Contextuality in Causal Scenarios

Shane Mansfield

UPMC SORBONNE UNIVERSITÉS

LIP6

Marie Skłodowska-Curie Actions

Logical Structures in Computation Workshop
Simons Institute
14th December 2017

Overview

1. Quick recap of contextuality
2. Example of experiment with non-trivial causal structure: double-slit
3. Causal measurement scenarios and causal empirical models
4. Unified description of various non-classical phenomena

$$\{\text{nonlocality}, \quad \text{contextuality}, \quad \text{violations of macrorealism}\}$$

Overview

1. Quick recap of contextuality
2. Example of experiment with non-trivial causal structure: double-slit
3. Causal measurement scenarios and causal empirical models
4. Unified description of various non-classical phenomena

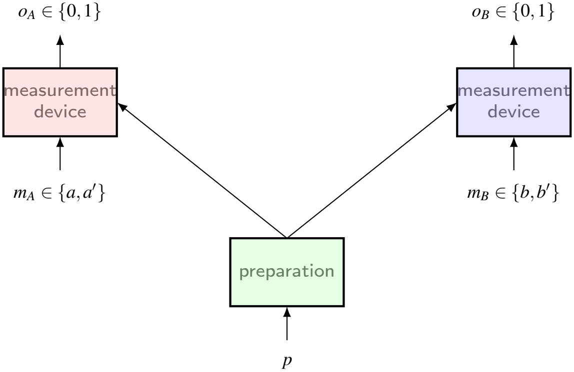
$$\{\text{nonlocality}, \quad \text{contextuality}, \quad \text{violations of macrorealism}\}$$

- Extends the *sheaf-theoretic framework* for nonlocality and contextuality initiated by Abramsky & Brandenburger
- Key additional ingredient: causal structure

Nonlocality & contextuality recap

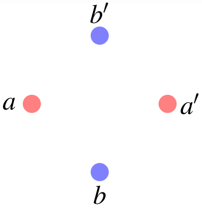
Empirical data (e.g. CHSH experiment)

	$(0,0)$	$(0,1)$	$(1,0)$	$(1,1)$
(a,b)	$1/2$	0	0	$1/2$
(a,b')	$3/8$	$1/8$	$1/8$	$3/8$
(a',b)	$3/8$	$1/8$	$1/8$	$3/8$
(a',b')	$1/8$	$3/8$	$3/8$	$1/8$



Measurement scenarios: CHSH

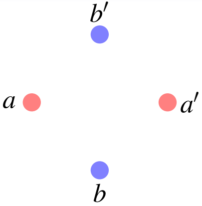
	$(0,0)$	$(0,1)$	$(1,0)$	$(1,1)$
(a,b)	$1/2$	0	0	$1/2$
(a,b')	$3/8$	$1/8$	$1/8$	$3/8$
(a',b)	$3/8$	$1/8$	$1/8$	$3/8$
(a',b')	$1/8$	$3/8$	$3/8$	$1/8$



A *measurement scenario* is a triple $\langle X, \mathscr{M}, O\rangle$ where:

Measurement scenarios: CHSH

	$(0,0)$	$(0,1)$	$(1,0)$	$(1,1)$
(a,b)	$1/2$	0	0	$1/2$
(a,b')	$3/8$	$1/8$	$1/8$	$3/8$
(a',b)	$3/8$	$1/8$	$1/8$	$3/8$
(a',b')	$1/8$	$3/8$	$3/8$	$1/8$



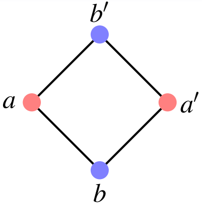
A *measurement scenario* is a triple $\langle X, \mathscr{M}, O \rangle$ where:

X a finite set of measurements — e.g.

$$X = \{a, a', b, b'\}$$

Measurement scenarios: CHSH

	$(0,0)$	$(0,1)$	$(1,0)$	$(1,1)$
(a,b)	$1/2$	0	0	$1/2$
(a,b')	$3/8$	$1/8$	$1/8$	$3/8$
(a',b)	$3/8$	$1/8$	$1/8$	$3/8$
(a',b')	$1/8$	$3/8$	$3/8$	$1/8$



A *measurement scenario* is a triple $\langle X, \mathcal{M}, O\rangle$ where:

X a finite set of measurements — e.g.

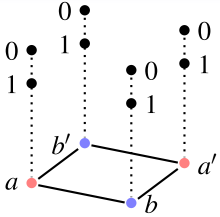
$$X = \{a, a', b, b'\}$$

$\mathcal{M}$ the (maximal) contexts — e.g.

$$\mathcal{M} = \{\{a,b\}, \quad \{a,b'\}, \quad \{a',b\}, \quad \{a',b'\}\}$$

Measurement scenarios: CHSH

	$(0,0)$	$(0,1)$	$(1,0)$	$(1,1)$
(a,b)	$1/2$	0	0	$1/2$
(a,b')	$3/8$	$1/8$	$1/8$	$3/8$
(a',b)	$3/8$	$1/8$	$1/8$	$3/8$
(a',b')	$1/8$	$3/8$	$3/8$	$1/8$



A *measurement scenario* is a triple $\langle X, \mathscr{M}, O\rangle$ where:

X a finite set of measurements — e.g.

$$X = \{a, a', b, b'\}$$

$\mathscr{M}$ the (maximal) contexts — e.g.

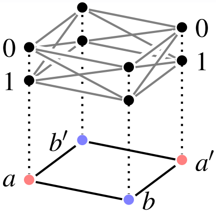
$$\mathscr{M} = \{\{a,b\}, \quad \{a,b'\}, \quad \{a',b\}, \quad \{a',b'\}\}$$

O a finite set of outcomes — e.g.

$$O = \{0,1\}$$

Empirical models

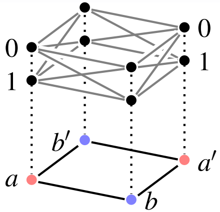
	$(0,0)$	$(0,1)$	$(1,0)$	$(1,1)$
(a,b)	$1/2$	0	0	$1/2$
(a,b')	$3/8$	$1/8$	$1/8$	$3/8$
(a',b)	$3/8$	$1/8$	$1/8$	$3/8$
(a',b')	$1/8$	$3/8$	$3/8$	$1/8$



- Fix a measurement scenario $\langle X, \mathcal{M}, O \rangle$

Empirical models

	$(0,0)$	$(0,1)$	$(1,0)$	$(1,1)$
(a,b)	$1/2$	0	0	$1/2$
(a,b')	$3/8$	$1/8$	$1/8$	$3/8$
(a',b)	$3/8$	$1/8$	$1/8$	$3/8$
(a',b')	$1/8$	$3/8$	$3/8$	$1/8$

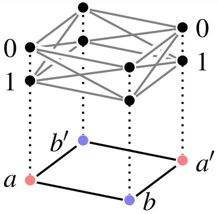


- Fix a measurement scenario $\langle X, \mathcal{M}, O \rangle$
- *Empirical model*: family $\{e_C\}_{C \in \mathcal{M}}$ where each $e_C \in \mathcal{D}(O^C)$
- i.e. a distribution for each context:

$$e_{\{a,b\}} = \text{prob}(o_a, o_b | a, b), \quad \ldots, \quad e_{\{a',b'\}} = \text{prob}(o_{a'}, o_{b'} | a', b')$$

Empirical models

	$(0,0)$	$(0,1)$	$(1,0)$	$(1,1)$
(a,b)	$1/2$	0	0	$1/2$
(a,b')	$3/8$	$1/8$	$1/8$	$3/8$
(a',b)	$3/8$	$1/8$	$1/8$	$3/8$
(a',b')	$1/8$	$3/8$	$3/8$	$1/8$



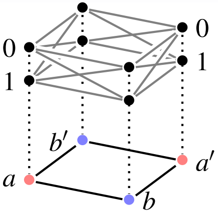
- Fix a measurement scenario $\langle X, \mathcal{M}, O \rangle$
- *Empirical model*: family $\{e_C\}_{C \in \mathcal{M}}$ where each $e_C \in \mathcal{D}(O^C)$
- i.e. a distribution for each context:

$$e_{\{a,b\}} = \text{prob}(o_a, o_b | a, b), \quad \ldots, \quad e_{\{a',b'\}} = \text{prob}(o_{a'}, o_{b'} | a', b')$$

- 'Local' consistency: $\forall C, C' \in \mathcal{M}. \quad e_C|_{C \cap C'} = e_{C'}|_{C \cap C'}$

Empirical models

	$(0,0)$	$(0,1)$	$(1,0)$	$(1,1)$
(a,b)	$1/2$	0	0	$1/2$
(a,b')	$3/8$	$1/8$	$1/8$	$3/8$
(a',b)	$3/8$	$1/8$	$1/8$	$3/8$
(a',b')	$1/8$	$3/8$	$3/8$	$1/8$



- Fix a measurement scenario $\langle X, \mathcal{M}, O \rangle$
- *Empirical model*: family $\{e_C\}_{C \in \mathcal{M}}$ where each $e_C \in \mathcal{D}(O^C)$
- i.e. a distribution for each context:

$$e_{\{a,b\}} = \text{prob}(o_a, o_b | a, b), \quad \ldots, \quad e_{\{a',b'\}} = \text{prob}(o_{a'}, o_{b'} | a', b')$$

- 'Local' consistency: $\forall C, C' \in \mathcal{M}. \quad e_C|_{C \cap C'} = e_{C'}|_{C \cap C'}$

$$\text{prob}(o_a | a, b) = \text{prob}(o_a | a, b') = \text{prob}(o_a | a), \text{ etc.}$$

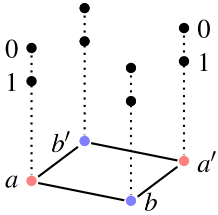
(generalised) no-signalling

Classical correlations

	$(0,0)$	$(0,1)$	$(1,0)$	$(1,1)$
(a,b)	$1/2$	0	0	$1/2$
(a,b')	$1/2$	0	0	$1/2$
(a',b)	$1/2$	0	0	$1/2$
(a',b')	0	$1/2$	$1/2$	0

Classical data arises as a convex combination of *global assignments*:

$$
\begin{gathered}
(a,a',b,b') \mapsto (0,0,0,0), \\
(a,a',b,b') \mapsto (0,0,0,1), \\
\ldots \quad , \\
(a,a',b,b') \mapsto (1,1,1,1)
\end{gathered}
$$

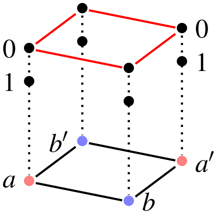


Classical correlations

	$(0,0)$	$(0,1)$	$(1,0)$	$(1,1)$
(a,b)	1	0	0	0
(a,b')	1	0	0	0
(a',b)	1	0	0	0
(a',b')	1	0	0	0

Classical data arises as a convex combination of *global assignments*:

$$
\begin{gathered}
(a,a',b,b') \mapsto (0,0,0,0), \\
(a,a',b,b') \mapsto (0,0,0,1), \\
\dots \quad , \\
(a,a',b,b') \mapsto (1,1,1,1)
\end{gathered}
$$

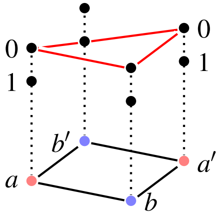


Classical correlations

	$(0,0)$	$(0,1)$	$(1,0)$	$(1,1)$
(a,b)	1	0	0	0
(a,b')	0	1	0	0
(a',b)	1	0	0	0
(a',b')	0	1	0	0

Classical data arises as a convex combination of *global assignments*:

$$(a,a',b,b') \mapsto (0,0,0,0),$$
$$(a,a',b,b') \mapsto (0,0,0,1),$$
$$\ldots \quad ,$$
$$(a,a',b,b') \mapsto (1,1,1,1)$$

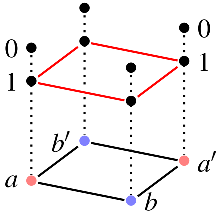


Classical correlations

	$(0,0)$	$(0,1)$	$(1,0)$	$(1,1)$
(a,b)	0	0	0	1
(a,b')	0	0	0	1
(a',b)	0	0	0	1
(a',b')	0	0	0	1

Classical data arises as a convex combination of *global assignments*:

$$(a,a',b,b') \mapsto (0,0,0,0),$$
$$(a,a',b,b') \mapsto (0,0,0,1),$$
$$\cdots \quad ,$$
$$(a,a',b,b') \mapsto (1,1,1,1)$$

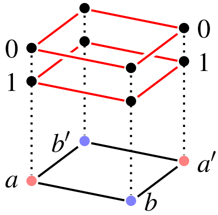


Classical correlations

	$(0,0)$	$(0,1)$	$(1,0)$	$(1,1)$
(a,b)	$1/2$	0	0	$1/2$
(a,b')	$1/2$	0	0	$1/2$
(a',b)	$1/2$	0	0	$1/2$
(a',b')	$1/2$	0	0	$1/2$

Classical data arises as a convex combination of *global assignments*:

$$(a,a',b,b') \mapsto (0,0,0,0),$$
$$(a,a',b,b') \mapsto (0,0,0,1),$$
$$\ldots \quad ,$$
$$(a,a',b,b') \mapsto (1,1,1,1)$$

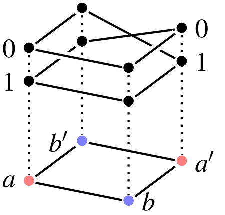


Classical correlations

	$(0,0)$	$(0,1)$	$(1,0)$	$(1,1)$
(a,b)	$1/2$	0	0	$1/2$
(a,b')	$1/2$	0	0	$1/2$
(a',b)	$1/2$	0	0	$1/2$
(a',b')	0	$1/2$	$1/2$	0

Classical data arises as a convex combination of *global assignments*:

$$
\begin{gathered}
(a,a',b,b') \mapsto (0,0,0,0), \\
(a,a',b,b') \mapsto (0,0,0,1), \\
\ldots \quad , \\
(a,a',b,b') \mapsto (1,1,1,1)
\end{gathered}
$$



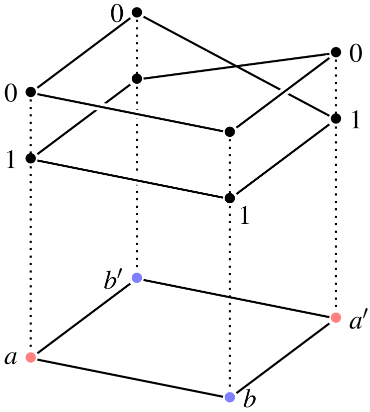
However, the above correlations *cannot* be obtained as a convex combination of global assignments!

PR box bundle diagram

Logical contextuality:
present at the level of possibilities
Strong contextuality:
no event can be extended to a global assignment

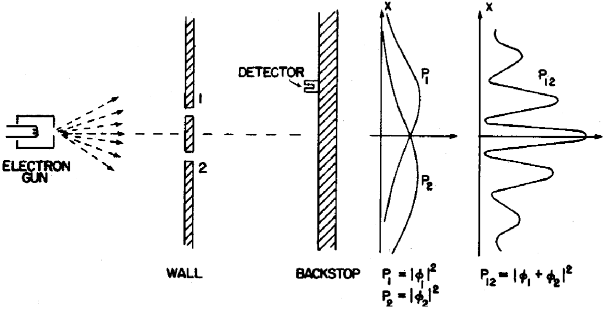
PR box possibility table:

	$(0,0)$	$(0,1)$	$(1,0)$	$(1,1)$
(a,b)	✓	×	×	✓
(a,b')	✓	×	×	✓
(a',b)	✓	×	×	✓
(a',b')	×	✓	✓	×



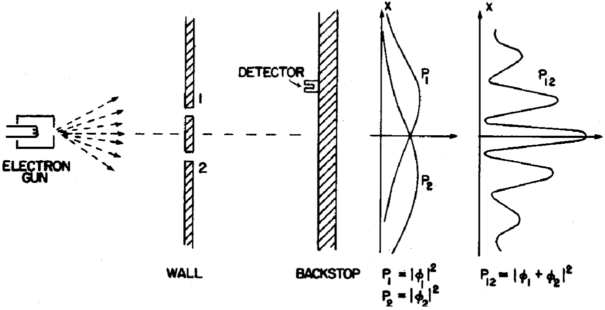
Example: the double-slit experiment

A classic experiment



- Turn down intensity to shoot one electron at a time

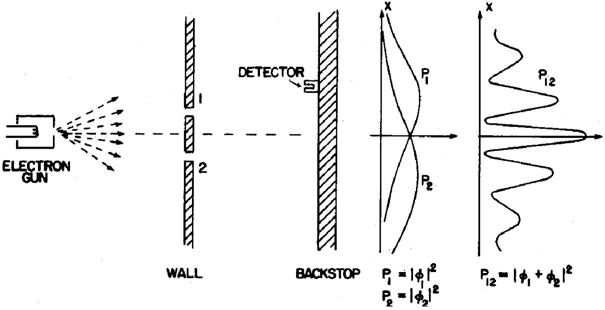
A classic experiment



- Turn down intensity to shoot one electron at a time

- Check which way electron passes: ballistic distribution $\frac{1}{2}(P_1 + P_2)$
- Don't check: interference distribution P_{12}

A classic experiment



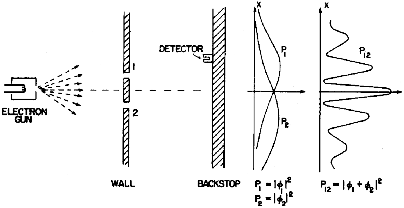
- Turn down intensity to shoot one electron at a time

- Check which way electron passes: ballistic distribution $\frac{1}{2}(P_1 + P_2)$
- Don't check: interference distribution P_{12}

- Wave-particle duality mysticism, etc. ...

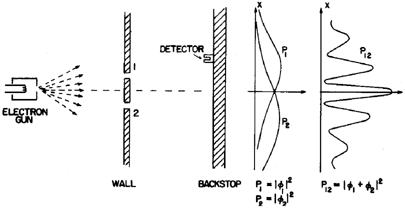
A classic experiment — empirical model?

Position detector in dark fringe



A classic experiment — empirical model?

Position detector in dark fringe

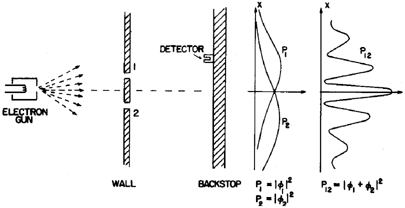


Measurements $X = \{x_e, x_w, x_d\}$,

1. x_e — *emission*, values in $\{0,1\}$
2. x_w — *which way* (non-demolition), values in $\{0,1,2\}$
3. x_d — *detection*, values in $\{0,1\}$

A classic experiment — empirical model?

Position detector in dark fringe



Measurements $X = \{x_e, x_w, x_d\}$,

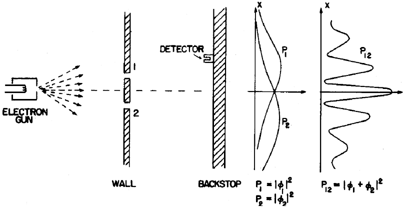
1. x_e — *emission*,
   values in $\{0,1\}$
2. x_w — *which way* (non-demolition),
   values in $\{0,1,2\}$
3. x_d — *detection*,
   values in $\{0,1\}$

Contexts,

$$\mathcal{M} = \quad \{\{x_e, x_w, x_d\}, \quad \{x_e, x_d\}\}$$

A classic experiment — empirical model?

Position detector in dark fringe



Measurements $X = \{x_e, x_w, x_d\}$,

1. x_e — *emission*, values in $\{0,1\}$
2. x_w — *which way* (non-demolition), values in $\{0,1,2\}$
3. x_d — *detection*, values in $\{0,1\}$

Contexts,

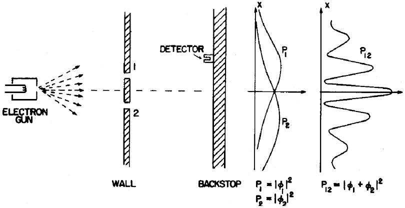
$$\mathscr{M} = \quad \{\{x_e, x_w, x_d\}, \quad \{x_e, x_d\}\}$$

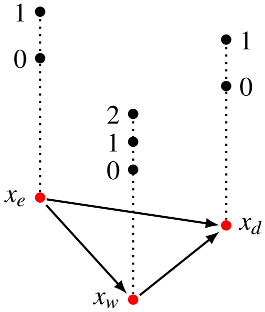
Outcomes,

$$O = \{0,1,2\}$$

A classic experiment — empirical model?

Position detector in dark fringe





Measurements $X = \{x_e, x_w, x_d\}$,

1. x_e — *emission*, values in $\{0,1\}$
2. x_w — *which way* (non-demolition), values in $\{0,1,2\}$
3. x_d — *detection*, values in $\{0,1\}$

Contexts,

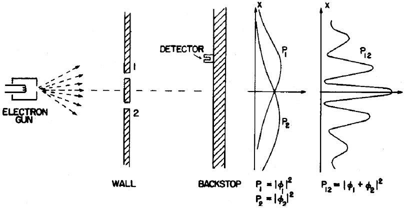
$$\mathscr{M} = \quad \{\{x_e, x_w, x_d\}, \quad \{x_e, x_d\}\}$$

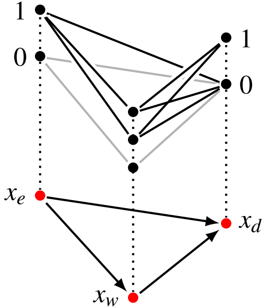
Outcomes,

$$O = \{0,1,2\}$$

A classic experiment — empirical model?

Position detector in dark fringe





Measurements $X = \{x_e, x_w, x_d\}$,

1. x_e — *emission*, values in $\{0,1\}$
2. x_w — *which way* (non-demolition), values in $\{0,1,2\}$
3. x_d — *detection*, values in $\{0,1\}$

Contexts,

$$\mathscr{M} = \quad \{\{x_e, x_w, x_d\}, \quad \{x_e, x_d\}\}$$

Outcomes,

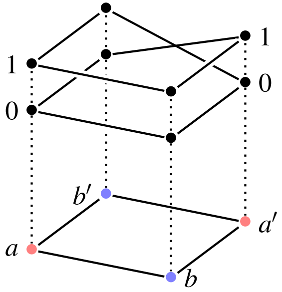
$$O = \{0,1,2\}$$

Trouble with marginals

- Empirical models satisfy 'local' consistency:

$$\text{prob}(o_1|a,b) = \text{prob}(o_1|a,b') = \text{prob}(o_1|a), \text{ etc.}$$

(generalised) no-signalling



Trouble with marginals

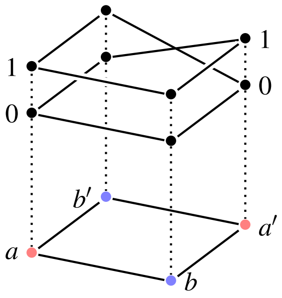
- Empirical models satisfy 'local' consistency:

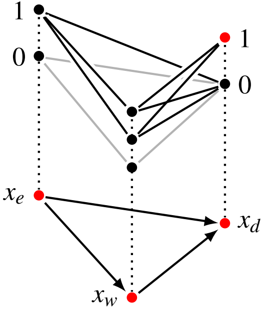
$$\text{prob}(o_1|a,b) = \text{prob}(o_1|a,b') = \text{prob}(o_1|a), \text{ etc.}$$

(generalised) no-signalling

- Not the case here:

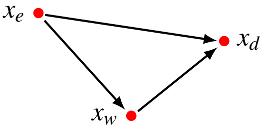
$$\text{prob}(1_d \mid x_d, x_e) \neq \text{prob}(1_d \mid x_d, x_w, x_e)$$





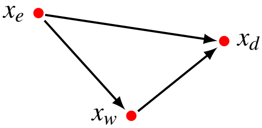
Solution: causal structure

- Only impose consistency for marginals that respect causal structure



Solution: causal structure

- Only impose consistency for marginals that respect causal structure



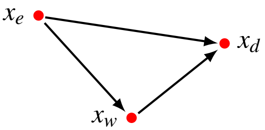
  - Given a poset $(X, \leq)$ define a partial order $\leq$ on $\mathscr{P}(X)$ by

$$A \leq B \quad \textit{iff} \quad (\downarrow A) \cap B = A$$

  i.e. B reduces to A via successive removals of maximal elements

Solution: causal structure

- Only impose consistency for marginals that respect causal structure



  - Given a poset $(X, \leq)$ define a partial order $\leq$ on $\mathscr{P}(X)$ by

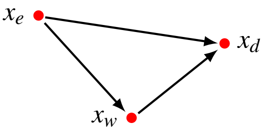
$$A \leq B \quad \textit{iff} \quad (\downarrow A) \cap B = A$$

    i.e. B reduces to A via successive removals of maximal elements

  - e.g. $\{x_e, x_w\} \leq \{x_e, x_w, x_d\}$ but $\{x_e, x_d\} \not\leq \{x_e, x_w, x_d\}$

Solution: causal structure

- Only impose consistency for marginals that respect causal structure



  - Given a poset $(X, \leq)$ define a partial order $\leq$ on $\mathscr{P}(X)$ by

$$A \leq B \quad \textit{iff} \quad (\downarrow A) \cap B = A$$

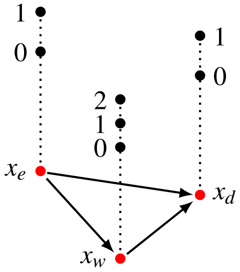
    i.e. B reduces to A via successive removals of maximal elements

  - e.g. $\{x_e, x_w\} \leq \{x_e, x_w, x_d\}$ but $\{x_e, x_d\} \not\leq \{x_e, x_w, x_d\}$

  - Intuition: *later* measurements cannot affect *earlier* outcomes
    Earlier measurements can affect *later* outcomes

Causal empirical models

Causal measurement scenarios



A *causal measurement scenario* is a tuple $\langle X, \leq, \mathscr{M}, O \rangle$ where:

- $(X, \leq)$ a poset of measurements — e.g.

$$X = \{x_e, x_w, x_d\}$$

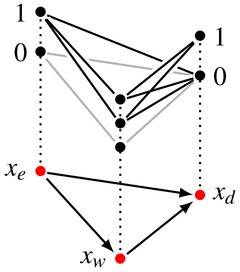
- $\mathscr{M}$ the maximal contexts (wrt '$\leq$') — e.g.

$$\mathscr{M} = \{\{x_e, x_d\}, \quad \{x_e, x_w, x_d\}\}$$

- O a finite set of outcomes — e.g.

$$O = \{0, 1, 2\}$$

Causal empirical models



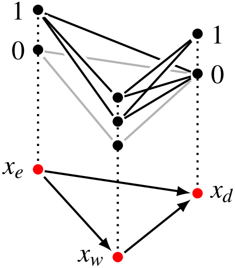
Measurement scenario $\langle X, \leq, \mathscr{M}, O \rangle$

Causal empirical model: family $\{e_C\}_{C \in \mathscr{M}}$ where each $e_C \in \mathscr{D}(O^C)$

- i.e. a distribution for each maximal context — e.g.

$$e_{\{x_e, x_w, x_d\}} = \mathrm{prob}(o_e, o_w, o_d \mid x_e, x_w, x_d), \qquad e_{\{x_e, x_d\}} = \mathrm{prob}(o_e, o_d \mid x_e, x_d)$$

Causal empirical models



Measurement scenario $\langle X, \leq, \mathscr{M}, O \rangle$

Causal empirical model: family $\{e_C\}_{C \in \mathscr{M}}$ where each $e_C \in \mathscr{D}(O^C)$

- i.e. a distribution for each maximal context — e.g.

$$e_{\{x_e, x_w, x_d\}} = \text{prob}(o_e, o_w, o_d \mid x_e, x_w, x_d), \qquad e_{\{x_e, x_d\}} = \text{prob}(o_e, o_d \mid x_e, x_d)$$

- 'Local' consistency of causal marginals:

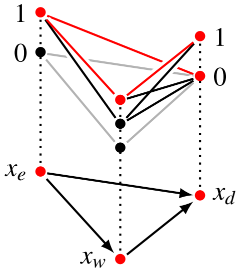
$$\text{prob}(o_e \mid x_e, x_w, x_d) = \text{prob}(o_e \mid x_e, x_w) = \text{prob}(o_e \mid x_e), \text{ etc.}$$

generalised no signalling (outside of forward light-cone)

Contextuality and macrorealism

Double-slit model is logically contextual

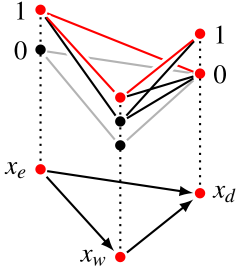
i.e. cannot be obtained as convex combination of global assignments



Contextuality and macrorealism

Double-slit model is logically contextual

i.e. cannot be obtained as convex combination of global assignments



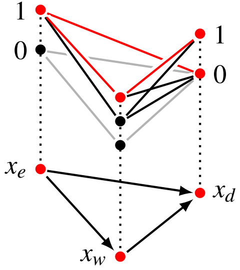
For totally ordered scenarios, *macrorealism* (Leggett–Garg) has 3 requirements (Maroney–Timpson):

1. Macrorealism *per se* — mixtures of deterministic hidden variables
2. Non-invasiveness — parameter-independence (forwards in time)
3. Induction — parameter-independence (backwards in time)

Contextuality and macrorealism

Double-slit model is logically contextual

i.e. cannot be obtained as convex combination of global assignments



For totally ordered scenarios, *macrorealism* (Leggett–Garg) has 3 requirements (Maroney–Timpson):

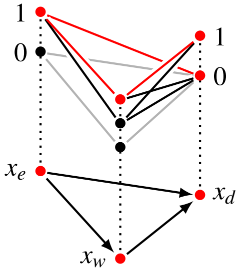
1. Macrorealism *per se*
2. Non-invasiveness
3. Induction

} convex mixtures of global assignments

Contextuality and macrorealism

Double-slit model is logically contextual

i.e. cannot be obtained as convex combination of global assignments



For totally ordered scenarios, *macrorealism* (Leggett–Garg) has 3 requirements (Maroney–Timpson):

1. Macrorealism *per se*
2. Non-invasiveness
3. Induction

} convex mixtures of global assignments

Macrorealism $\longleftrightarrow$ non-contextuality

Special causal scenarios

1. (Standard/static) empirical scenarios:
   '$\leq$' is empty *or* each maximal context contains only pairwise incomparable elements

2. Leggett-Garg-type scenarios:
   '$\leq$' is a total order

3. Temporal Bell-type scenarios (Fritz):
   X can be partitioned into time steps as $(X_t)_{t \in \mathbb{N}}$, such that:
   - $\mathscr{M} = \prod X_t$
     (contexts contain one measurement per time step)
   - For all $x \in X_t, y \in X_{t'}$ we have $x \leq y$ iff $t \leq t'$
     (causal order is induced by the partition)

Conclusion

- Unified approach to

$$\{\text{nonlocality}, \quad \text{contextuality}, \quad \text{violations of macrorealism}\}$$

Representative examples:

$$\{\text{Bell}, \quad \text{Kochen-Specker}, \quad \text{Leggett-Garg}\}$$

Conclusion

- Unified approach to

$$\{\text{nonlocality}, \quad \text{contextuality}, \quad \text{violations of macrorealism}\}$$

  Representative examples:

$$\{\text{Bell}, \quad \text{Kochen-Specker}, \quad \text{Leggett-Garg}\}$$

Possible future directions:

- ‘Causal contextuality’ in non-physical settings?
- Quantifiying ‘causal contextuality’
- Process calculus
- Advantages over classical causal models in informatic tasks
- Extension to indefinite causal structures