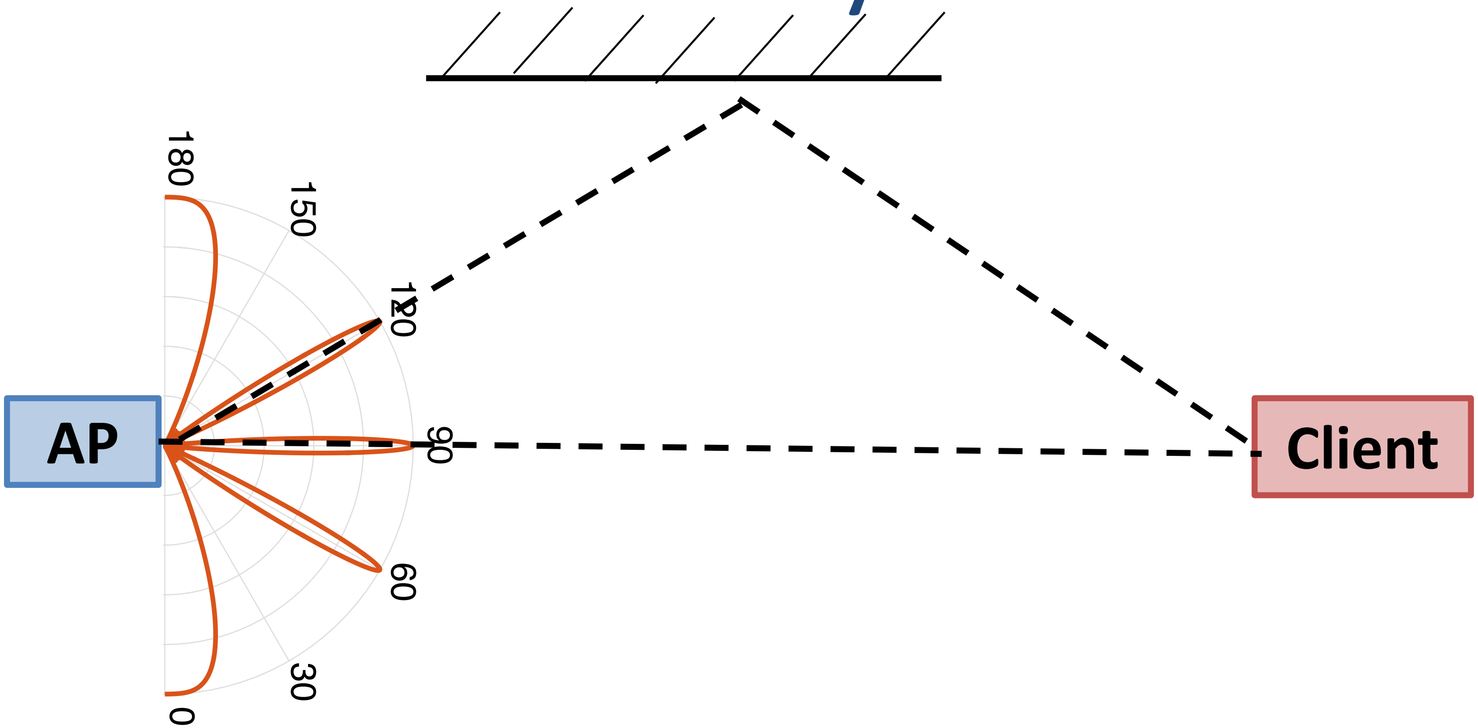
$\vec{a}^t =$ 



1. How can we generate multi-armed beams?

2. What is the best choice of multi-armed beams to quickly find the right direction?

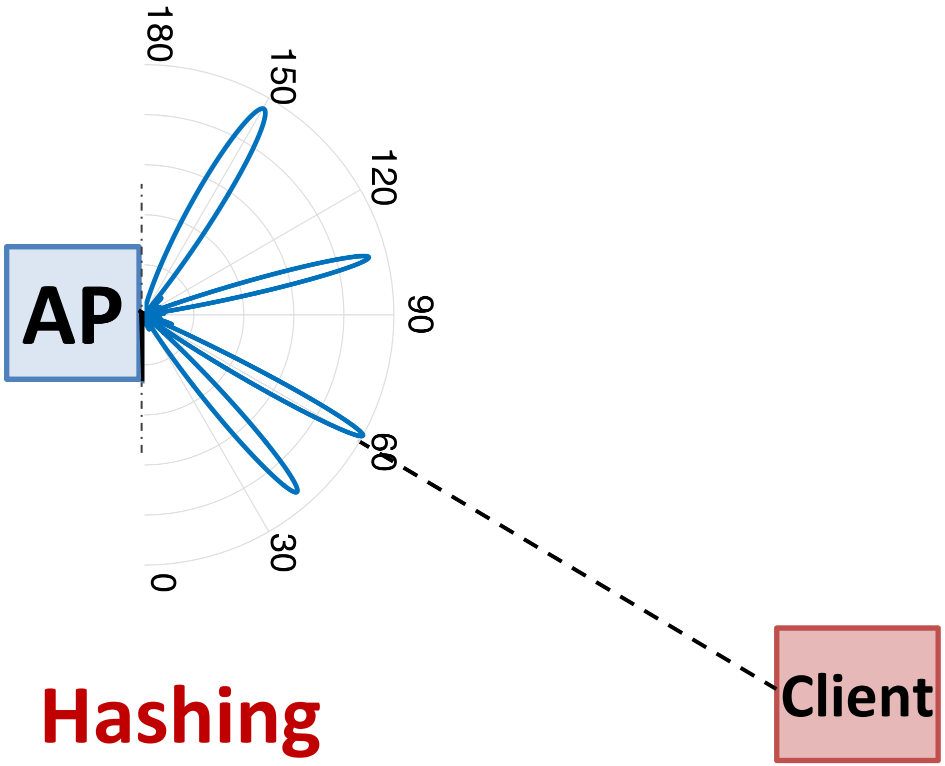
Why do we need to choose the beams carefully?

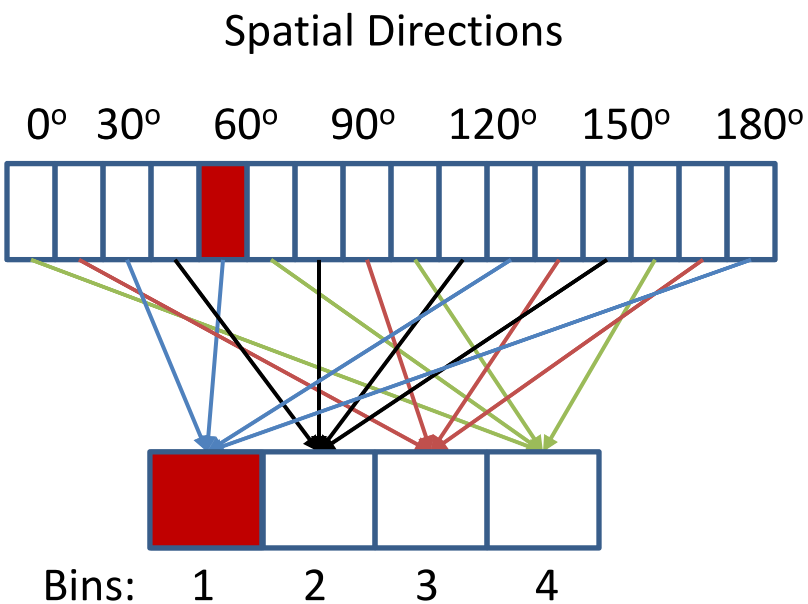


Signals traveling along different paths can cancel each other

What is the best choice of multi-armed beams?

Multi-Armed Beam ⟷ **Random Hash**





Hashing

- Pick multi-armed beams to create random hash functions

Voting

- Estimate the true direction using voting

1. How can we generate multi-armed beams?

2. What is the best choice of multi-armed beams to quickly find the right direction?

Theoretical Results

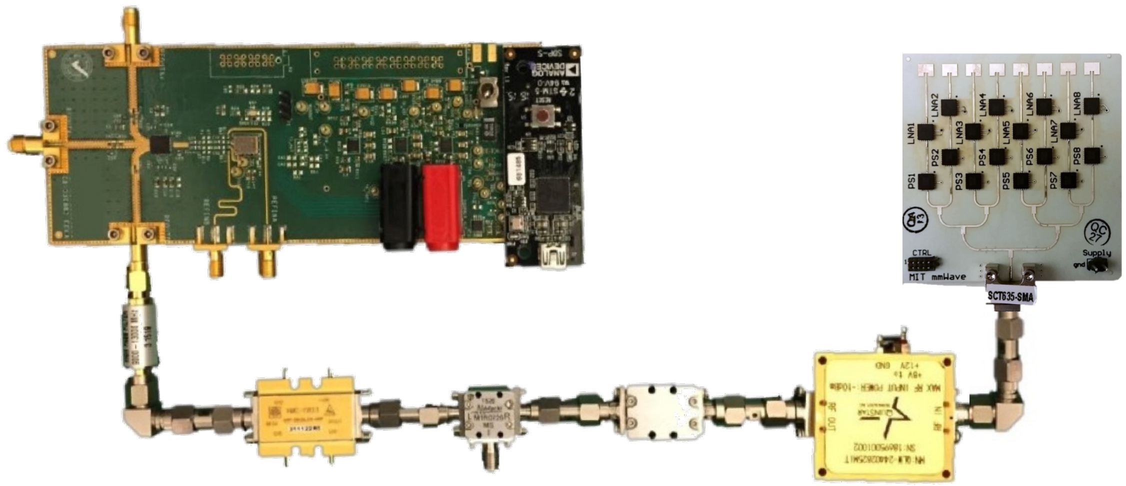
- n: # of spatial directions, k: # of signal paths
- Number of packets needed to discover direction of alignment:

Exhaustive Scan	802.11ad	Sparse FFT
$O(n^2)$	$O(n)$	$O(k \log n)$

Can find the best alignment in without scanning the space using $O(k \log n)$ measurements and $O(nk \log n)$ computations

Implementation and Evaluation

Built a Millimeter Wave Radio with a Phased Array.

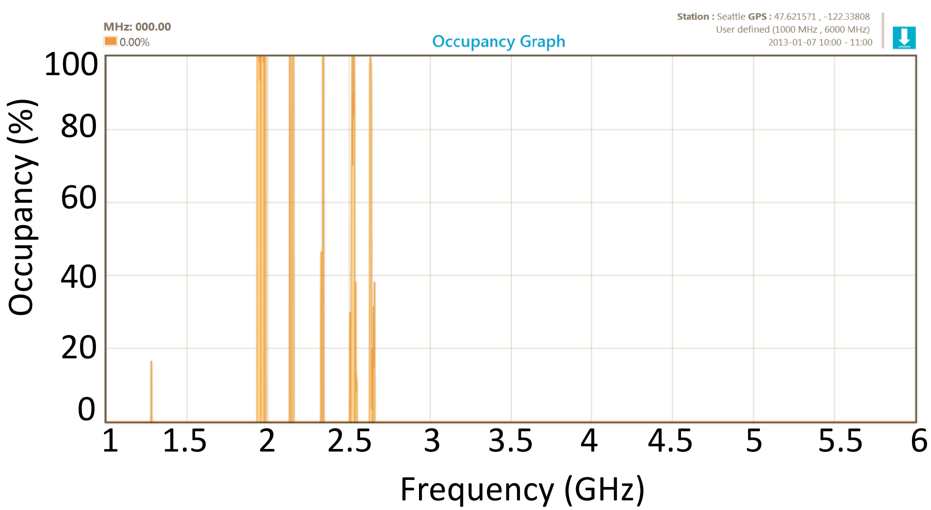


Achieves fast beam alignment: less than 1ms

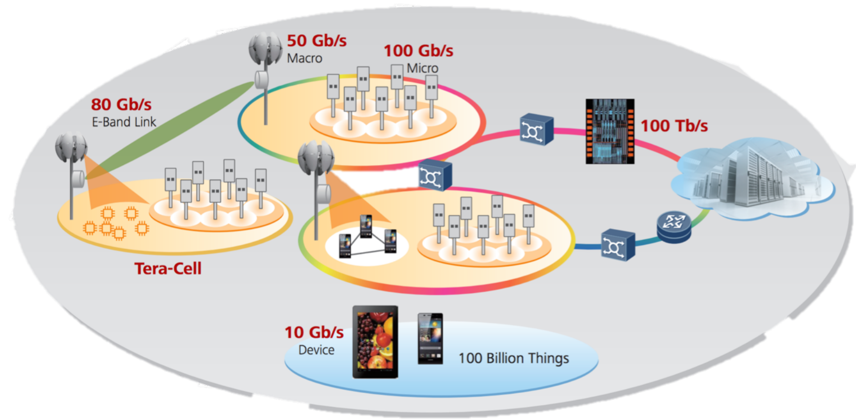
For $n = 256, 100\times$ faster than 802.11ad and $10000\times$ faster than exhaustive search

Real-time Applications to Wireless Networks

Spectrum Sensing & Acquisition



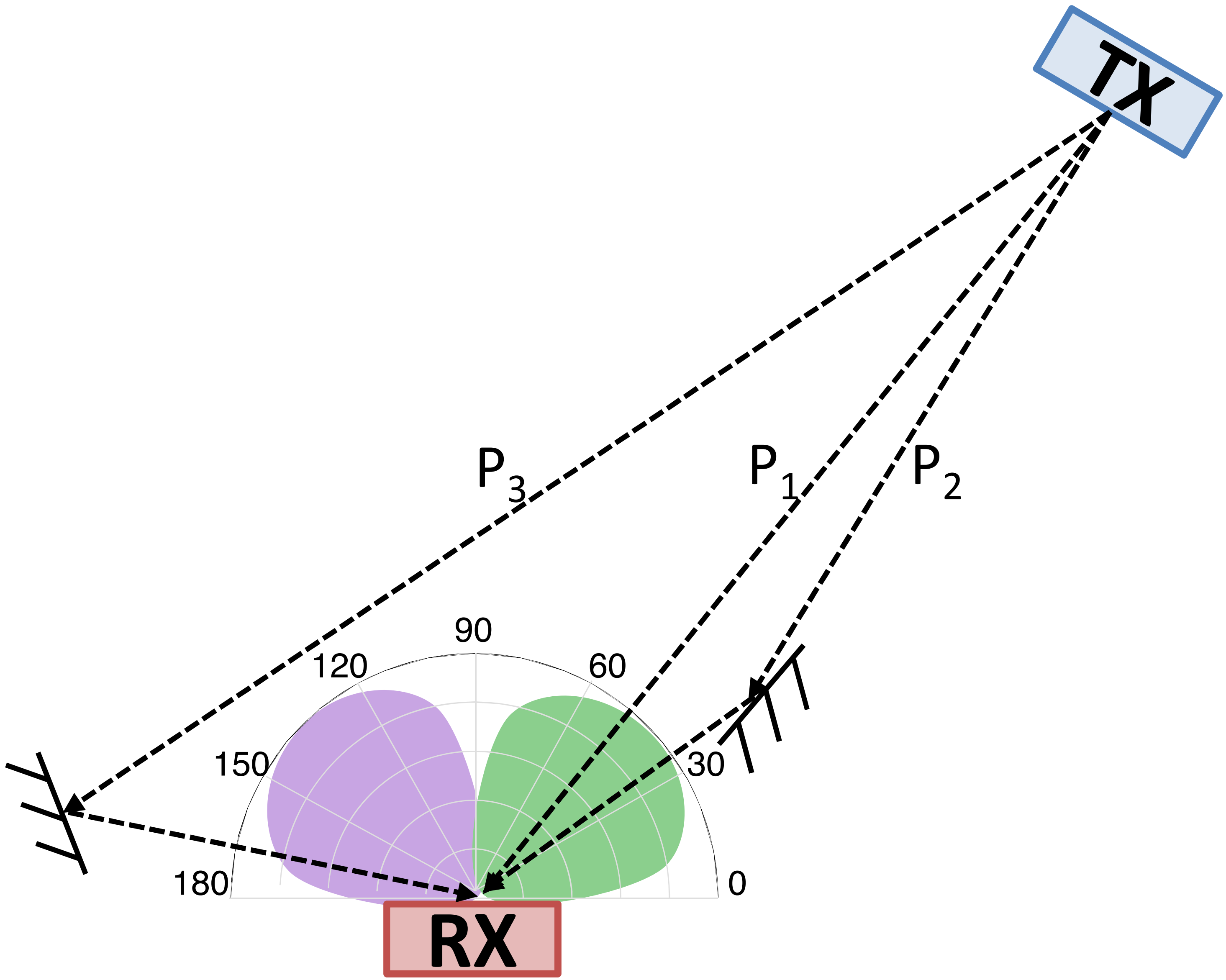
Millimeter Wave Networks



With Sparse Fourier Transform

Capture and sense GHz spectrum in real-time

Align the beams & quickly establish link


TX
P3
P1
P2
90
120
60
150
30
180
0
RX