LOCALLY CONSISTENT EQUATIONS,
THE STRUCTURE OF SOLUTION SPACES,
AND QUANTUM INFORMATION GAMES

Albert Atserias
Universitat Politècnica de Catalunya

based on joint with

Kolaitis, Ochremiak, Roberson, Severini

Talk plan

1. What I worked on before the program
2. What I learned and worked on at the program
3. What I worked on after the program

Homomorphism and Isomorphism Problems

$$G \xrightarrow{\text{hom}} H$$

$$G \overset{\text{iso}}{\equiv} H$$

Part I

LOGICO-COMBINATORIAL RELAXATIONS

Logico-Combinatorial Relaxations

$$G \xrightarrow{\mathsf{hom}} H \quad \Longleftrightarrow \quad G \xrightarrow{\mathsf{E}_+} H \quad \Longrightarrow \quad G \xrightarrow{\mathsf{E}_+^k} H$$

E_+ : existential-positive first-order logic, i.e., atoms, $\wedge, \exists$.
E_+^k : k-variable fragment of E_+

Logico-Combinatorial Relaxations

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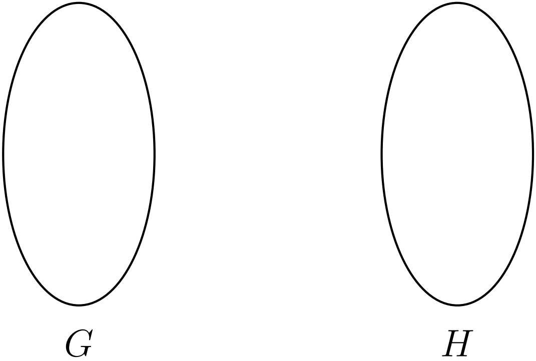
E_+^k : k-variable fragment of E_+

$$G \overset{\mathsf{iso}}{\equiv} H \iff G \overset{\mathsf{C}}{\equiv} H \implies G \overset{\mathsf{C}^k}{\equiv} H$$

C : counting logic, i.e., atoms, $\neg, \wedge, \exists^{\geq 1}, \exists^{\geq 2}, \ldots$
C^k : k-variable fragment of C.

Ehrenfeucht-Fraïssé-type k-pebble games



E^k_+: existential-positive k-pebble game [KV95]
C^k: bijective k-pebble game [H96]

Counterexamples to reverse implication for E_+^k

An easy counterexample:

$$K_{k+1} \overset{\text{hom}}{\not\to} K_k$$
$$K_{k+1} \overset{\mathsf{E}_+^k}{\longrightarrow} K_k$$

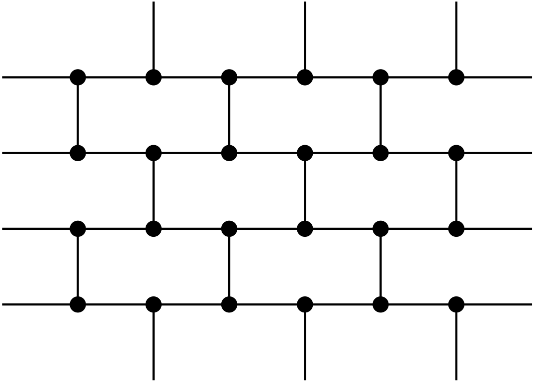
A stronger counterexample from [A05]:

$$\text{TSEITIN}_{k,\text{odd}} \overset{\text{hom}}{\not\to} \text{3-XOR}$$
$$\text{TSEITIN}_{k,\text{odd}} \overset{\mathsf{E}_+^k}{\longrightarrow} \text{3-XOR}$$

where

3-XOR $=$ template for parity equations, i.e., $(\{\pm 1\}, xyz = \pm 1)$
TSEITIN$_{k,\text{odd}}$ $=$ certain system of parity eqns on the k^2-wall

The m-wall graph

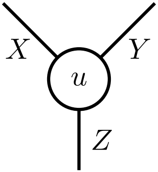


Tseitin system of parity equations

Construction of TSEITIN(G, σ):

G is an undirected graph.
$\sigma : V(G) \to \{\pm 1\}$ is a ± 1 labelling of the nodes of G.

There is a variable at every edge.
There is an equation at every node:



$$\Longrightarrow \quad XYZ = \sigma(u)$$

Counterexamples to reverse implication for C^k

A counterexample from [CFI95]:

$$\mathsf{CFI}_k^+ \overset{\text{iso}}{\not\equiv} \mathsf{CFI}_k^-$$
$$\mathsf{CFI}_k^+ \overset{\mathsf{C}^k}{\equiv} \mathsf{CFI}_k^-$$

A reinterpretation of CFI from [ABD07]:

$$\mathsf{TSEITIN}_{k,\text{even}}^{\times 2} \overset{\text{iso}}{\not\equiv} \mathsf{TSEITIN}_{k,\text{odd}}^{\times 2}$$
$$\mathsf{TSEITIN}_{k,\text{even}}^{\times 2} \overset{\mathsf{C}^k}{\equiv} \mathsf{TSEITIN}_{k,\text{odd}}^{\times 2}$$

Part II

LINEAR AND SEMIDEFINITE PROGRAMMING RELAXATIONS

Part II

OR, BY DUALITY, SHERALI-ADAMS AND LASSERRE/SUMS-OF-SQUARES REFUTATIONS

Hom and Iso as systems of polynomial equations

Variables:

$X_{u,v}$: a variable for each $u \in V(G)$ and $v \in V(H)$

Equations:

$\sum_v X_{u,v} - 1 = 0$ for all u

$X_{u,v} X_{u',v'} = 0$ for all $(u, u') \in E(G)$ and $(v, v') \notin E(H)$

$X_{u,v} X_{u',v'} = 0$ for all $u = u'$ and $v \neq v'$

$X_{u,v}^2 - X_{u,v} = 0$ for all u and v

Hom and Iso as systems of polynomial equations

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$X_{u,v} X_{u',v'} = 0$	for all $u = u'$ and $v \neq v'$
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$X_{u,v}^2 - X_{u,v} = 0$	for all u and v

Nullstellensatz, Sherali-Adams, and Lasserre/SOS

Systems of polynomial equations over $\{0,1\}^n$:

$$X_1^2 - X_1 = 0, \ldots, X_n^2 - X_n = 0$$
$$P_1(X) = 0, \ldots, P_m(X) = 0$$

Nullstellensatz refutation of degree k:

$$\sum_{j=1}^{t} P_{i_j} Q_j = -1$$

where $Q_1, \ldots, Q_t$ are arbitrary polynomials of total degree $\leq k$.

Nullstellensatz, Sherali-Adams, and Lasserre/SOS

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Sherali-Adams refutation of degree k:

$$\sum_{j=1}^{t} P_{i_j} Q_j + Q_0 = -1$$

where $Q_1, \ldots, Q_t$ are arbitrary polynomials
and

$$Q_0 = \sum_i c_i^2 \prod_{i \in I} X_i \prod_{i \in J} (1 - X_i),$$

all of total degree $\leq k$.

Nullstellensatz, Sherali-Adams, and Lasserre/SOS

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and

$$Q_0 = \sum_i Q_{0,i}^2,$$

all of total degree $\leq k$.

Proof complexity relaxations

$$G \xrightarrow{\text{hom}} H \implies G \xrightarrow{\text{SOS}^k} H \implies G \xrightarrow{\text{SA}^k} H \implies G \xrightarrow{\text{NS}^k} H$$

$$G \overset{\text{iso}}{\equiv} H \implies G \overset{\text{SOS}^k}{\equiv} H \implies G \overset{\text{SA}^k}{\equiv} H \implies G \overset{\text{NS}^k}{\equiv} H$$

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Implication diagram

$$G \overset{\mathsf{SOS}^k}{\equiv} H \quad \Longrightarrow \quad G \overset{\mathsf{SA}^k}{\equiv} H \quad \Longrightarrow \quad G \overset{\mathsf{NS}^k}{\equiv} H$$

Implication diagram

$$G \overset{\mathsf{SOS}^k}{\equiv} H \implies \boxed{\begin{array}{c} G \overset{\mathsf{SA}^k}{\equiv} H \\ \Big\Updownarrow \\ G \overset{\mathsf{C}^k}{\equiv} H \end{array}} \implies G \overset{\mathsf{NS}^k}{\equiv} H$$

[AM11,M11,GO12]

Implication diagram

$$G \overset{\mathsf{SOS}^k}{\equiv} H \quad \Longrightarrow \quad G \overset{\mathsf{SA}^k}{\equiv} H \quad \Longrightarrow \quad G \overset{\mathsf{NS}^k}{\equiv} H$$

$$G \overset{\mathsf{SA}^k}{\equiv} H \quad \Updownarrow \quad G \overset{\mathsf{C}^k}{\equiv} H$$

[AM11,M11,GO12]

$$G \overset{\mathsf{C}^k}{\equiv} H \quad \Leftrightarrow \quad G \overset{\mathsf{mon\text{-}PC}^k}{\equiv} H$$

[BG15]

Implication diagram

$G \overset{\mathsf{SOS}^k}{\equiv} H \quad \Longrightarrow \quad G \overset{\mathsf{SA}^k}{\equiv} H \quad \overset{\not\Longleftarrow}{\Longrightarrow} \quad G \overset{\mathsf{NS}^k}{\equiv} H$

[GP17]

$G \overset{\mathsf{SA}^k}{\equiv} H \quad \Big\Updownarrow \quad G \overset{\mathsf{C}^k}{\equiv} H$

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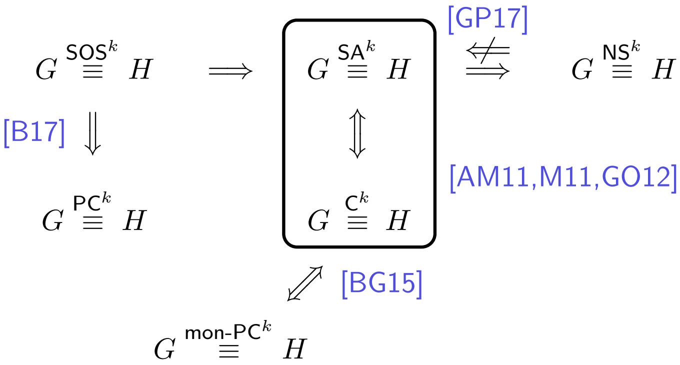

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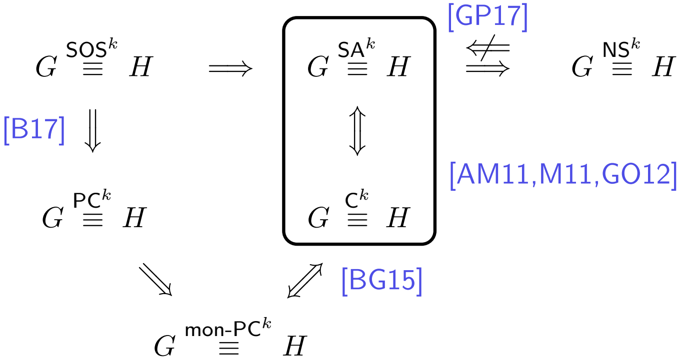

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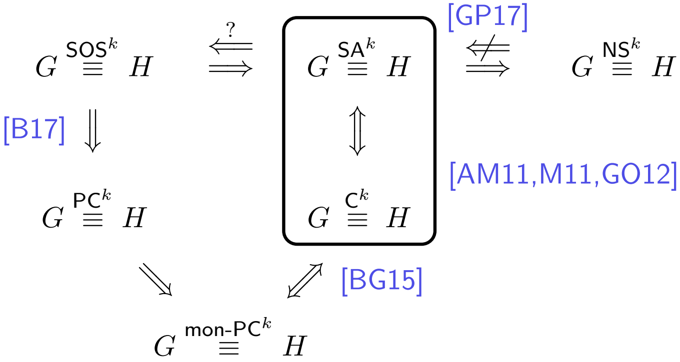

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Proof complexity of graph iso (an incomplete survey)

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  - NS^k by [GP17],
  - mon-PC^k by [GP17b],
  - width-k and Horn resolution by [GP17b] (but see also [A02]),
  - SA^k by [ADH15],
  - PC^k and SOS^k?

Part III

QUANTUM RELAXATIONS

An Important Example: Mermin-Peres Magic Square

Nine variables, six equations:

$$
\begin{aligned}
X_{11} X_{12} X_{13} &= +1 \\
X_{21} X_{22} X_{23} &= +1 \\
X_{31} X_{32} X_{33} &= +1 \\
X_{11} X_{21} X_{31} &= +1 \\
X_{12} X_{22} X_{32} &= +1 \\
X_{13} X_{23} X_{33} &= -1
\end{aligned}
$$

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\shortparallel & \shortparallel & \shortparallel & \\
+1 & +1 & -1 &
\end{array}$$

Proof of unsatisfiability (over $\mathbb{R}$)

$$X_{11}X_{12}X_{13}X_{21}X_{22}X_{23}X_{31}X_{32}X_{33} = +1$$
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Remark:

Relies heavily on the fact that product commutes.

Indeed ...

There is a solution in 4x4 complex matrices

$$\begin{array}{cccccc} I \otimes Z & Z \otimes I & Z \otimes Z & = & +I \\ X \otimes I & I \otimes X & X \otimes X & = & +I \\ X \otimes Z & Z \otimes X & Y \otimes Y & = & +I \\ \| & \| & \| & & \\ +I & +I & -I & & \end{array}$$

where X, Y, Z are the Pauli matrices:

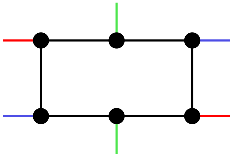
$$X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad Y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \qquad Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Guess what ...

MERMIN SQUARE = TSEITIN($K_{3,3}$, odd)

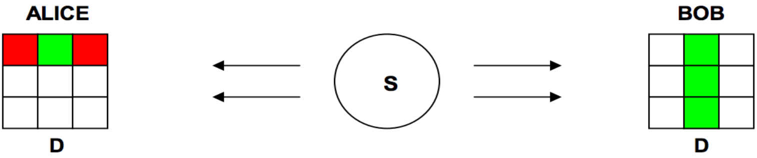
and

$K_{3,3}$ = twisted 6-wall



Where does this come from? Quantum entanglement

[Einstein-Podolsky-Rosen 1935], [Bell 1964], [Mermin 1990]



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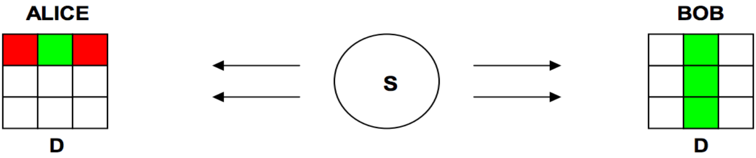

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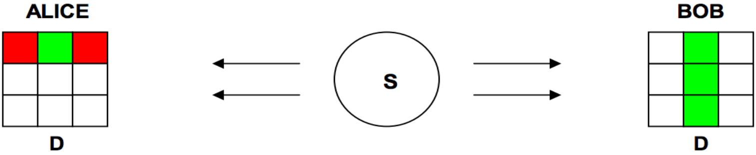

$\hat{p}_{ij,ab}$:= "empirical probability that ij lights as ab"

- Not explained by **classical probability**: $\hat{p}_{ij,ab} \neq \mu(ab|ij)$
- Explained by **quantum entanglement**: $\hat{p}_{ij,ab} = \langle\psi|P_{ij,ab}|\psi\rangle$.

Quantum homomorphisms and isomorphisms

Quantum homomorphisms defined in [MR12]
Quantum isomorphisms defined in [AMRŠSV17]
Both defined in terms of **non-local games**

Here we define them **algebraically**
(equivalences are proved in the papers)

Quantum isomorphism

Variables:

$X_{u,v}$: a variable for each $u \in V(G)$ and $v \in V(H)$

Equations:

$$
\begin{array}{ll}
\sum_v X_{u,v} - 1 = 0 & \text{for all } u \\
\sum_u X_{u,v} - 1 = 0 & \text{for all } v \\
X_{u,v} X_{u',v'} = 0 & \text{for all } u, u', v, v' \text{ s.t. } \mathsf{atp}_G(u,u') \neq \mathsf{atp}_G(v,v') \\
X_{u,v}^2 - X_{u,v} = 0 & \text{for all } u \text{ and } v
\end{array}
$$

Subject to:

Each $X_{u,v}$ is a self-adjoint linear operator of a Hilbert space.

Quantum relaxation of isomorphism

$$G \overset{\text{iso}}{\equiv} H \implies G \overset{\text{qiso}}{\equiv} H \implies G \overset{\mathsf{C}^3}{\equiv} H$$

$$\overset{?}{\Longleftarrow} \qquad \overset{\text{[MRV17]}}{\not\Longleftarrow}$$

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Fact [AMRŠSV17]:

$$\mathsf{CFI}_k^+ \overset{\mathsf{qiso}}{\equiv} \mathsf{CFI}_k^- \quad \text{but} \quad \mathsf{CFI}_k^+ \overset{\mathsf{iso}}{\not\equiv} \mathsf{CFI}_k^- .$$

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Proof.

- View CFI as TSEITIN in disguise (by [ABD07])
- Build the matrix solution from the one for Mermin's square.

A fundamental-looking problem

A fundamental-looking problem

Find a logic $\mathcal{L}$ for which

$$G \overset{\text{qiso}}{\equiv} H \text{ iff } G \overset{\mathcal{L}}{\equiv} H$$

Part IV

QUANTUM SATISFIABILITY

Schaefer's framework for generalized satisfiability

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Boolean domain: $\{\pm 1\}$ with $+1 =$ *false* and and $-1 =$ *true*;

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$R(X_1, \ldots, X_r) = 0$

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Examples:

OR	disjunctions of literals
LIN	linear equations over $\mathbb{Z}_2$
1-IN-3	triples with one -1 and two $+1$ components
NAE	triples with not-all-equal components

Generalized Satisfiability Problems: SAT(A)

$$\exists X_1 \cdots X_n (C_1 \wedge \cdots \wedge C_m)$$

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in A

X_i's or ± 1

Examples:

3-SAT	1-IN-3-SAT
HORN-SAT	NAE-SAT
LIN-SAT	...

[Schaefer 1978]

… via Operator Assignments [CM14]

$$\exists X_1 \cdots X_n (C_1 \wedge \cdots \wedge C_m)$$

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and
$X_i^2 = I$ for all $i \in [n]$

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$Y_i Y_j = Y_j Y_i$ for all $i, j \in [r]$
and
$X_i^2 = I$ for all $i \in [n]$

SAT(A)	satisfiability over Boolean domain
SAT*(A)	satisfiability over some finite-dimensional $\mathcal{H}$
SAT**(A)	satisfiability over some arbitrary $\mathcal{H}$

Gap Instances

Mermin-Peres Magic Square

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X_{11} & X_{12} & X_{13} & = +1 \\
X_{21} & X_{22} & X_{23} & = +1 \\
X_{31} & X_{32} & X_{33} & = -1 \\
\parallel & \parallel & \parallel & \\
+1 & +1 & +1 &
\end{array}
$$

Unsatisfiable SAT-instance of LIN

Satisfiable SAT*-instance of LIN

↓

a SAT-vs-SAT* gap for LIN

Gap Instances

Mermin-Peres Magic Square

$$\begin{array}{ccc} X_{11} X_{12} X_{13} = +1 \\ X_{21} X_{22} X_{23} = +1 \\ X_{31} X_{32} X_{33} = -1 \\ \| \quad \| \quad \| \\ +1 \quad +1 \quad +1 \end{array}$$

Unsatisfiable SAT-instance of LIN

Satisfiable SAT*-instance of LIN

↓

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SAT-vs-SAT*	gap of the first kind
SAT-vs-SAT**	gap of the second kind
SAT*-vs-SAT**	gap of the third kind

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Gaps of first kind for LIN exist [Mermin 1990]

Gaps of third kind for LIN exist [Slofstra 2017]

Gaps of first kind for 2-SAT or HORN do not exist [Ji 2014]

Classification

Theorem:
For every Boolean constraint language A,

1. either gaps of every kind for A exist,
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iff
LIN is **not** pp-definable from A

Comparison with Schaefer's dichotomy for tractability

	Tractable	Gaps exist
0-valid/1-valid	YES	NO
Horn/dual-Horn	YES	NO
bijunctive	YES	NO
linear	YES	YES
anythingelse	NO	YES

Primitive Positive Definitions

$$R(Y_1, \ldots, Y_r) \quad \equiv \quad \exists Z_1 \cdots \exists Z_s (C_1 \wedge \cdots \wedge C_t)$$

auxiliary variables

constraints on the Y's and Z's

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Example:

$$\mathsf{NAE}(X, Y, Z) \;\; \equiv \;\; (X \vee Y \vee Z) \wedge (\overline{X} \vee \overline{Y} \vee \overline{Z})$$

Proof technique

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Ingredient 1: gap preserving reductions

Lemma:
If A is pp-definable from B,
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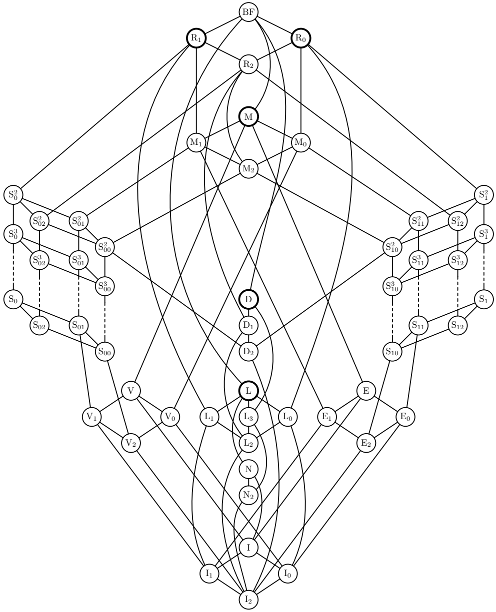
then gaps for B imply gaps for A.

Ingredient 2: Post's Lattice of Boolean co-clones

Theorem [Post 1941]:
There are countably many Boolean constraint languages up to pp-definability, **and we know them**.

Post's Lattice



More on Primitive Positive Definability

$$R(Y_1, \ldots, Y_r) \quad \equiv \quad \exists Z_1 \cdots \exists Z_s (C_1 \wedge \cdots \wedge C_t)$$

pp-def Z_i's range over $B(\mathbb{C})$ (i.e., over $\{\pm 1\}$ by $Z_i^2 = I$)
pp*-def Z_i's range over $B(\mathcal{H})$, for some finite-dim $\mathcal{H}$
pp**-def Z_i's range over $B(\mathcal{H})$, for some arbitrary $\mathcal{H}$

A Conservativity Theorem

Theorem:
For every two constraint languages A and B,
the following statements are equivalent.

1. every relation in A is pp-definable from B
2. every relation in A is pp*-definable from B

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Corollary: OR is **not** pp*-definable from LIN

Closure Operations via Operators

R is invariant under $F : B(\mathcal{H}_1) \times \cdots \times B(\mathcal{H}_s) \to B(\mathcal{H})$ if

$$\frac{\begin{matrix} R(& A_{1,1} & ,\cdots, & A_{1,r} &) = 0 \text{ and commute} \\ & \vdots & \ddots & \vdots & \\ R(& A_{s,1} & ,\cdots, & A_{s,r} &) = 0 \text{ and commute} \end{matrix}}{R(F(\mathbf{A}_{*,1}), \cdots, F(\mathbf{A}_{*,r})) = 0 \text{ and commute}}$$

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Lemma: If A is invariant under $F : \{\pm 1\}^s \to \{\pm 1\}$, then every $R \subseteq \{\pm 1\}^r$ pp*-definable from A is invariant under

$$F^*(X_1, \ldots, X_s) := \sum_{S \subseteq [s]} \widehat{F}(S) \bigotimes_{i=1}^{s} X_i^{S(i)}$$

Proof by Example

Operator composition doesn't work:

$$
\begin{array}{cccl}
X_{11} & X_{12} & X_{13} & = +1 \\
X_{21} & X_{22} & X_{23} & = +1 \\
X_{31} & X_{32} & X_{33} & = +1 \\
\parallel & \parallel & \parallel & \\
+1 & +1 & -1 & \neq +1 \text{ (!)}
\end{array}
$$

Proof by Example

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$$\begin{array}{cccl} X_{11} & X_{12} & X_{13} & = +1 \\ X_{21} & X_{22} & X_{23} & = +1 \\ X_{31} & X_{32} & X_{33} & = +1 \\ \parallel & \parallel & \parallel & \\ +1 & +1 & -1 & \neq +1 \text{ (!)} \end{array}$$

Operator tensoring works:

$$(X_{11} \otimes X_{21} \otimes X_{31})(X_{12} \otimes X_{22} \otimes X_{32})(X_{13} \otimes X_{23} \otimes X_{33}) =$$
$$(X_{11}X_{12}X_{13}) \otimes (X_{21}X_{22}X_{23}) \otimes (X_{31}X_{32}X_{33}) =$$
$$(+I) \otimes (+I) \otimes (+I) =$$
$$+I$$

Future Work

Question 1:

Are $\mathsf{SAT}^*(\mathsf{LIN})$ and QISO^* decidable?

Question 2:

Is pp**-definability $=$ pp-definability also?

Question 3:

Find a logic $\mathcal{L}$ for which

$G \overset{\text{qiso}}{\equiv} H$ iff $G \overset{\mathcal{L}}{\equiv} H$

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