

# A Complete Axiomatisation of the ZX-Calculus for Clifford+T Quantum Mechanics

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## 1 Introduction

## 2 ZX-Calculus

ZX-Calculus

Diagram Transformations

Universality, Soundness, Completeness

## 3 Completion for the First Approximately Universal Fragment

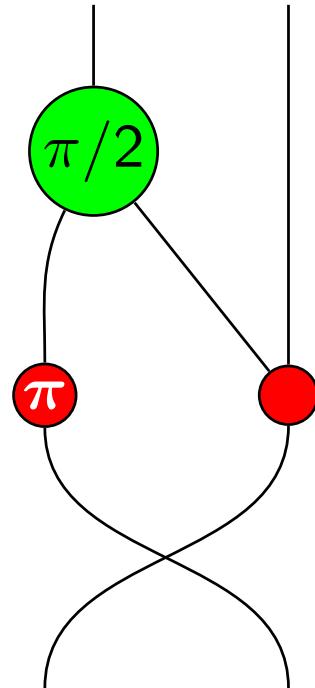
The ZW-Calculus

ZX-rules from the ZW-Calculus

Completeness of the  $\frac{\pi}{4}$ -fragment

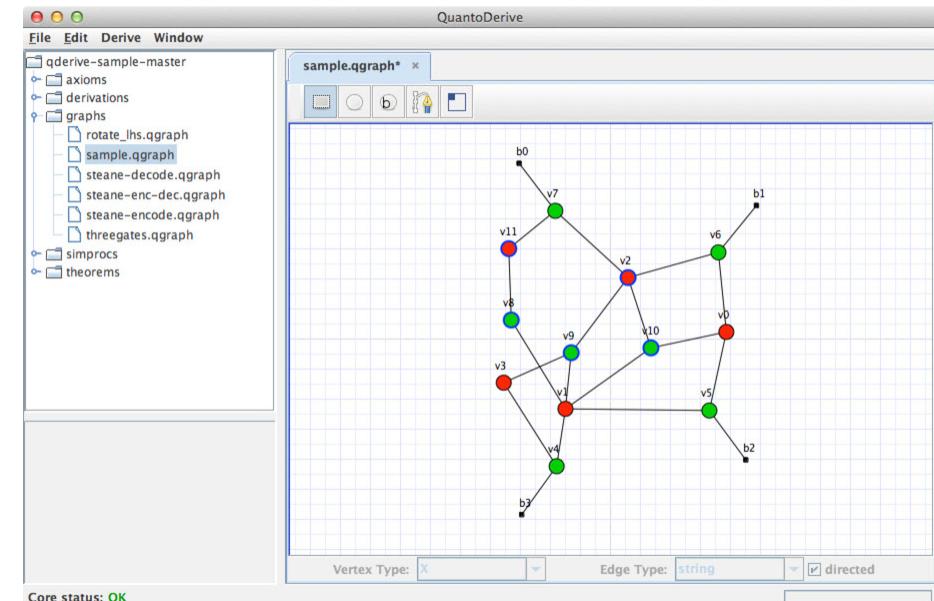
## 4 Conclusion

# ZX-calculus [Coecke, Duncan]



Foundations of quantum mechanics.

Tool for QIP: Prove protocols, develop quantum error correcting codes, intermediate language...



[quantomatic.github.io](https://quantomatic.github.io)

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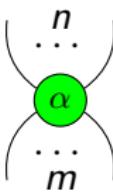
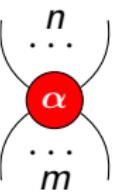
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# ZX-Calculus Generators and Semantics

[Coecke,Duncan'08]

with  $n, m \in \mathbb{N}$  and  $\alpha \in \mathbb{R}$

# ZX-Calculus Generators and Semantics

[Coecke,Duncan'08]

The standard interpretation  $\llbracket \cdot \rrbracket :$

	$\mapsto \begin{pmatrix} 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & e^{i\alpha} \end{pmatrix}$		$\mapsto \left[ \begin{array}{c c} \text{---} & \text{---} \end{array} \right]^{\otimes m} \circ \left[ \begin{array}{c c} \text{---} & \text{---} \end{array} \right] \left[ \begin{array}{c c} \text{---} & \text{---} \end{array} \right] \circ \left[ \begin{array}{c c} \text{---} & \text{---} \end{array} \right]^{\otimes n}$
	$\mapsto \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$		$\mapsto (1)$
	$\mapsto \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$		$\mapsto (1 \ 0 \ 0 \ 1)$
	$\mapsto \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$		$\mapsto \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix}$

with  $n, m \in \mathbb{N}$  and  $\alpha \in \mathbb{R}$

# ZX-Calculus Compositions

$$\left[ \begin{array}{c|c} \dots & \dots \\ \hline D_1 & D_2 \\ \hline \dots & \dots \end{array} \right] = \left[ \begin{array}{c|c} \dots & \dots \\ \hline D_1 & \\ \hline \dots & \dots \end{array} \right] \otimes \left[ \begin{array}{c|c} \dots & \dots \\ \hline & D_2 \\ \hline \dots & \dots \end{array} \right] \text{ and } \left[ \begin{array}{c|c} \dots & \dots \\ \hline D_1 & \\ \hline \dots & \dots \\ \hline D_2 & \\ \hline \dots & \dots \end{array} \right] = \left[ \begin{array}{c|c} \dots & \dots \\ \hline & D_2 \\ \hline \dots & \dots \end{array} \right] \circ \left[ \begin{array}{c|c} \dots & \dots \\ \hline & D_1 \\ \hline \dots & \dots \end{array} \right]$$

## ZX-Calculus Compositions

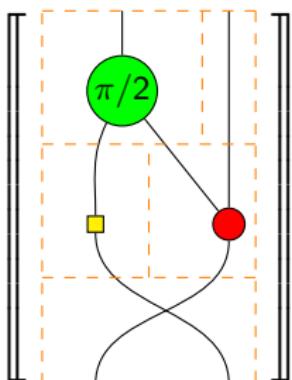
$$\left[ \begin{array}{c|c} \dots & \dots \\ \hline D_1 & D_2 \end{array} \right] = \left[ \begin{array}{c|c} \dots & \dots \\ \hline D_1 & \dots \end{array} \right] \otimes \left[ \begin{array}{c|c} \dots & \dots \\ \hline \dots & D_2 \end{array} \right] \text{ and } \left[ \begin{array}{c|c} \dots & \dots \\ \hline D_1 & \dots \\ \hline \dots & D_2 \end{array} \right] = \left[ \begin{array}{c|c} \dots & \dots \\ \hline \dots & D_2 \end{array} \right] \circ \left[ \begin{array}{c|c} \dots & \dots \\ \hline D_1 & \dots \end{array} \right]$$

E.g.

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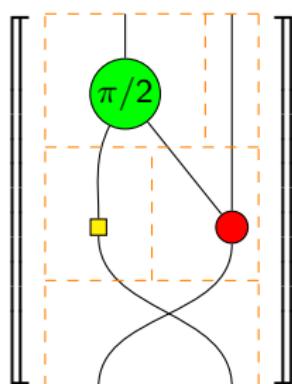
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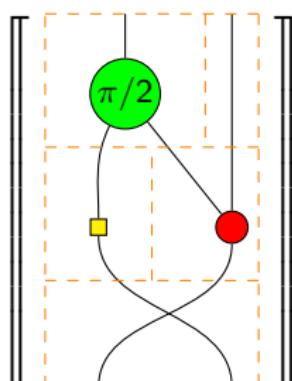
E.g.

$$\left[ \begin{array}{c|c|c} & & \\ \hline & \text{---} & \text{---} \\ \hline \text{---} & \text{---} & \text{---} \\ \hline & \text{---} & \text{---} \\ \hline \end{array} \right] = \left[ \begin{array}{c} \times \\ \diagdown \diagup \end{array} \right] \circ \left( \left[ \begin{array}{c|c} & \\ \hline & \square \end{array} \right] \otimes \left[ \begin{array}{c} \vee \\ \diagup \diagdown \end{array} \right] \right) \circ \left( \left[ \begin{array}{c} \pi/2 \\ \text{---} \end{array} \right] \otimes \left[ \begin{array}{c} | \\ \text{---} \end{array} \right] \right)$$


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E.g.

$$\begin{aligned}
 \left[ \begin{array}{c|c} \text{---} & \text{---} \\ \hline \text{---} & \text{---} \end{array} \right] &= \left[ \begin{array}{c} \text{---} \\ \hline \text{---} \end{array} \right] \circ \left( \left[ \begin{array}{c} \text{---} \\ \hline \text{---} \end{array} \right] \otimes \left[ \begin{array}{c} \text{---} \\ \hline \text{---} \end{array} \right] \right) \circ \left( \left[ \begin{array}{c} \text{---} \\ \hline \text{---} \end{array} \right] \otimes \left[ \begin{array}{c} \text{---} \\ \hline \text{---} \end{array} \right] \right) \\
 &= \frac{1}{2} \begin{pmatrix} 1 & 0 & 0 & i \\ 1 & 0 & 0 & -i \\ 0 & 1 & i & 0 \\ 0 & 1 & -i & 0 \end{pmatrix}
 \end{aligned}$$


# Only Connectivity Matters

$$\text{ ↗ } = | = \text{ ↘ } \quad \text{ ↗ } = | \quad \text{ ↗ } = \cap \quad \text{ ↗ } = \cup$$

$$\text{ ↗ } = \text{ ↗ } \quad \text{ ↗ } = \text{ ↗ } \quad \text{ ↗ } = \text{ ↗ }$$

+ colour-swapped versions

## ZX-Calculus Axioms [Jeandel,Perdrix,Vilmart,Wang'17]

$$\begin{array}{c} \text{Diagram showing two green nodes } \alpha \text{ and } \beta \text{ connected by a dashed line, with other wires labeled } \dots \end{array} = \begin{array}{c} \text{Diagram showing a single green node } \alpha + \beta \text{ with multiple outgoing wires labeled } \dots \end{array} \text{ (S1)}$$

$$\begin{array}{c} \text{Diagram showing a vertical wire with a green dot at the top} \end{array} = \begin{array}{c} \text{Diagram showing a vertical line with a green circle at the top} \end{array} \text{ (S2)}$$

$$\begin{array}{c} \text{Diagram showing a horizontal wire with a green dot at the right end} \end{array} = \begin{array}{c} \text{Diagram showing a horizontal wire with a green circle at the right end} \end{array} \text{ (S3)}$$

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# ZX-Calculus Axioms [Jeandel,Perdrix,Vilmart,Wang'17]

$$\begin{array}{c} \text{Diagram showing two green nodes } \alpha \text{ and } \beta \text{ connected by wires, with a dashed wire connecting them.} \\ = \\ \text{Diagram showing a single green node labeled } \alpha + \beta \text{ with multiple outgoing wires.} \end{array} \quad (\text{S1})$$

$$\begin{array}{c} \text{Diagram showing a vertical wire with a green dot at the top.} \\ = \\ \text{Diagram showing a vertical wire with a black vertical bar in the middle.} \end{array} \quad (\text{S2})$$

$$\begin{array}{c} \text{Diagram showing a horizontal wire with a green dot at the left end.} \\ = \\ \text{Diagram showing a horizontal wire with a green dot at the right end.} \end{array} \quad (\text{S3})$$

$$\begin{array}{c} \text{Diagram showing a red dot above a green dot, which is connected to a wire.} \\ = \\ \text{Diagram showing two separate red dots on vertical wires.} \end{array} \quad (\text{B1})$$

$$\begin{array}{c} \text{Diagram showing four nodes: two red (top) and two green (bottom), connected by various wires.} \\ = \\ \text{Diagram showing a red dot above a green dot, which is connected to a wire.} \end{array} \quad (\text{B2})$$

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$$\begin{array}{c} \text{Diagram showing two green nodes } \alpha \text{ and } \beta \text{ connected by wires, with a dashed wire connecting them.} \\ = \\ \text{Diagram showing a single green node labeled } \alpha + \beta \text{ with multiple outgoing wires.} \end{array} \quad (\text{S1})$$

$$\begin{array}{c} \text{Diagram showing a single green node with a vertical wire passing through it.} \\ = \\ \text{Diagram showing a vertical line with a hole in the middle.} \end{array} \quad (\text{S2})$$

$$\begin{array}{c} \text{Diagram showing a curved wire connecting two nodes.} \\ = \\ \text{Diagram showing a green node with a curved wire attached to it.} \end{array} \quad (\text{S3})$$

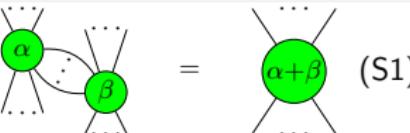
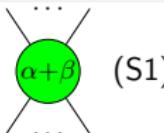
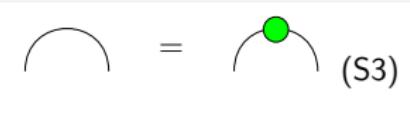
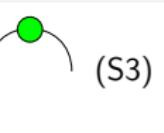
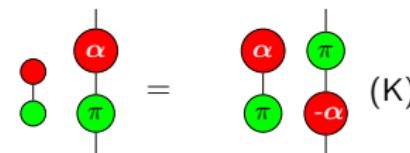
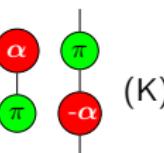
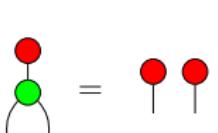
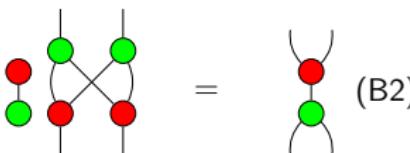
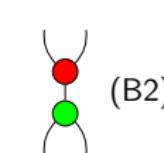
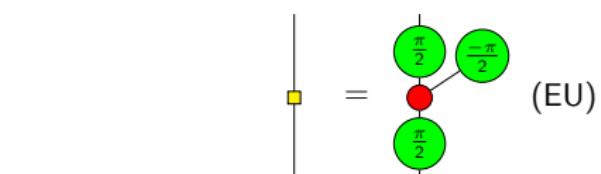
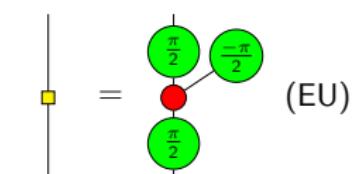
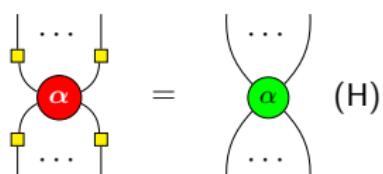
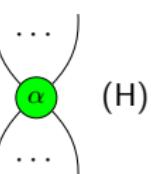
$$\begin{array}{c} \text{Diagram showing two red nodes } \alpha \text{ and } \pi \text{ connected by a vertical wire, with a green node } \pi \text{ below it.} \\ = \\ \text{Diagram showing a red node } \alpha \text{ connected to a green node } \pi \text{ by a vertical wire, with a red node } -\alpha \text{ below the green node.} \end{array} \quad (\text{K})$$

$$\begin{array}{c} \text{Diagram showing two red nodes connected by a vertical wire, with a green node below the bottom one.} \\ = \\ \text{Diagram showing two red nodes connected by a vertical wire.} \end{array} \quad (\text{B1})$$

$$\begin{array}{c} \text{Diagram showing four nodes: two red and two green, arranged in a square-like pattern with crossing wires.} \\ = \\ \text{Diagram showing a red node connected to a green node by a vertical wire, with a red node below the green node.} \end{array} \quad (\text{B2})$$

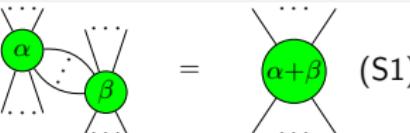
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# ZX-Calculus Axioms [Jeandel,Perdrix,Vilmart,Wang'17]

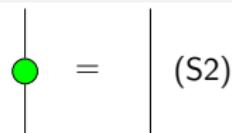
	=		(S1)		=		(S2)		=		(S3)
	=		(K)		=		(B1)		=		(B2)
	=		(EU)		=		(H)				

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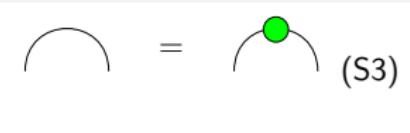
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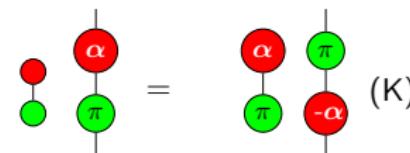
$$= \quad (\text{S1})$$



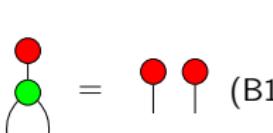
$$= \quad (\text{S2})$$



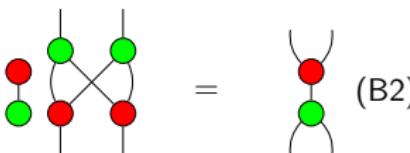
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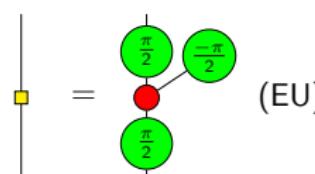
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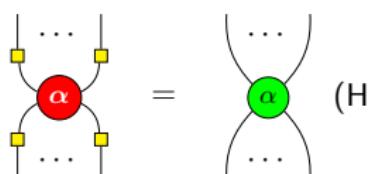
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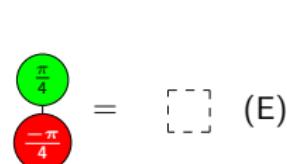
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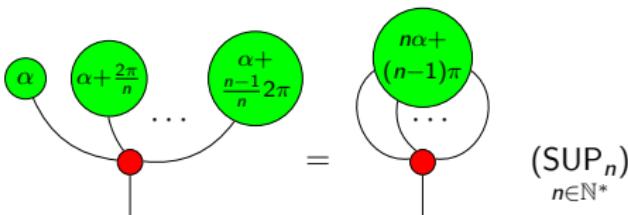
$$= \quad (\text{EU})$$



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$$= \quad [\quad] \quad (\text{E})$$



$$= \quad (\text{SUP}_n)_{n \in \mathbb{N}^*}$$

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# Universality, Soundness, Completeness

- ZX-Diagrams are universal:

$$\forall M \in \mathcal{M}_{2^n \times 2^m}(\mathbb{C}) \quad \exists D \in ZX, \quad \llbracket D \rrbracket = M$$

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$\frac{\pi}{p}$ -fragment: the restriction to angles multiple of  $\frac{\pi}{p}$

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not uni-  
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$$(\llbracket D_1 \rrbracket = \llbracket D_2 \rrbracket) \Rightarrow (ZX \vdash D_1 = D_2)$$

- not universal {
- ✓  $\pi$ -fragment. [Duncan, Perdrix'13]
  - ✓  $\frac{\pi}{2}$ -fragment. [Backens'13]
  - ✓ single-qubit  $\frac{\pi}{4}$ -fragment. [Backens'14]
  - ?  $\frac{\pi}{4}$ -fragment ( $\equiv$  Clifford+T): open question.

$\frac{\pi}{p}$ -fragment: the restriction to angles multiple of  $\frac{\pi}{p}$

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univ. ✗ No in general (with angles in  $\mathbb{R}$ ). [de Witt, Zamdzhev'14]

## Theorem (Approx. Universality)

$$\forall M \in \mathcal{M}_{2^n \times 2^m}(\mathbb{C}), \forall \epsilon > 0, \exists D \in ZX_{\pi/4} \text{ s.t. } \| [D] - M \|_2 < \epsilon.$$

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$$\mathbb{Q}[e^{i\pi/4}] = \{a + b e^{i\pi/4} + c(e^{i\pi/4})^2 + d(e^{i\pi/4})^3, a, b, c, d \in \mathbb{Q}\}.$$

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## Theorem

$ZX_{\pi/4}$  is universal for matrices over  $\mathbb{D}[e^{i\pi/4}]$ .

## 1 Introduction

## 2 ZX-Calculus

ZX-Calculus

Diagram Transformations

Universality, Soundness, Completeness

## 3 Completion for the First Approximately Universal Fragment

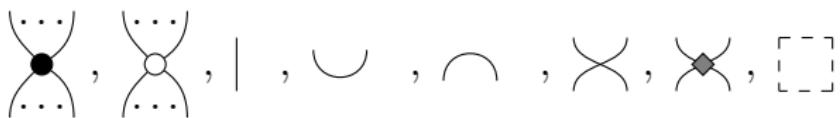
The ZW-Calculus

ZX-rules from the ZW-Calculus

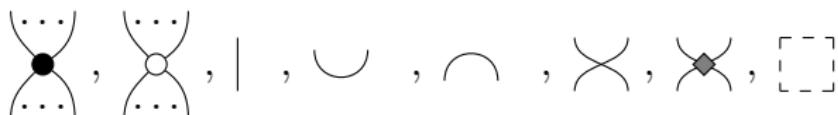
Completeness of the  $\frac{\pi}{4}$ -fragment

## 4 Conclusion

## ZW-calculus [Coecke&Kissinger, Hadzihasanovic]



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**Motivation:** diagrammatic classification of multipartite entanglement.



$|000\rangle + |111\rangle$   
(GHZ-state)



$|001\rangle + |010\rangle + |100\rangle$   
(W-state)

$$[\![D_1 \otimes D_2]\!] := [\![D_1]\!] \otimes [\![D_2]\!] \quad [\![D_2 \circ D_1]\!] := [\![D_2]\!] \circ [\![D_1]\!]$$

$$[\![\times]\!] := \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad [\![\times_{\diamond}]\!] := \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

$$[\![\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}]\!] := (1) \quad [\![|\ ]\!] := \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad [\![\cup]\!] := (1 \ 0 \ 0 \ 1)$$

$$[\![\curvearrowleft]\!] := \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix} \quad [\![\bullet|]\!] := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad [\![\vee]\!] := \begin{pmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$

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$$\llbracket D_1 \otimes D_2 \rrbracket := \llbracket D_1 \rrbracket \otimes \llbracket D_2 \rrbracket \quad \llbracket D_2 \circ D_1 \rrbracket := \llbracket D_2 \rrbracket \circ \llbracket D_1 \rrbracket$$

$$\llbracket \times \rrbracket := \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad \llbracket \times \diamond \rrbracket := \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

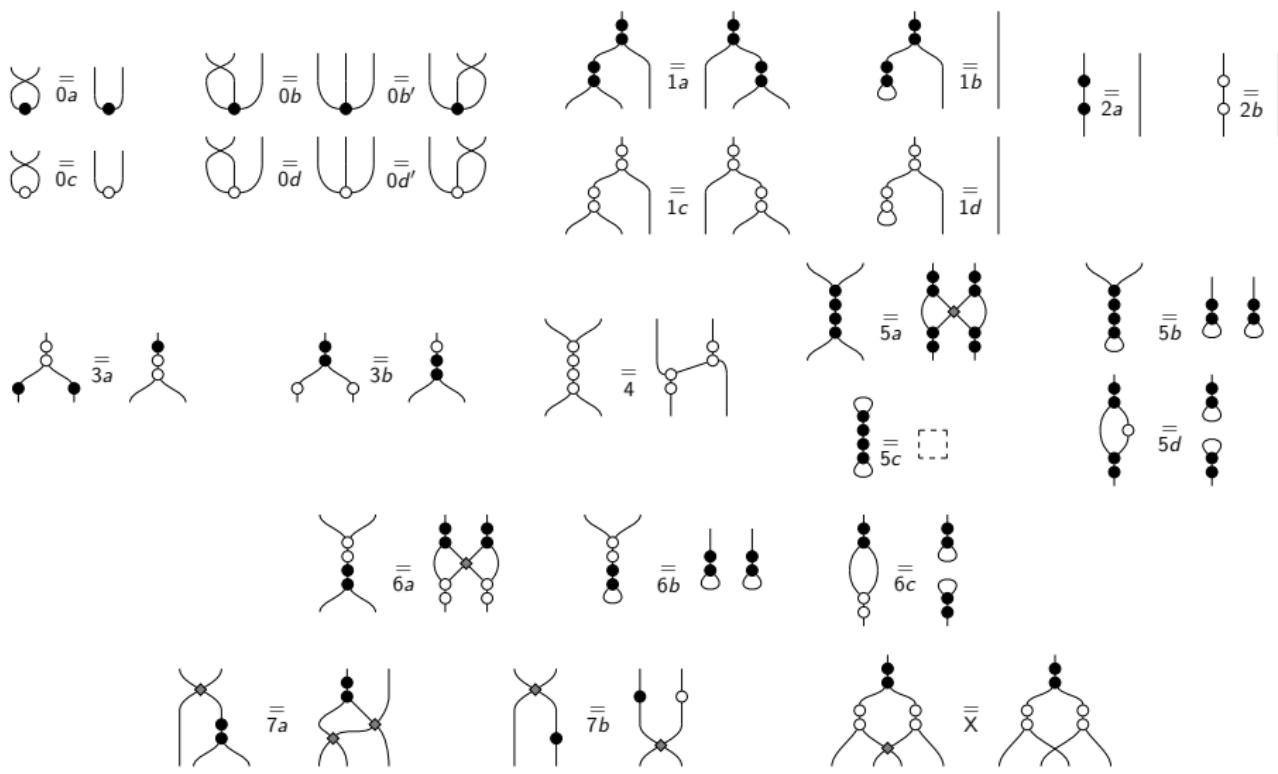
$$\llbracket \begin{smallmatrix} \square & \square \\ \square & \square \end{smallmatrix} \rrbracket := (1) \quad \llbracket | \rrbracket := \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \llbracket \cup \rrbracket := (1 \ 0 \ 0 \ 1)$$

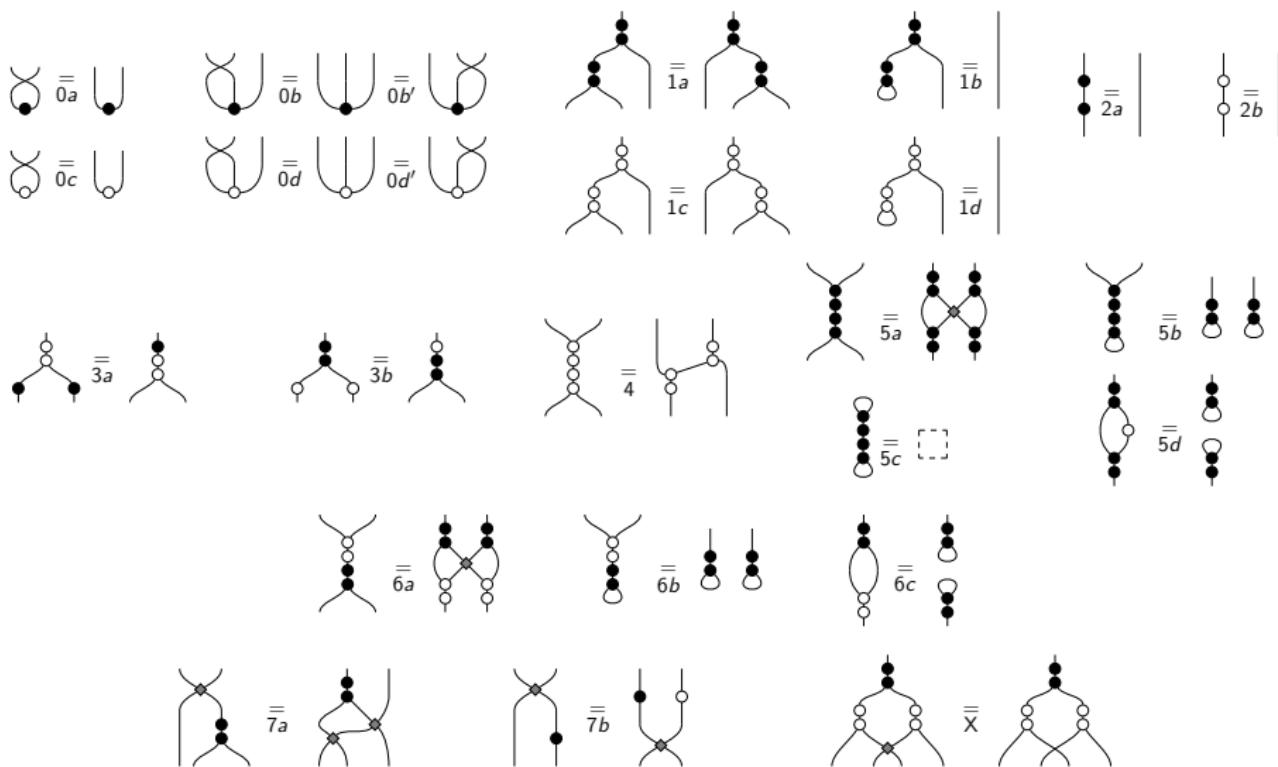
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Theorem [Hadzihasanovic'15]

ZW-Calculus is universal for integer matrices.





Theorem [Hadzihasanovic'15]

ZW-Calculus is complete for integer matrices.

Extended as “ZW<sub>1/2</sub>-Calculus”:

- Additional node:  $\star$  such that  $[\![\star]\!] = 1/2$

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## Lemma

ZW<sub>1/2</sub> is complete for matrices over  $\mathbb{D}$ .

$$\forall D_1, D_2 \in ZX_{\pi/4}, \quad \llbracket D_1 \rrbracket = \llbracket D_2 \rrbracket \qquad \qquad \not\Rightarrow \qquad \qquad ZX_{\pi/4} \vdash D_1 = D_2$$

$$\begin{array}{ccc} ZX_{\pi/4} & & ZW_{1/2} \\ \llbracket . \rrbracket \downarrow & & \downarrow \llbracket . \rrbracket \\ \mathcal{M}(\mathbb{D}[e^{i\pi/4}]) & & \mathcal{M}(\mathbb{D}) \end{array}$$

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$$\downarrow$$

$$\psi \llbracket D_1 \rrbracket = \psi \llbracket D_2 \rrbracket$$

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 & \downarrow & \\
 \psi [\![D_1]\!] = \psi [\![D_2]\!] & \Updownarrow \textcircled{1} & \\
 & \Downarrow & \\
 [\![\![D_1]\!]_{XW}\!] = [\![\![D_2]\!]_{XW}\!]
 \end{array}$$

$$\begin{array}{ccc}
 ZX_{\pi/4} & \xrightarrow{\quad [\![\cdot]\!]_{XW} \quad} & ZW_{1/2} \\
 \downarrow [\![\cdot]\!] & & \downarrow [\![\cdot]\!] \\
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①  $\[\![\cdot]\!]_{XW} = \psi([\![\cdot]\!])$

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 [\![\![D_1]\!]_{XW}\!] = [\![\![D_2]\!]_{XW}\!] & \Rightarrow & ZW_{1/2} \vdash [\![D_1]\!]_{XW} = [\![D_2]\!]_{XW}
 \end{array}$$

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 ZX_{\pi/4} & \xrightarrow{[\![\cdot]\!]_{XW}} & ZW_{1/2} \\
 \downarrow [\![\cdot]\!] & & \downarrow [\![\cdot]\!] \\
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 \psi \llbracket D_1 \rrbracket = \psi \llbracket D_2 \rrbracket & & ZX_{\pi/4} \vdash \llbracket \llbracket D_1 \rrbracket_{XW} \rrbracket_{WX} = \llbracket \llbracket D_2 \rrbracket_{XW} \rrbracket_{WX} \\
 \Updownarrow \textcircled{1} & & \uparrow \textcircled{2} \\
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 \end{array}$$

$$\begin{array}{ccccc}
 & & \llbracket \cdot \rrbracket_{WX} & & \\
 & \swarrow & & \searrow & \\
 ZX_{\pi/4} & & \llbracket \cdot \rrbracket_{XW} & & ZW_{1/2} \\
 \llbracket \cdot \rrbracket \downarrow & & & & \downarrow \llbracket \cdot \rrbracket \\
 \mathcal{M}(\mathbb{D}[e^{i\pi/4}]) & \xrightarrow{\psi} & & & \mathcal{M}(\mathbb{D})
 \end{array}$$

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 & \downarrow & \uparrow \textcolor{red}{3} \\
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 \llbracket \cdot \rrbracket \downarrow & & \downarrow \llbracket \cdot \rrbracket & & \downarrow \llbracket \cdot \rrbracket \\
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## Encoding $\mathbb{D}[e^{i\frac{\pi}{4}}]$ in $\mathbb{D}$

$$\begin{aligned}\mathbb{D}[e^{i\frac{\pi}{4}}] &\rightarrow \mathbb{D}^4 \\ a + b e^{i\frac{\pi}{4}} + c(e^{i\frac{\pi}{4}})^2 + d(e^{i\frac{\pi}{4}})^3 &\mapsto \begin{pmatrix} a & b & c & d \\ -d & a & b & c \\ -c & -d & a & b \\ -b & -c & -d & a \end{pmatrix}\end{aligned}$$

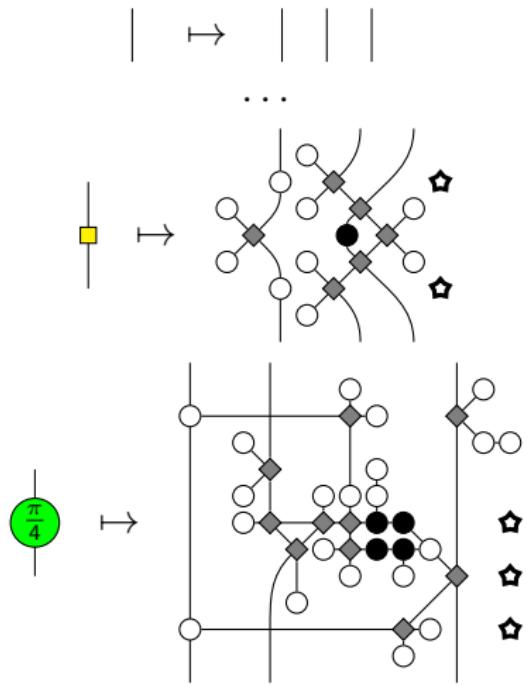
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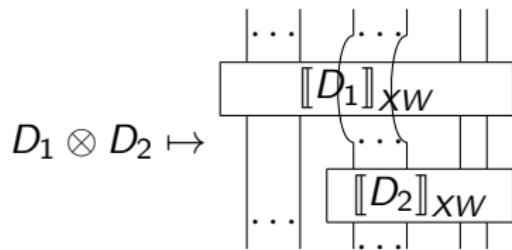
Extension to matrices:

$$\psi : \mathcal{M}_{2^n \times 2^m}(\mathbb{D}[e^{i\frac{\pi}{4}}]) \rightarrow \mathcal{M}_{2^{n+2} \times 2^{m+2}}(\mathbb{D})$$

# The Interpretation $\llbracket \cdot \rrbracket_{XW}$



$$D_1 \circ D_2 \mapsto \llbracket D_1 \rrbracket_{XW} \circ \llbracket D_2 \rrbracket_{XW}$$

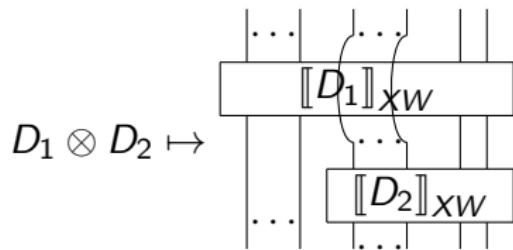
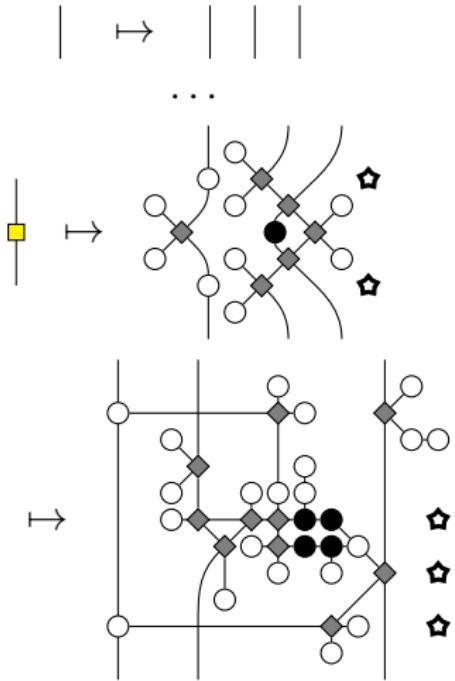


$$\begin{array}{c} n \\ \cdots \\ m \end{array} \text{ } \left( \begin{array}{c} \text{---} \\ | \\ \text{---} \end{array} \right) \mapsto \left( \begin{array}{c} \text{---} \\ | \\ \text{---} \end{array} \right) \circ \left( \llbracket \begin{array}{c} n \\ \cdots \\ m \end{array} \rrbracket_{XW} \right)^k \circ \left( \begin{array}{c} \text{---} \\ | \\ \text{---} \end{array} \right)$$

$$\begin{array}{c} n \\ \cdots \\ m \end{array} \text{ } \left( \begin{array}{c} \text{---} \\ | \\ \text{---} \end{array} \right)^{\otimes m} \mapsto \llbracket \left( \begin{array}{c} \text{---} \\ | \\ \text{---} \end{array} \right)^{\otimes m} \rrbracket_{XW} \circ \llbracket \begin{array}{c} n \\ \cdots \\ m \end{array} \rrbracket_{XW} \circ \llbracket \left( \begin{array}{c} \text{---} \\ | \\ \text{---} \end{array} \right)^{\otimes n} \rrbracket_{XW}$$

# The Interpretation $\llbracket \cdot \rrbracket_{XW}$

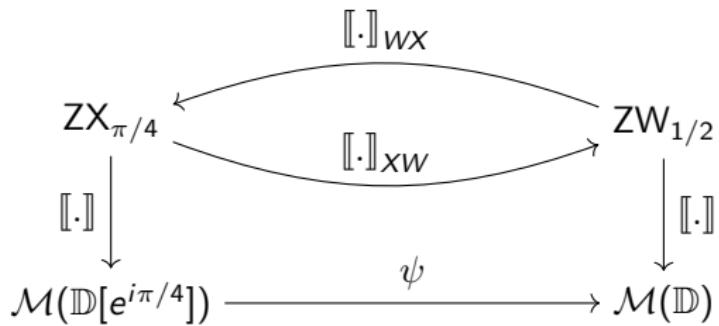
$$D_1 \circ D_2 \mapsto \llbracket D_1 \rrbracket_{XW} \circ \llbracket D_2 \rrbracket_{XW}$$



$$\begin{array}{c} n \\ \cdots \\ m \end{array} \text{ } \xrightarrow{\quad} \left( \begin{array}{c} \text{ } \\ \vdots \\ \text{ } \end{array} \middle| \right) \circ \left( \llbracket \begin{array}{c} \text{ } \\ \vdots \\ \frac{n}{4} \end{array} \rrbracket_{XW} \right)^k \circ \left( \begin{array}{c} \text{ } \\ \vdots \\ \text{ } \end{array} \middle| \right)$$

$$\begin{array}{c} n \\ \cdots \\ m \end{array} \text{ } \xrightarrow{\quad} \llbracket \left( \begin{array}{c} \text{ } \\ \vdots \\ \text{ } \end{array} \right)^{\otimes m} \rrbracket_{XW} \circ \begin{array}{c} n \\ \cdots \\ m \end{array} \text{ } \xrightarrow{\quad} \llbracket \left( \begin{array}{c} \text{ } \\ \vdots \\ \text{ } \end{array} \right)^{\otimes n} \rrbracket_{XW}$$

Prop:  $\llbracket \cdot \rrbracket_{XW} = \psi(\cdot)$

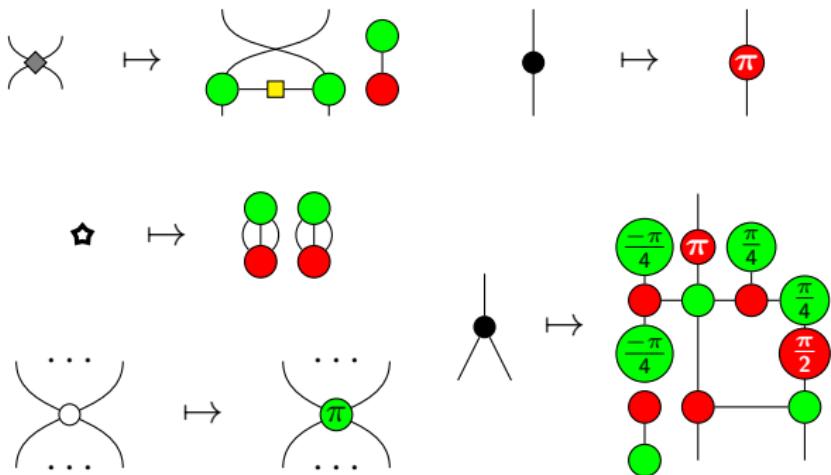


- ①  $\llbracket \llbracket \cdot \rrbracket_{XW} \rrbracket = \psi(\llbracket \cdot \rrbracket)$
- ②  $ZW_{1/2} \vdash D_1 = D_2 \Rightarrow ZX_{\pi/4} \vdash \llbracket D_1 \rrbracket_{WX} = \llbracket D_2 \rrbracket_{WX}$
- ③  $ZX_{\pi/4} \vdash \llbracket \llbracket D_1 \rrbracket_{XW} \rrbracket_{WX} = \llbracket \llbracket D_2 \rrbracket_{XW} \rrbracket_{WX} \Rightarrow ZX_{\pi/4} \vdash D_1 = D_2$

# From $ZW_{1/2}$ - to $ZX_{\pi/4}$ -Calculus

$\llbracket \cdot \rrbracket_{wx}:$

$$\begin{array}{ccc} \square & \mapsto & \square \\ | & \mapsto & | \\ \circ & \mapsto & \circ \\ \cup & \mapsto & \cup \\ \times & \mapsto & \times \end{array}$$



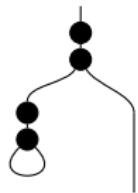
Lemma

For any  $D \in ZW_{1/2}$ ,  $\llbracket \llbracket D \rrbracket_{wx} \rrbracket = \llbracket D \rrbracket$ .

With a complete set of rules  $ZX_{\pi/4}$ , for any axiom  $D_1 = D_2$  of  $ZW$ ,

$$ZX_{\pi/4} \vdash \llbracket D_1 \rrbracket_{WX} = \llbracket D_2 \rrbracket_{WX}$$

e.g.



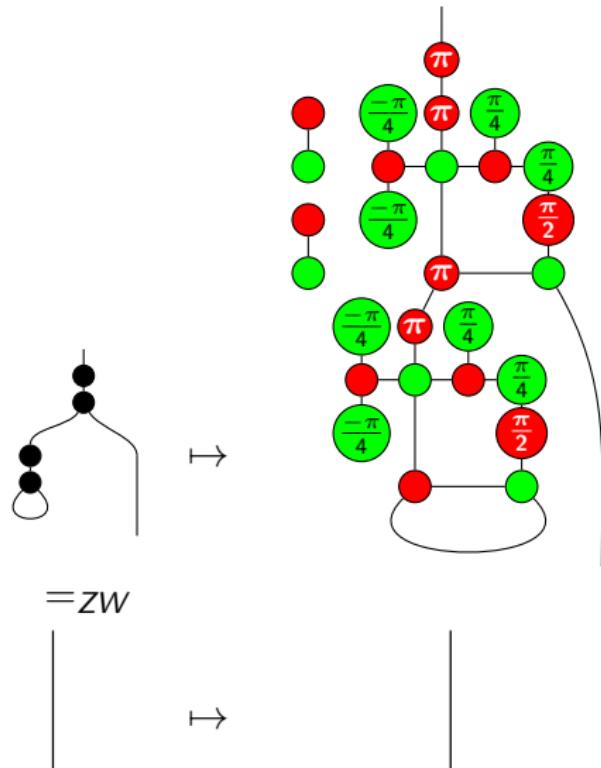
$$=_{ZW}$$



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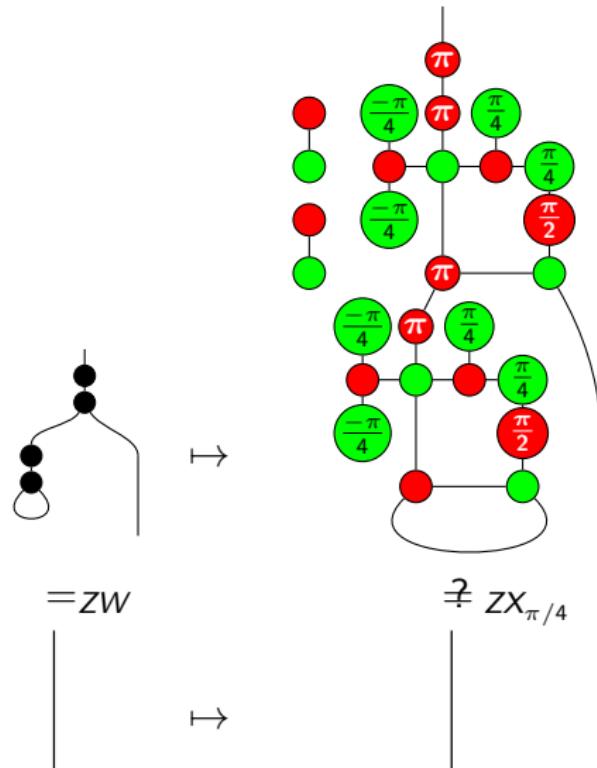
e.g.



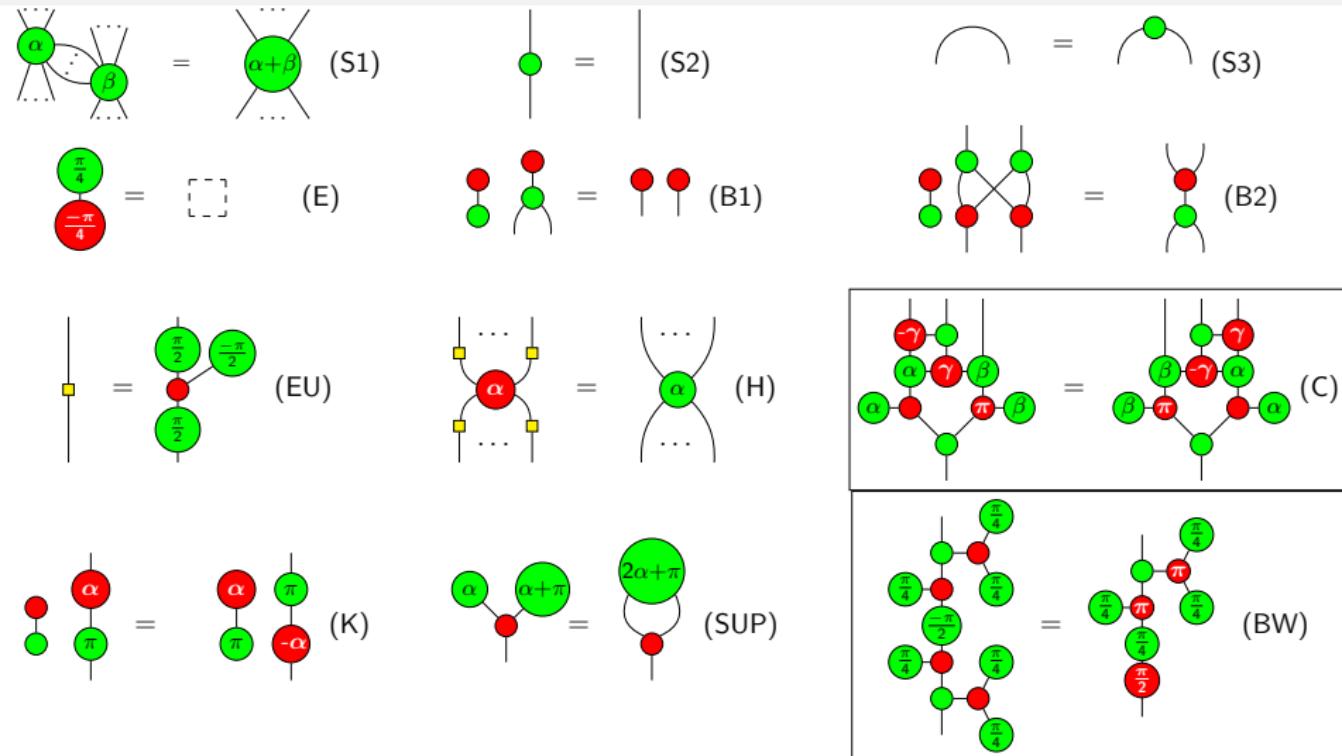
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e.g.



# Set of Rules $ZX_{\pi/4}$ [Jeandel, Perdrix, Vilmart'17]



Theorem (Completeness)

$$\forall D_1, D_2 \in ZX_{\pi/4}, \quad \llbracket D_1 \rrbracket = \llbracket D_2 \rrbracket \Rightarrow ZX_{\pi/4} \vdash D_1 = D_2$$

## 1 Introduction

## 2 ZX-Calculus

ZX-Calculus

Diagram Transformations

Universality, Soundness, Completeness

## 3 Completion for the First Approximately Universal Fragment

The ZW-Calculus

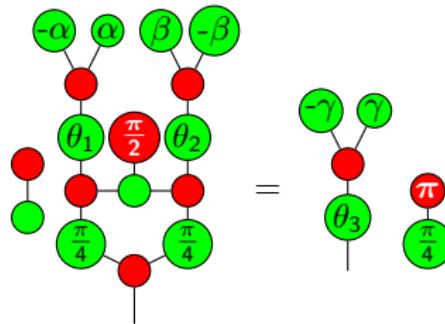
ZX-rules from the ZW-Calculus

Completeness of the  $\frac{\pi}{4}$ -fragment

## 4 Conclusion

- First completeness result for an approx. universal fragment of the ZX-Calculus
- This fragment represents exactly the matrices over  $\mathbb{D}[e^{i\frac{\pi}{4}}]$
- Lead to a completion in general (for angles in  $\mathbb{R}$ , and matrices over  $\mathbb{C}$ )  
[Kang Feng Ng,Wang'17 – Jeandel,Perdrix,Vilmart'17]

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$$\sqrt{2}e^{i\theta_3} \cos(\gamma) = e^{i\theta_1} \cos(\alpha) + e^{i\theta_2} \cos(\beta)$$

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Further questions:

- Normal form for the ZX-Calculus (can we do better than the interpretation of the normal form of the ZW-Calculus?)
- Proof of the necessity of the additional axioms