Automata minimization and glueing of categories

13 12 2017 Berkeley

Thomas Colcombet
joint work with Daniela Petrişan

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[MFCS 2017] & [Informal presentation in SIGLOG column]

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Description of the situation

Automata

Automata

An deterministic automaton is

$$\langle Q, i, f, (\delta_a)_{a \in A} \rangle$$

where

Q is a set of states,

$i : 1 \to Q$ is the initial map

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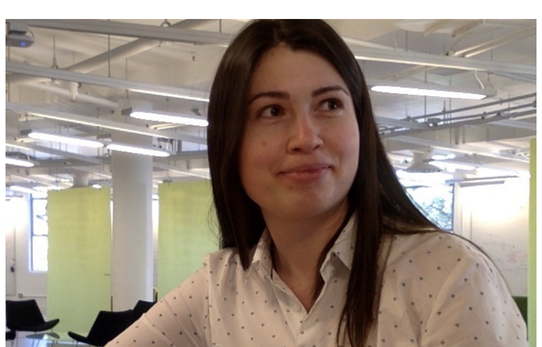

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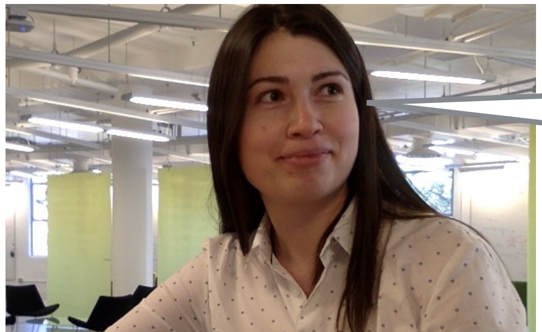

These data can modeled as a functor.

Example

$$L_{\texttt{Vec}}(u) = \begin{cases} 2^{|u|_a} & \text{if } |u|_b \text{ is even, and } |u|_c = 0 \\ 0 & \text{otherwise} \end{cases}$$

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$$\delta_a(x, y) = (2x, 2y)$$
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Is it possible to do better?

A better implementation

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Informally: use one bit for the parity to the number of b's.

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$$Q = (\{\texttt{odd}\} \times \mathbb{R}) \cup (\{\texttt{even}\} \times \mathbb{R})$$

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Why is it a better implementation?

Is there a good notion of such automata?

What are their properties (e.g. minimization) ?

A definition via categories

Automata in a category

Automata in a category

A (C,I,F)-automaton is

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where

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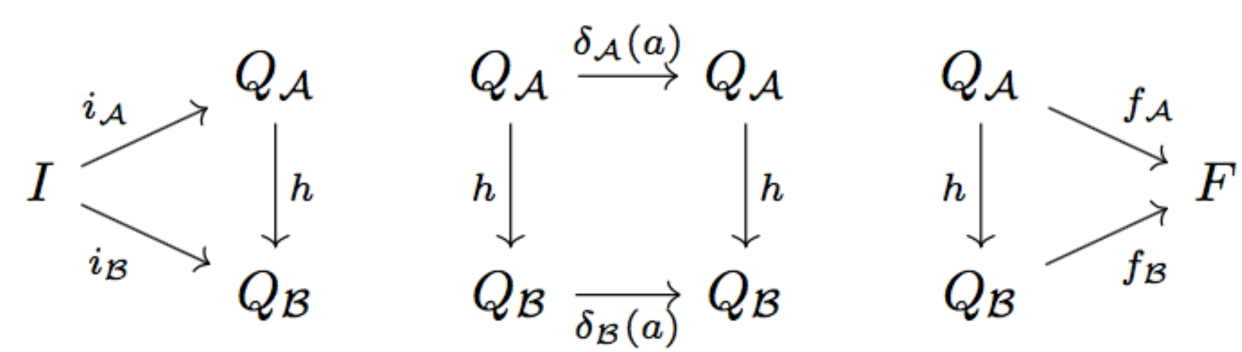

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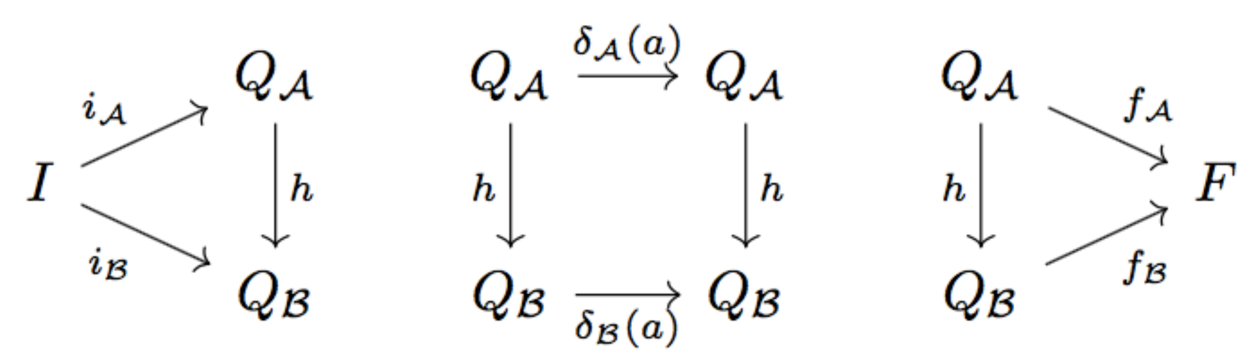

Rk: Morphisms preserve the language.

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- (Set,1,2)-automata are deterministic automata
- (Rel,1,1)-automata are non-deterministic automata
- (Vec,K,K)-automata are automata weighted over a field K. (more generally semi-modules)
- …

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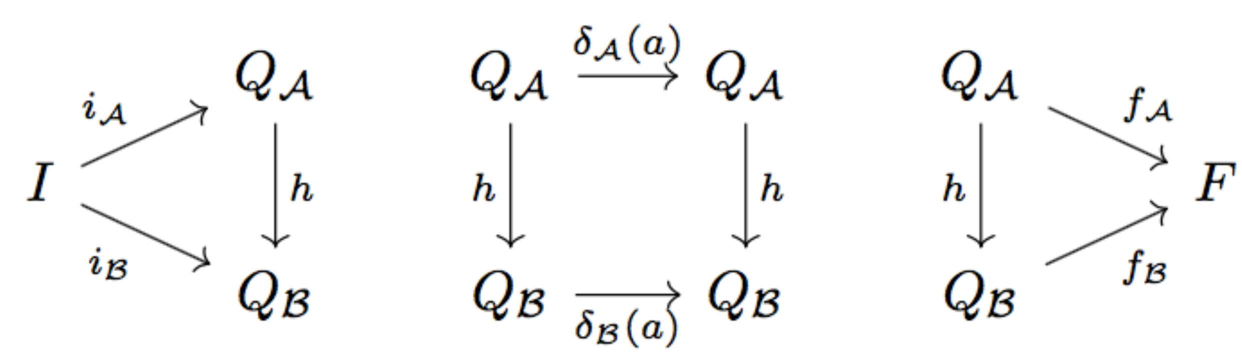

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(free co-product completion of **Vec**)

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A morphism from $(I, (V_i)_{i \in I})$ to $(J, (W_i)_{i \in J})$ is the pair of:

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Remark: Vec is a subcategory of Duvs.

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What can be done?

Minimizing automata via categories

Ingredients for the existence of a minimal automaton

Questions:

Given a (C,I,F)-automaton,

- what does it mean to be minimal?
- at what condition there exists a minimal automaton for a language?
- what do we need to effectively compute it?

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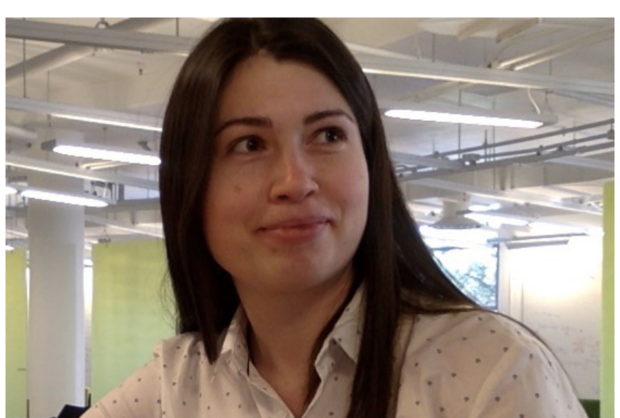

It suffices to have
1. an initial automaton
2. a final automaton
3. a factorization system

Factorization systems

A pair of families of arrows $(\mathcal{E}, \mathcal{M})$ is a factorization system if:

Factorization systems

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$$f = m \circ e$$

for some $e\colon X \to Z$ in $\mathcal{E}$ and $m\colon Z \to Y$ in $\mathcal{M}$.

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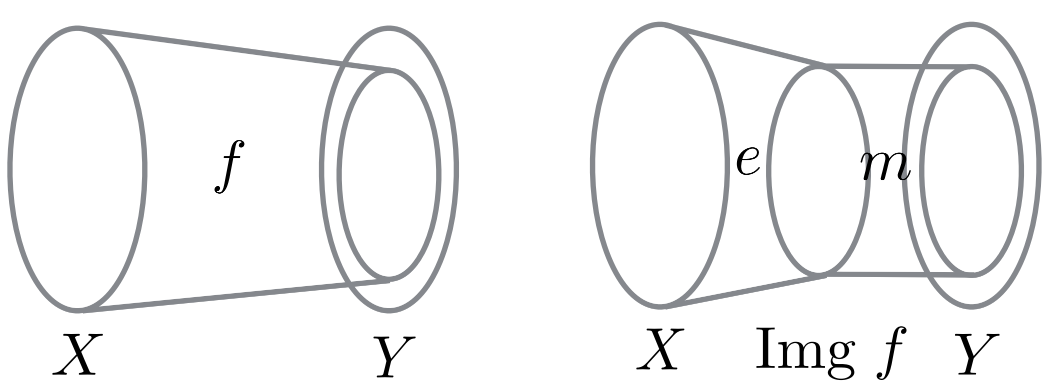

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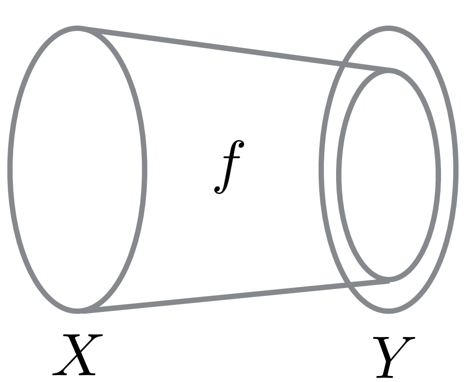

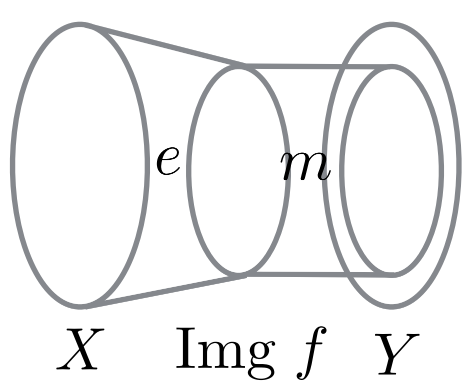

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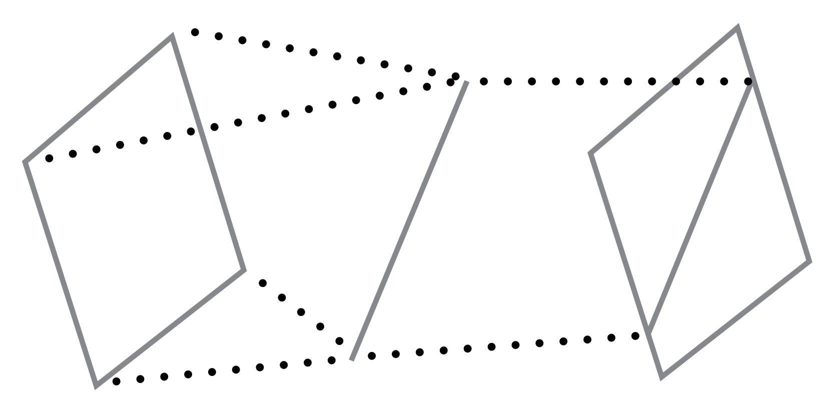

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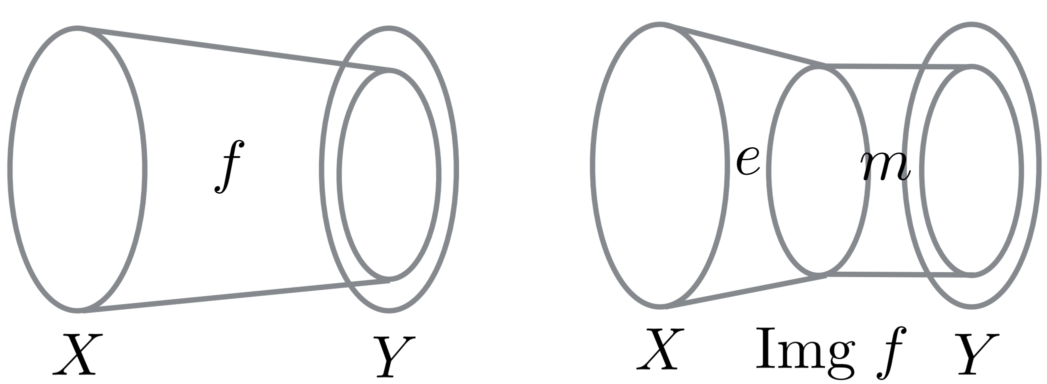

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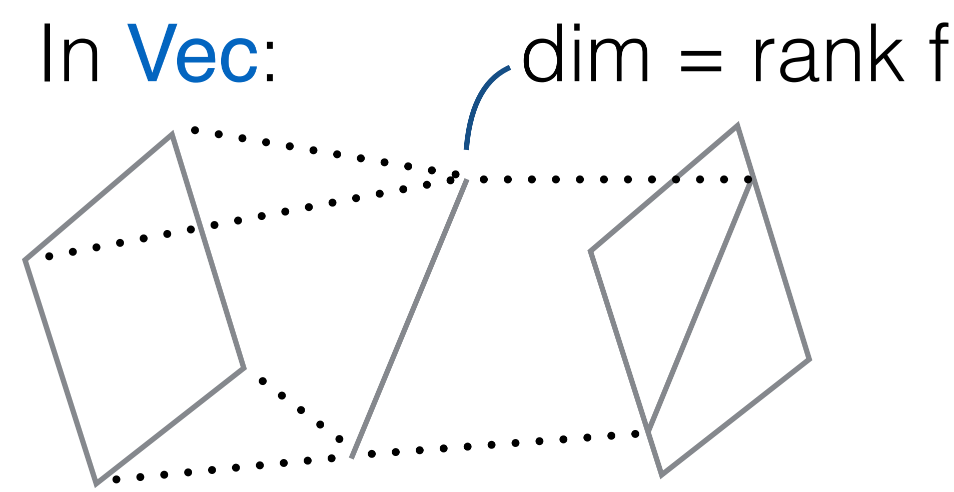

Factorization system for automata

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Lemma: If there is a factorization system $(\mathcal{E}, \mathcal{M})$ in a category $\mathcal{C}$ then it can be lifted to the category of $\mathcal{C}$-automata for a language: these automata morphisms that belong to $\mathcal{E}$ (resp. $\mathcal{M}$) as arrows in $\mathcal{C}$.

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Definition:

- an $\mathcal{M}$-subobject X of Y is such that there is an $\mathcal{M}$-arrow $m\colon X \to Y$,
- an $\mathcal{E}$-quotient X of Y is such that there is an $\mathcal{E}$-arrow $e\colon Y \to X$,
- X $(\mathcal{E}, \mathcal{M})$-divides Y if it is a $\mathcal{E}$-quotient of an $\mathcal{M}$-subobject of Y.

Minimization !

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Lemma: In a category with initial object, final object, and a factorization system $(\mathcal{E},\mathcal{M})$ then:

- there exists an object $\mathtt{Min}$ that $(\mathcal{E},\mathcal{M})$-divides all objects,
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where

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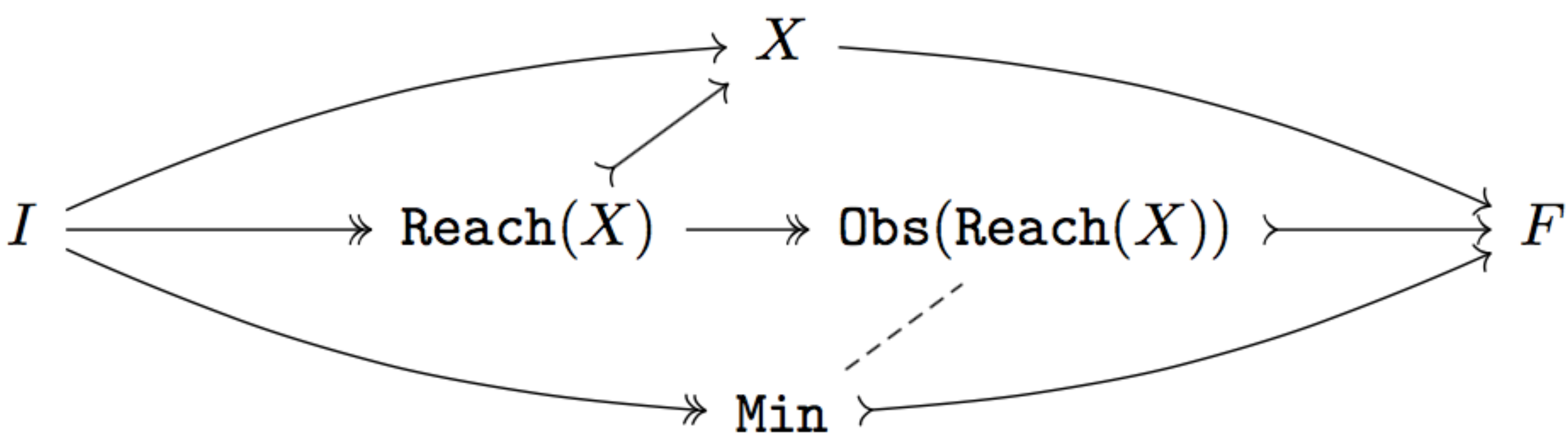

At this point…

We know that:

- C-automata and C-languages can be defined generally in a category C, yielding a

  category Auto(L) of « C-automata for the language L »

- for having a minimal object in a category, it is sufficient to have:
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At this point…

We know that:

- C-automata and C-languages can be defined generally in a category C, yielding a

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But, what about minimizing duvs-automata?

Minimization of Duvs automata (wrong version)

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However, for the definition of factorization system that works (epi,mono), the minimal automaton for

$$L_{\texttt{Vec}}(u) = \begin{cases} 2^{|u|_a} & \text{if } |u|_b \text{ is even, and } |u|_c = 0 \\ 0 & \text{otherwise} \end{cases}$$

has state space

$$Q = \mathbb{R}^2$$

and not

$$Q = (\{\texttt{odd}\} \times \mathbb{R}) \cup (\{\texttt{even}\} \times \mathbb{R})$$

Glueings

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Vec-automaton

$Q = \mathbb{R}^2$

$i(x) = (x, 0)$

$f(x, y) = x$

$\delta_a(x, y) = (2x, 2y)$
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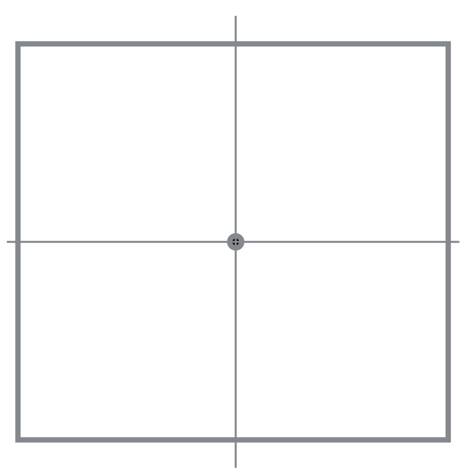

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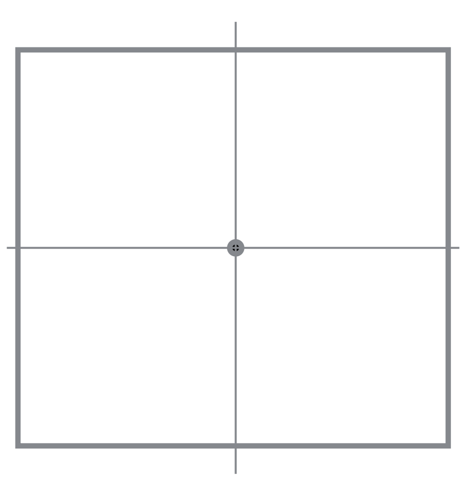

Duvs-automaton

$Q = \{\texttt{odd}, \texttt{even}\} \times \mathbb{R}$

$i(x) = (\texttt{even}, x)$

$f(\texttt{even}, x) = x$
$f(\texttt{odd}, x) = 0$

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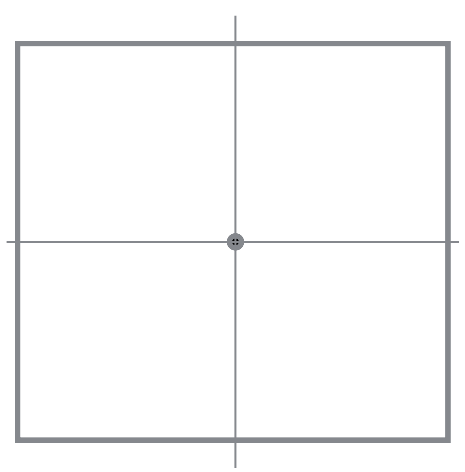

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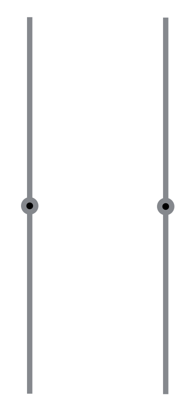

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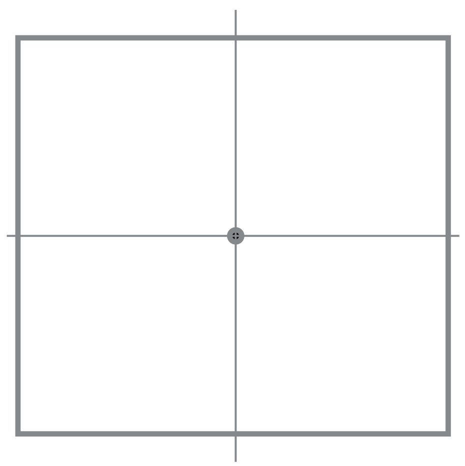

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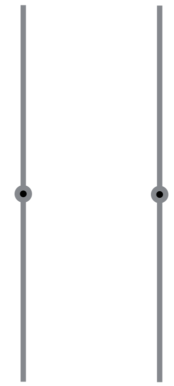

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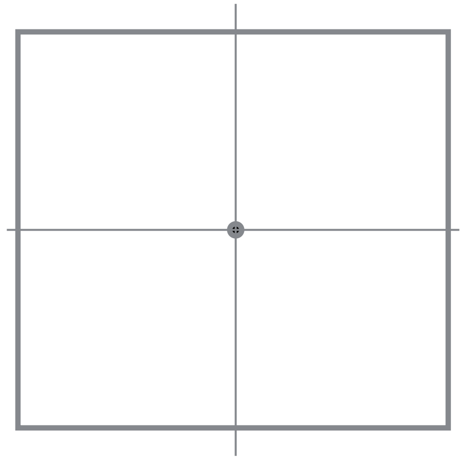

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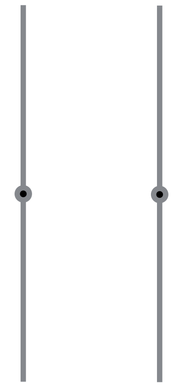

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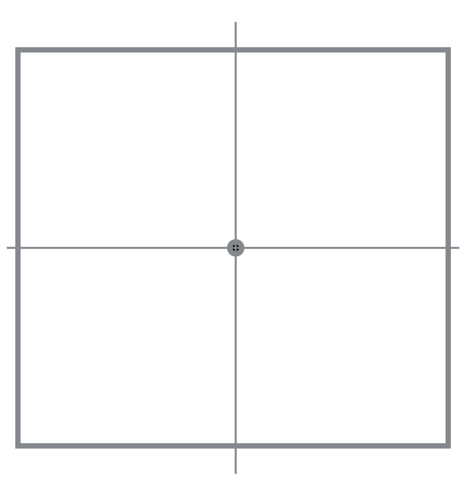

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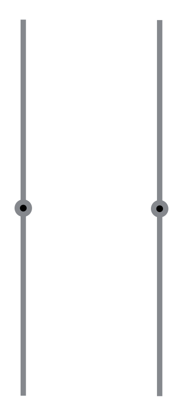

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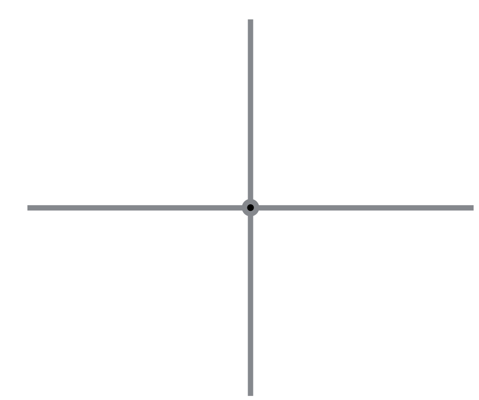

Defining Glue(Vec)

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A glueing of vector space is

- a disjoint union of vector spaces
- together with an equivalence relation which:
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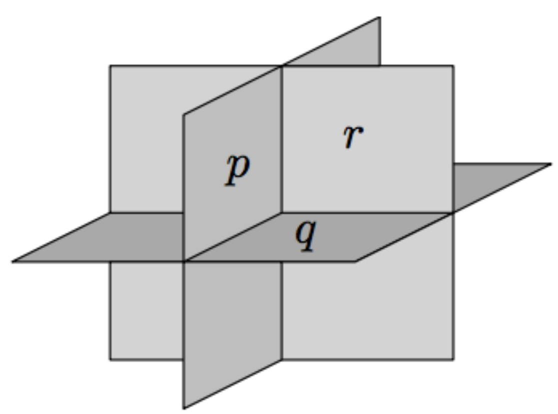

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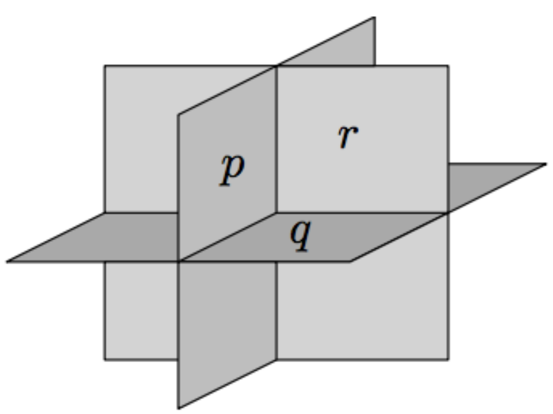

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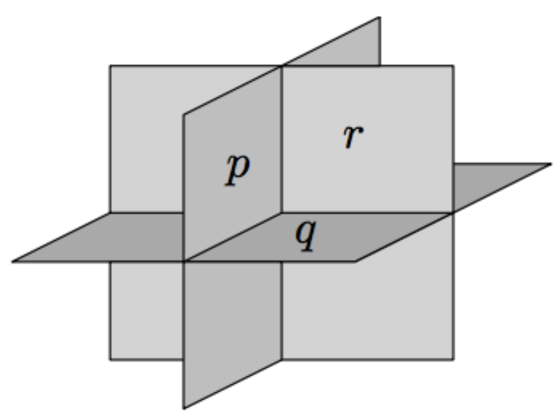

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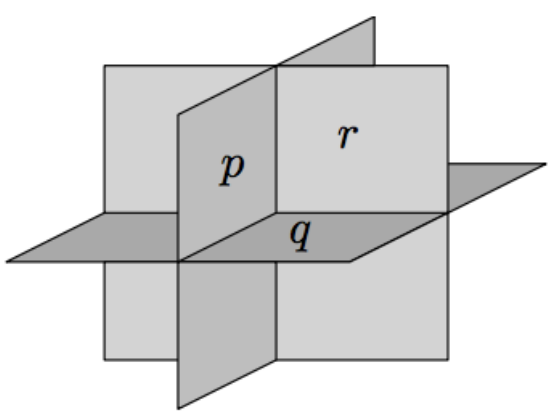

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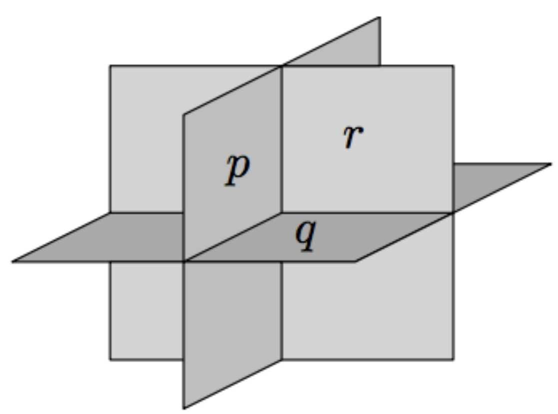

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The category of glueings of vector spaces is the restriction of the co-completion of Vec to some specific colimits: mono-colimits.

The advantage is that the concepts are well known, definition properly stated, and this can be applied to other categories than Vec.

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Consider a category that already has colimits (for instance Vec)

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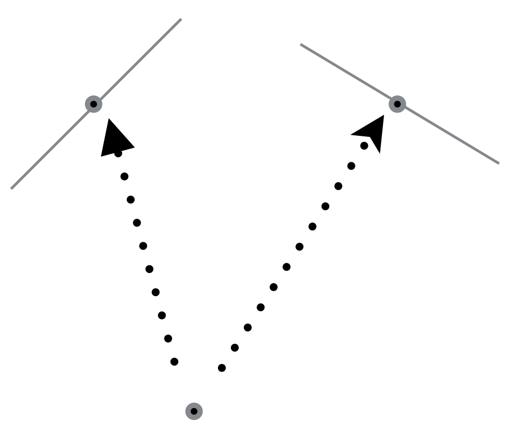

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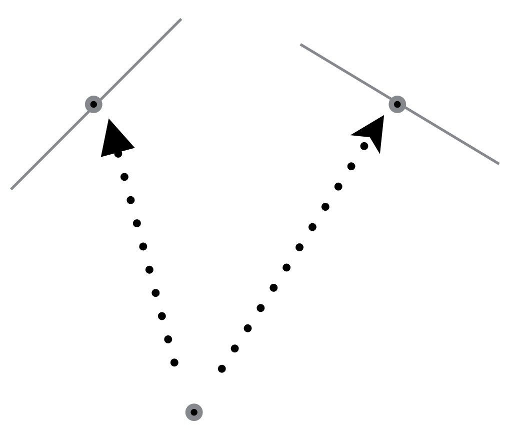

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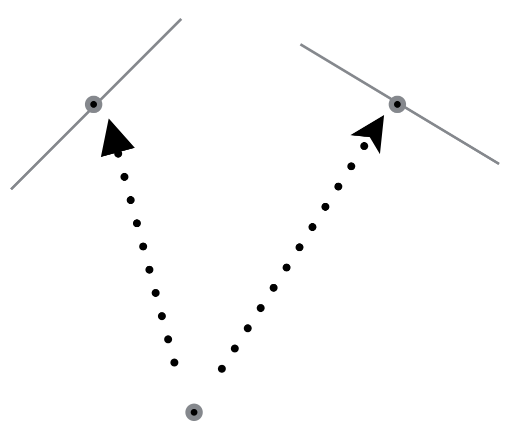

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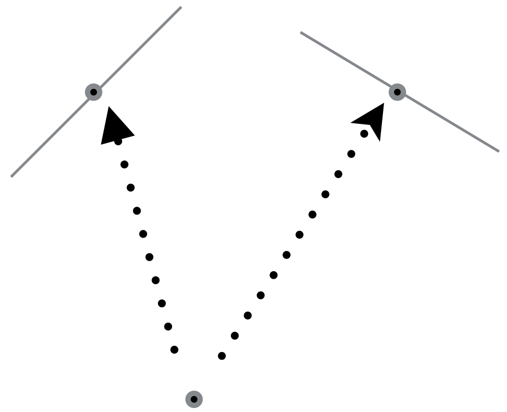

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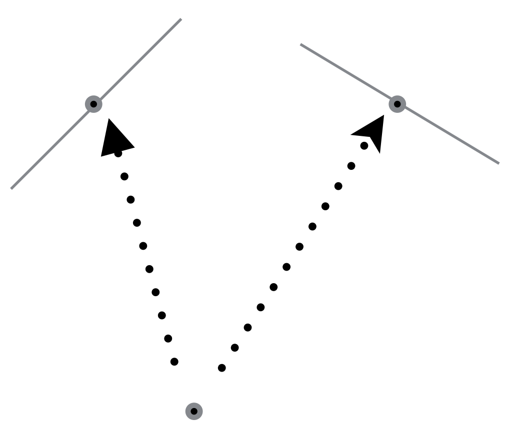

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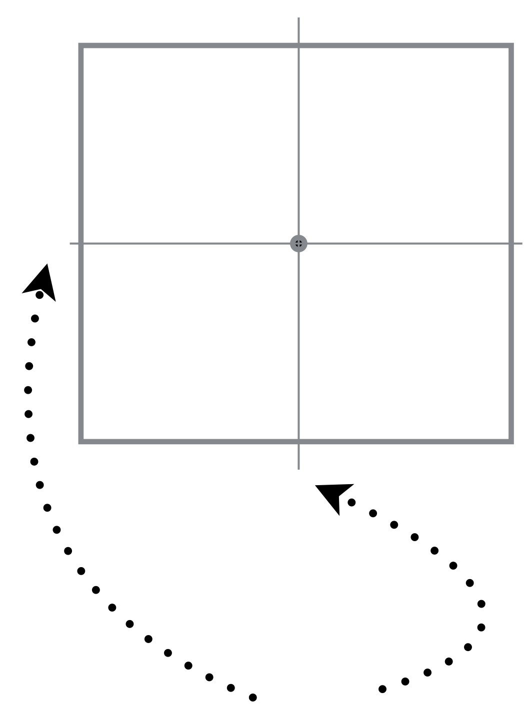

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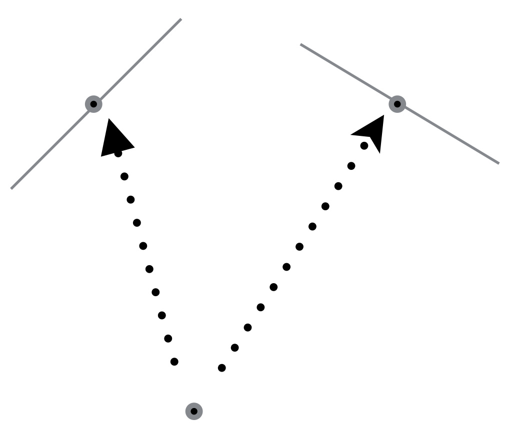

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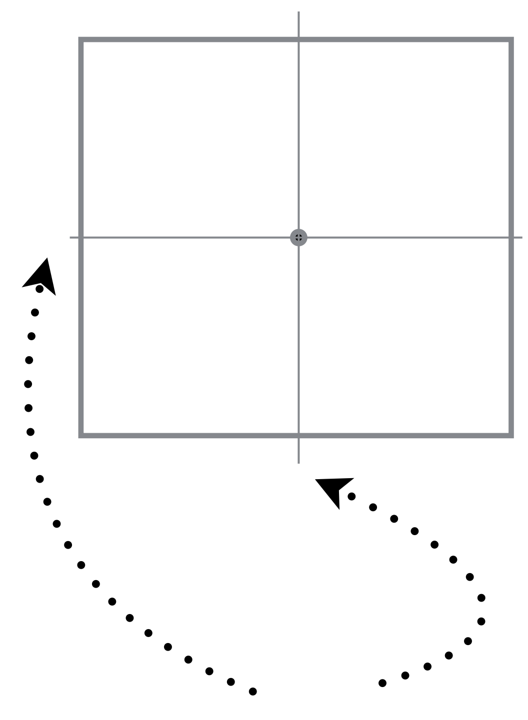

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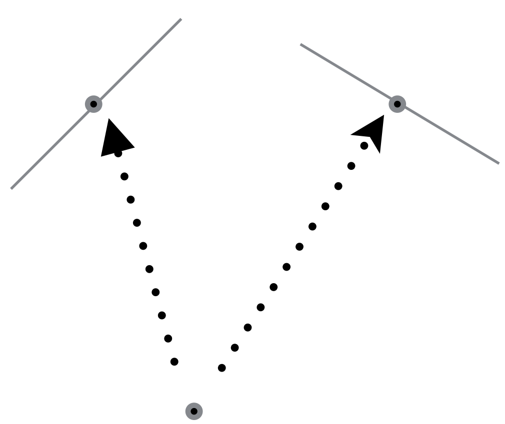

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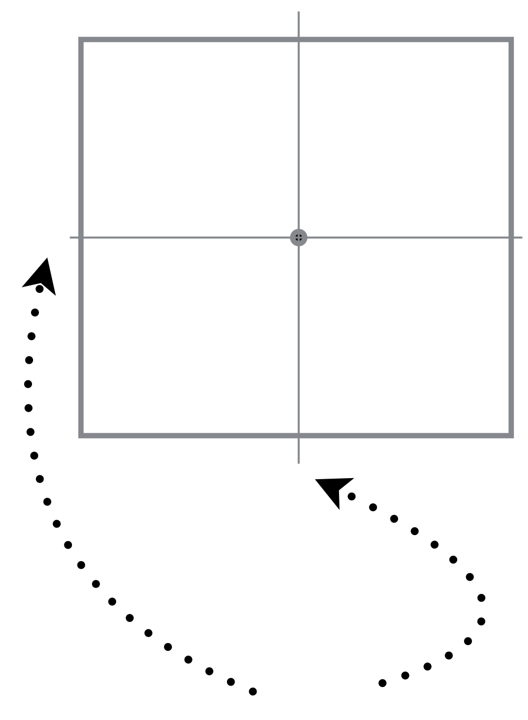

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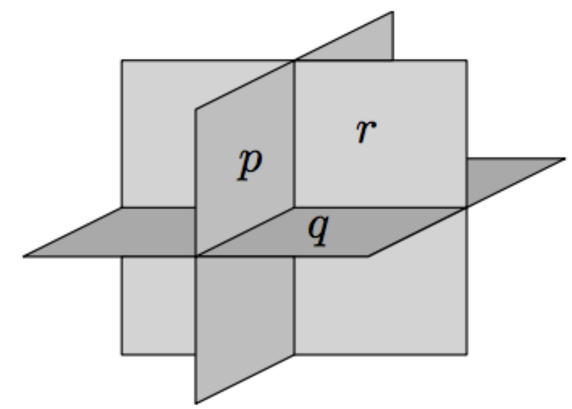

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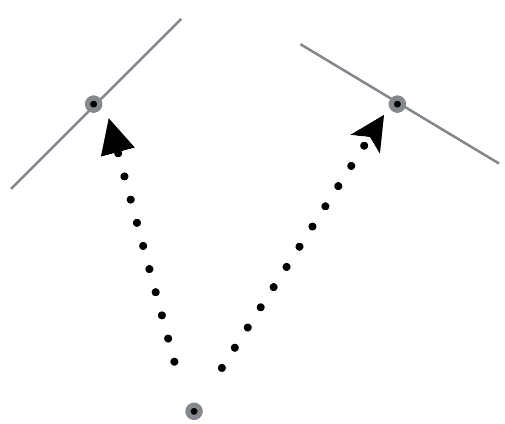

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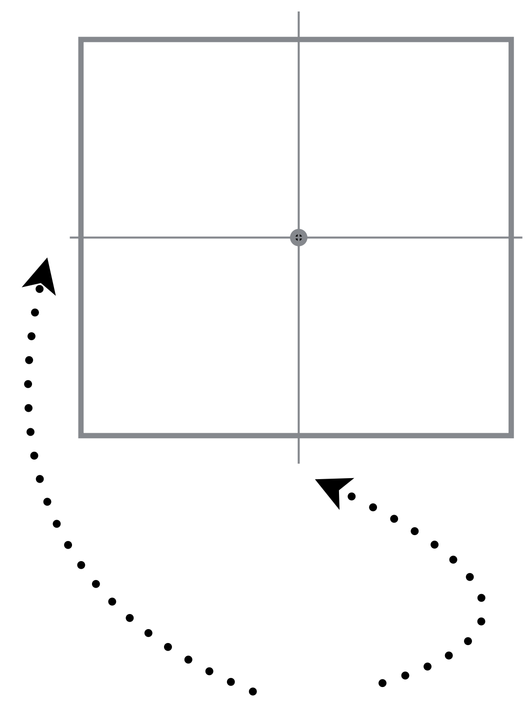

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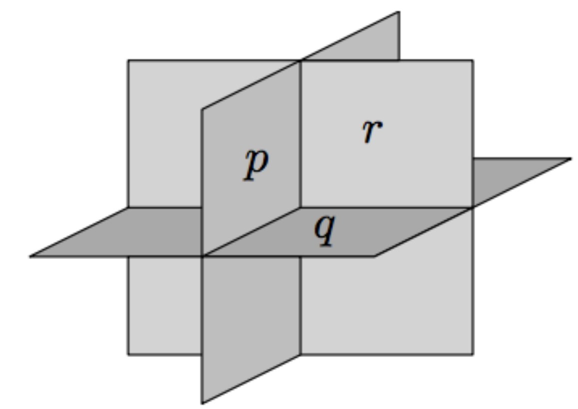

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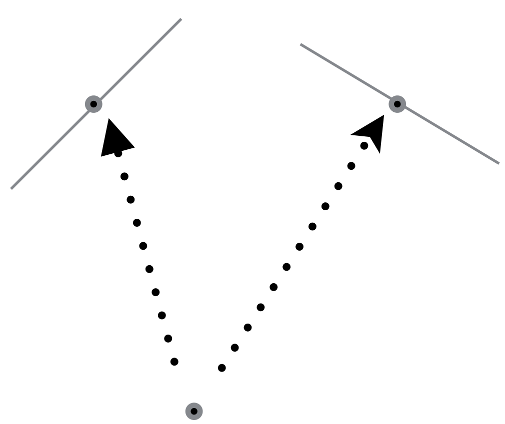

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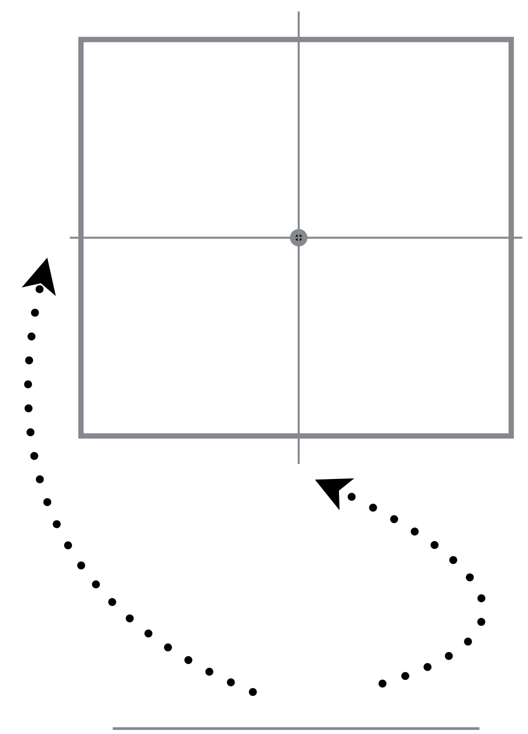

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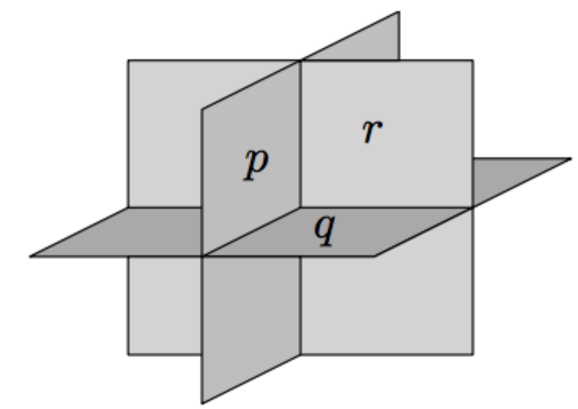

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Definition:
The glueings of a category is its free completion under mono-co-limits

Example: continued

The minimal automaton for our example is:

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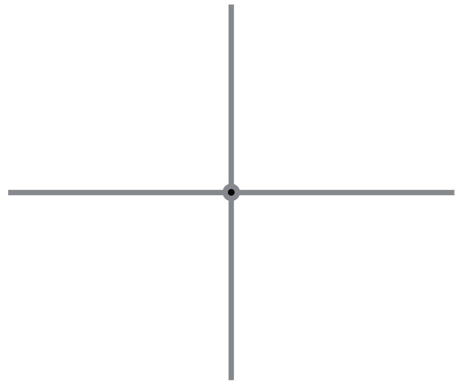

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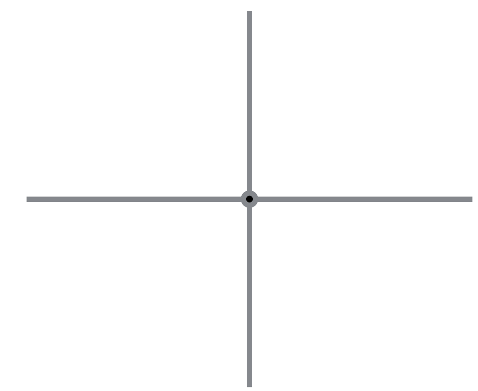

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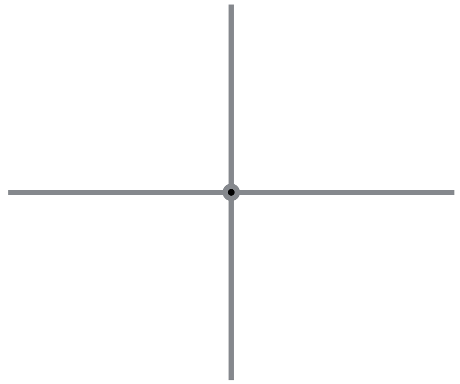

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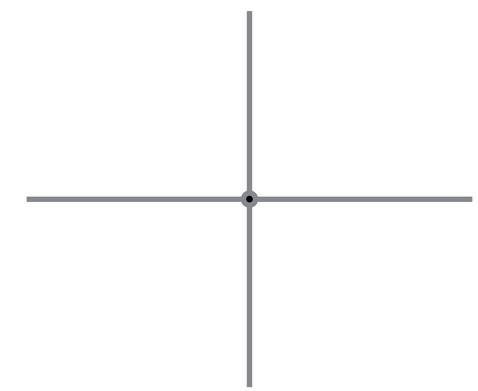

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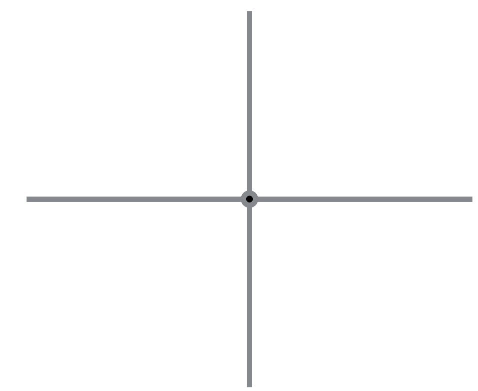

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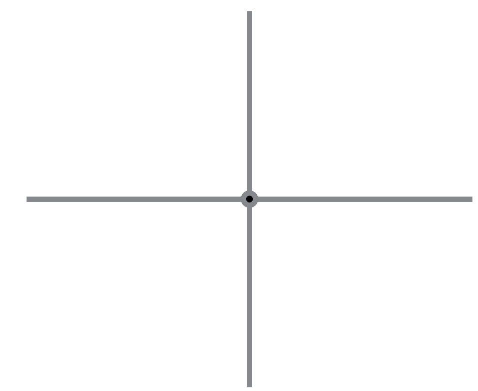

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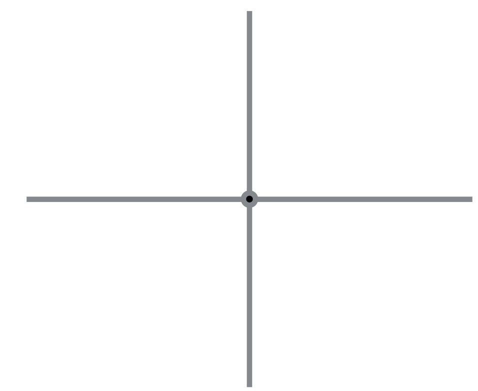

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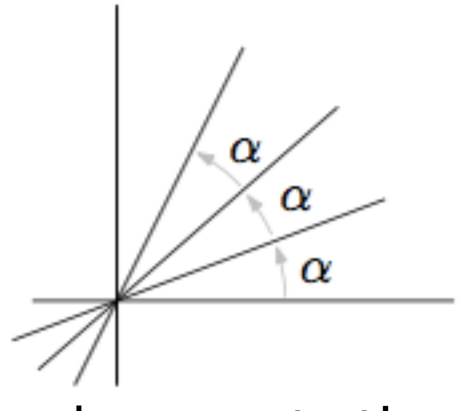

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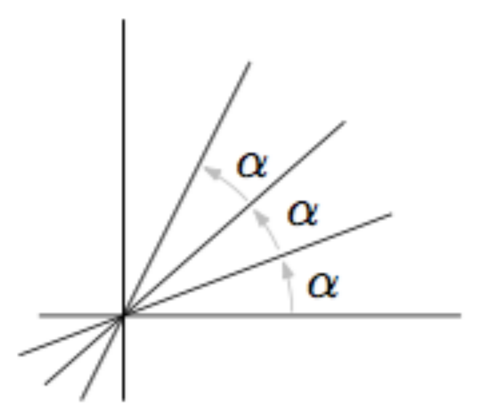

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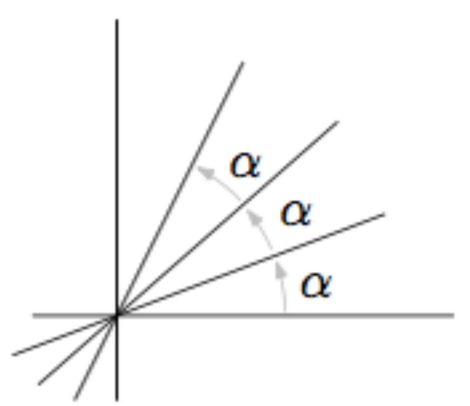

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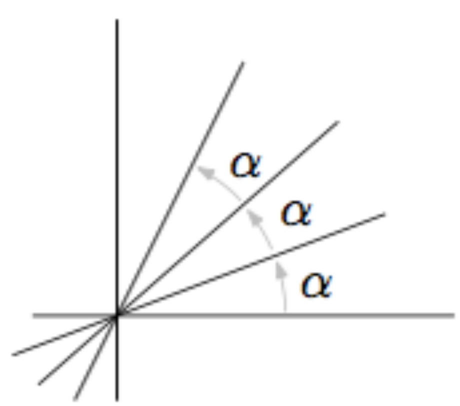

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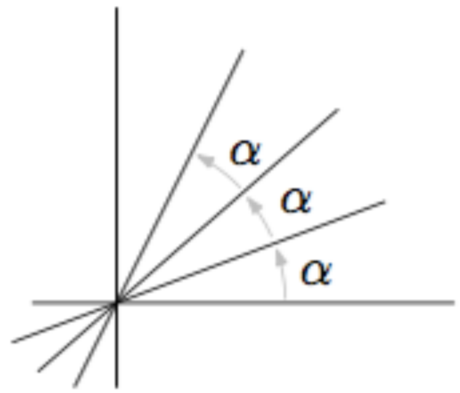

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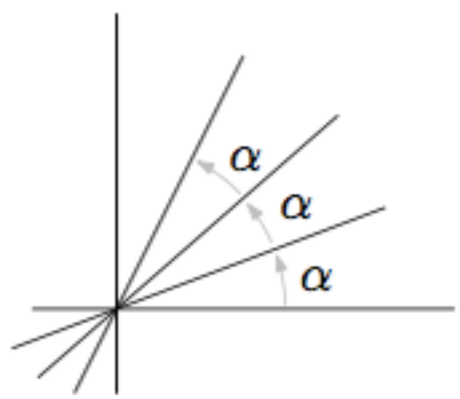

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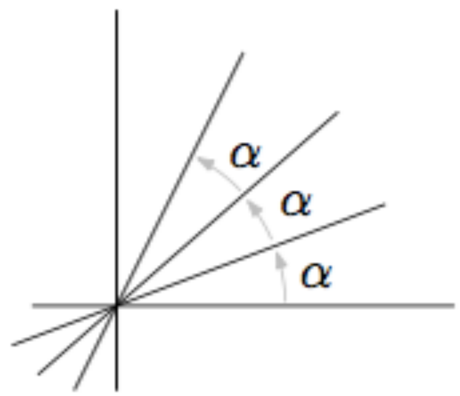

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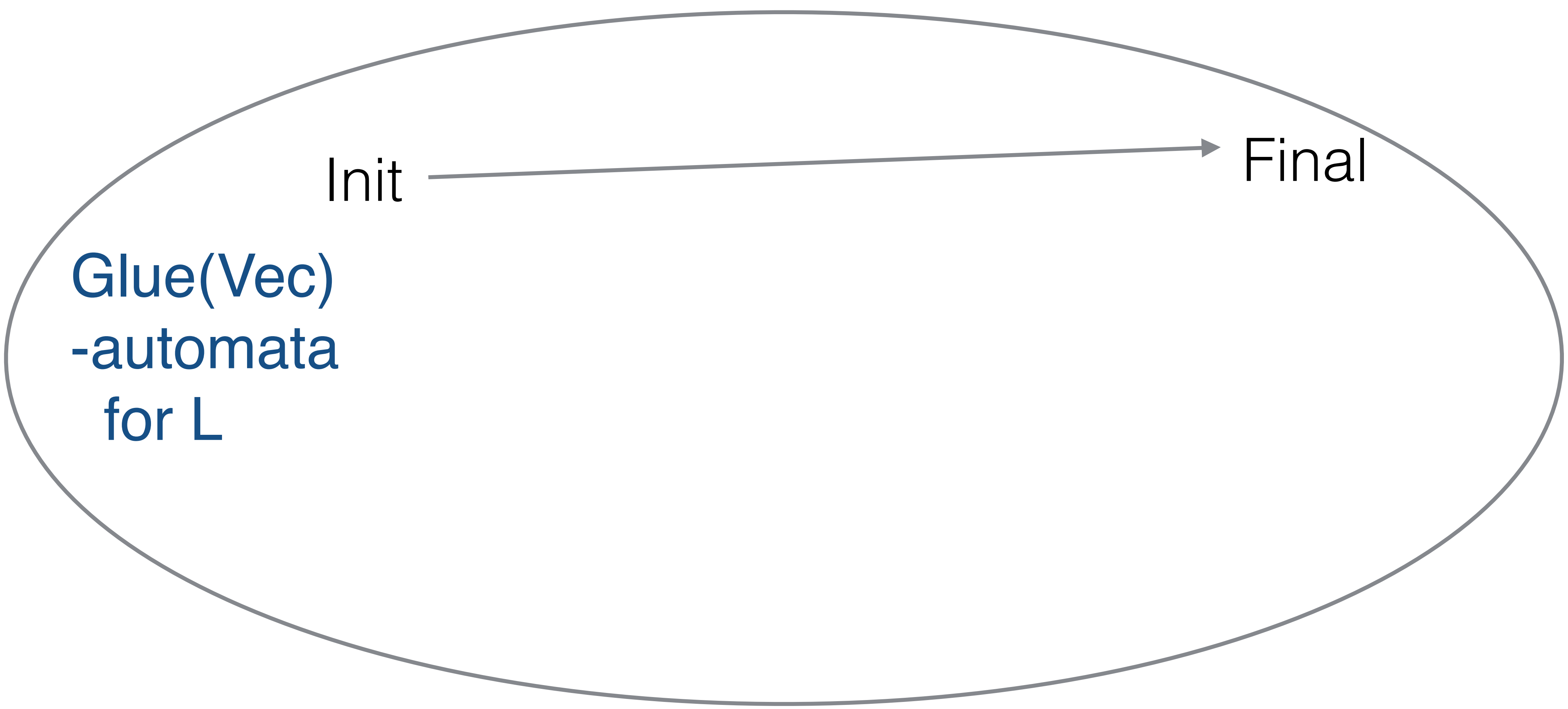

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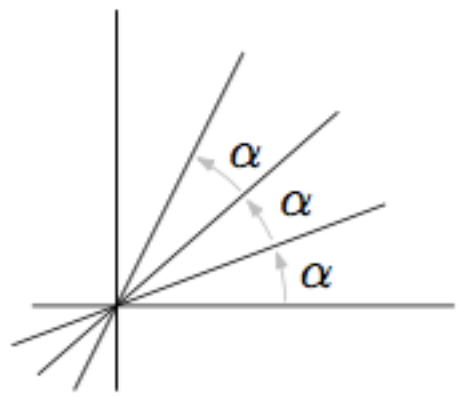

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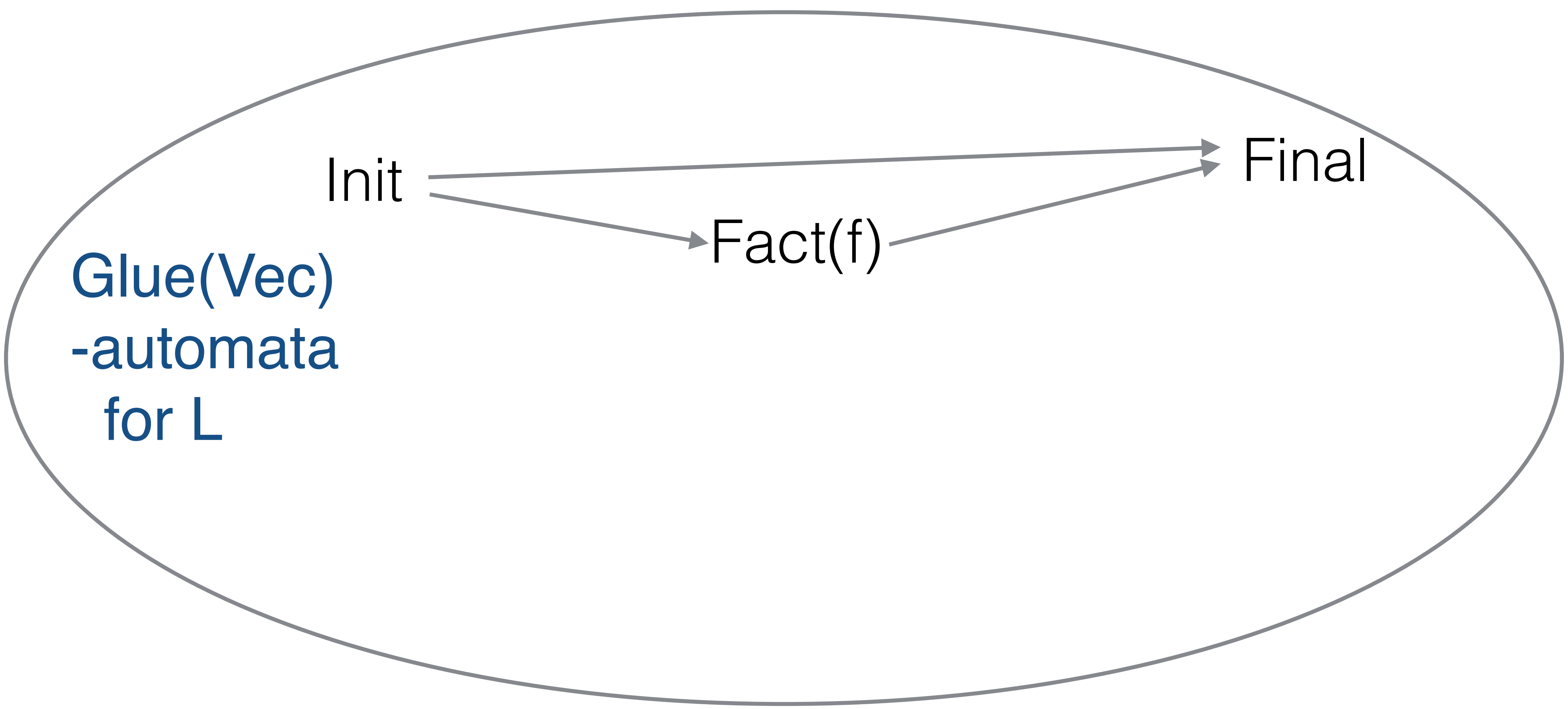

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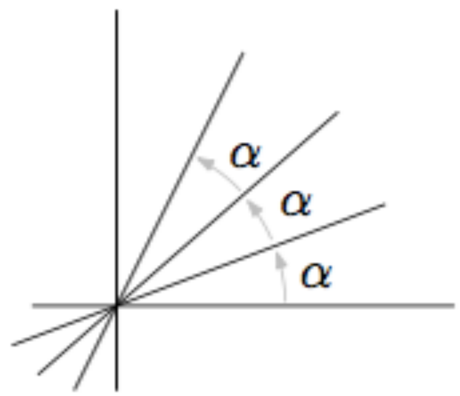

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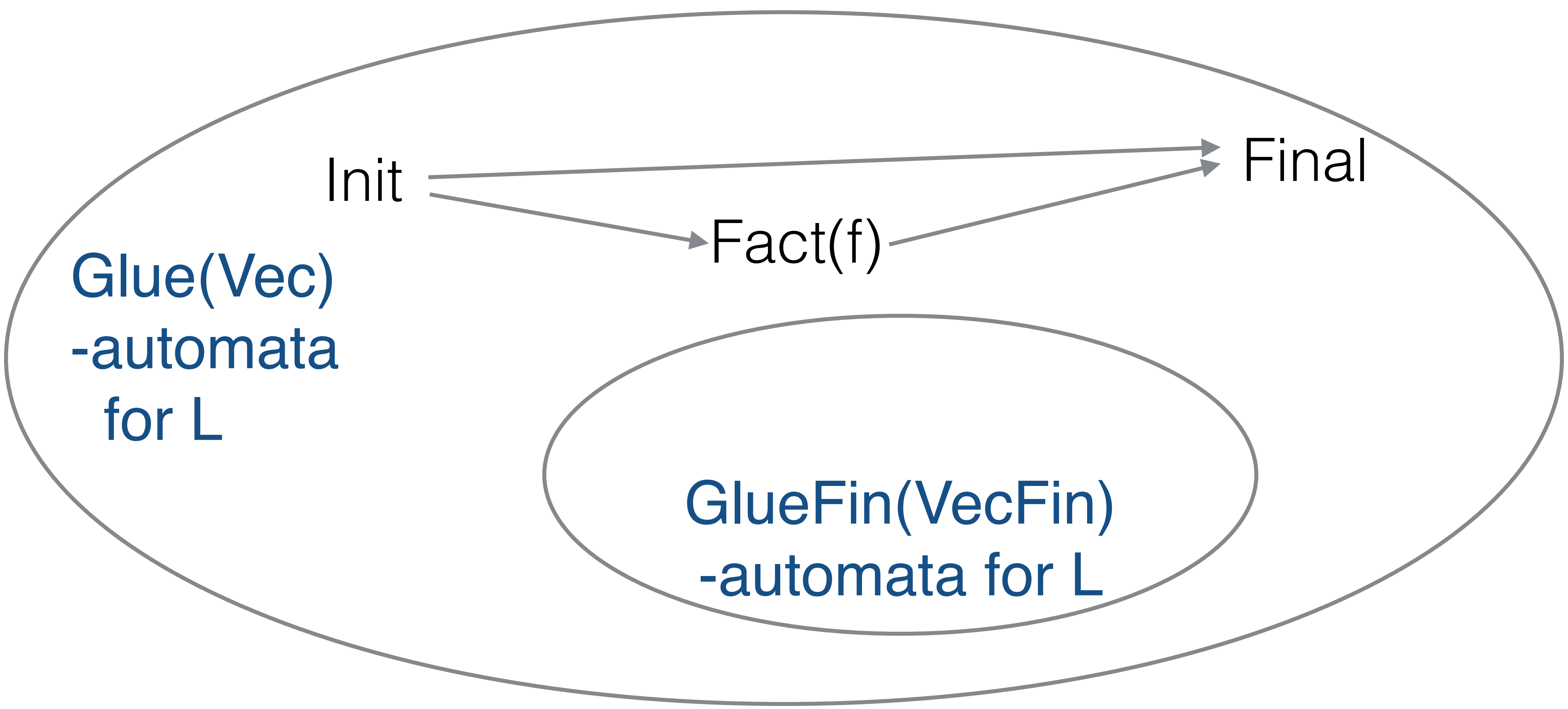

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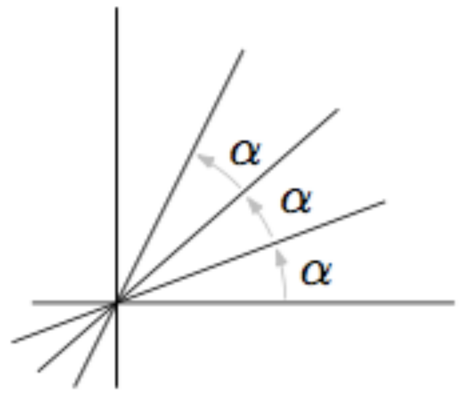

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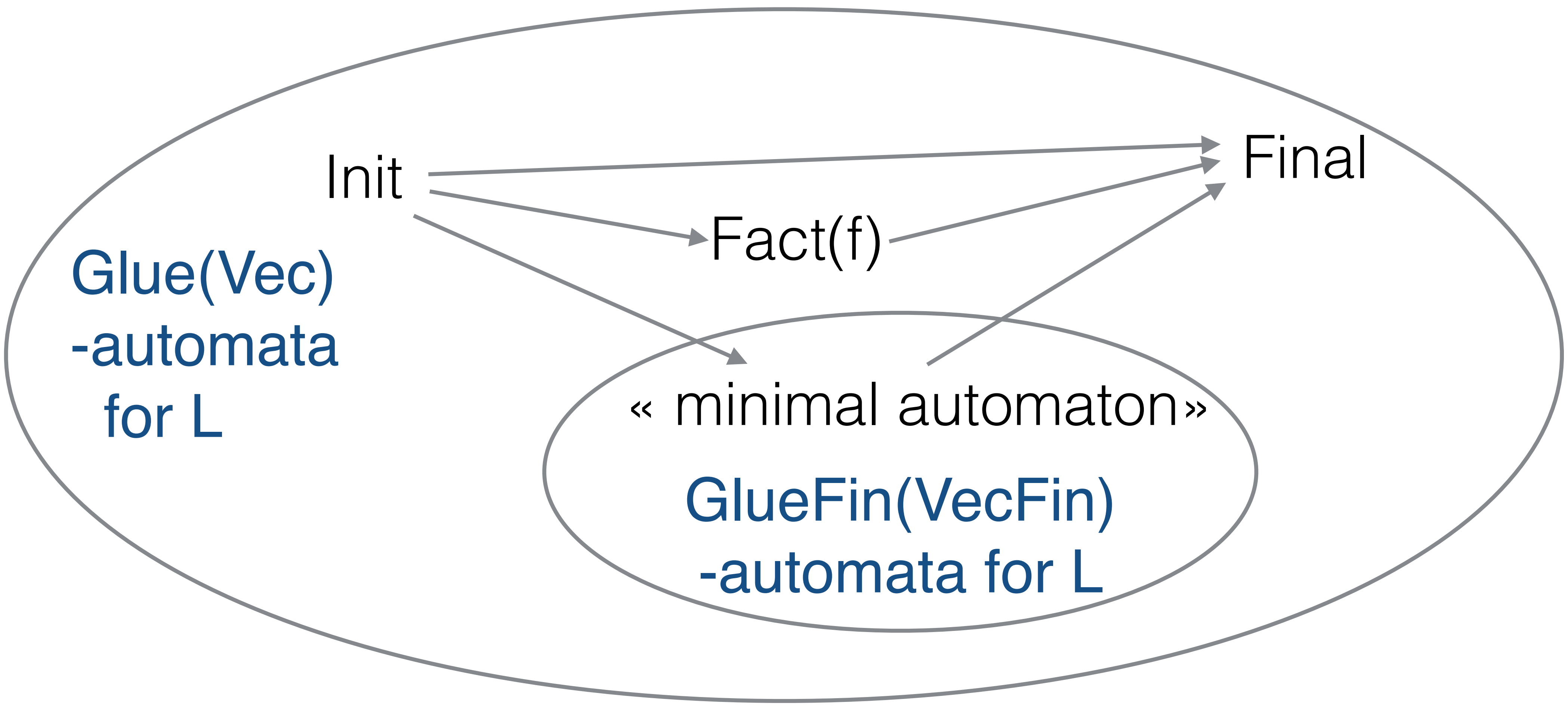

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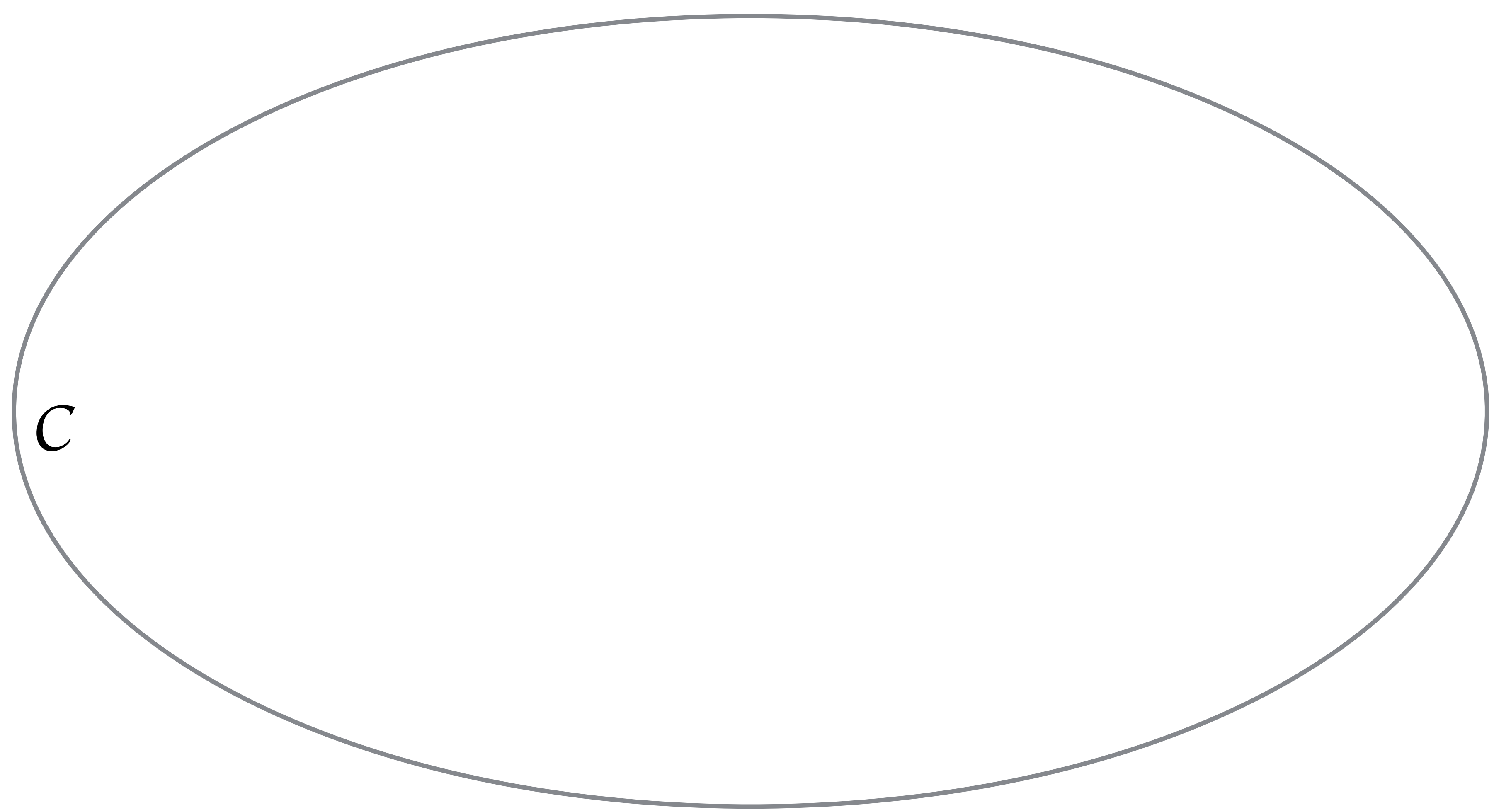

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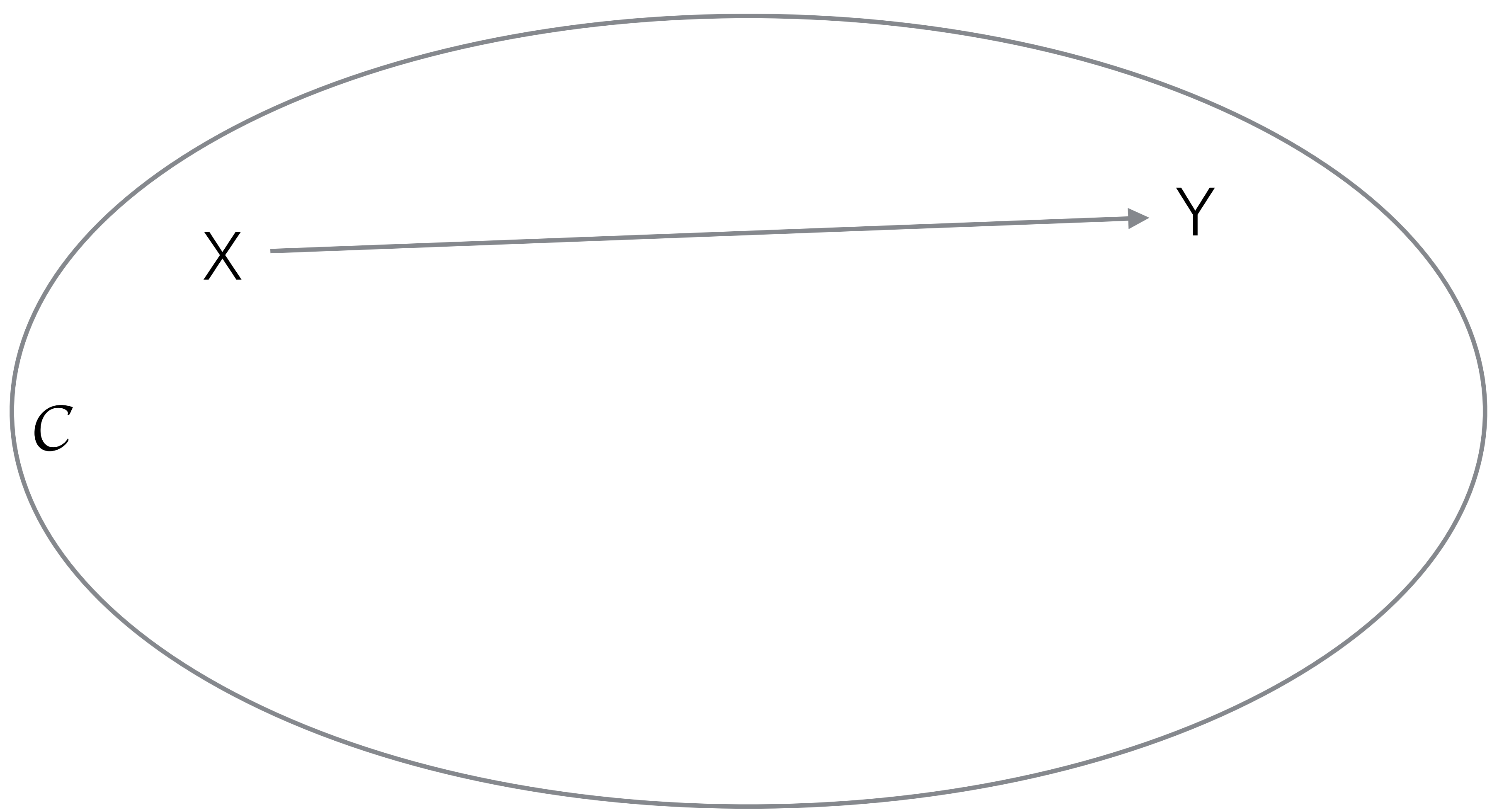

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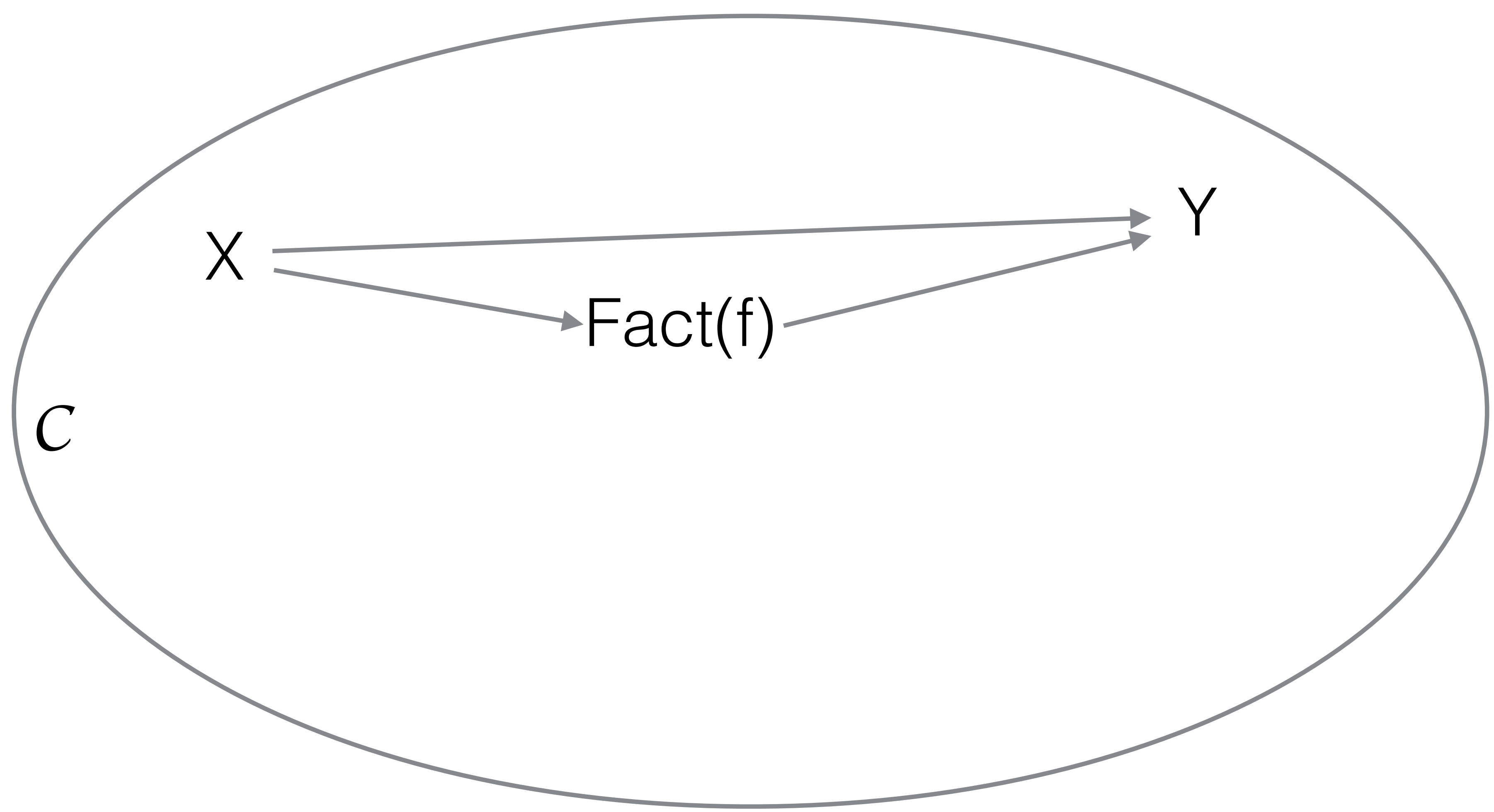

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consists of classes $(\mathcal{E},\mathcal{M})$ such that:

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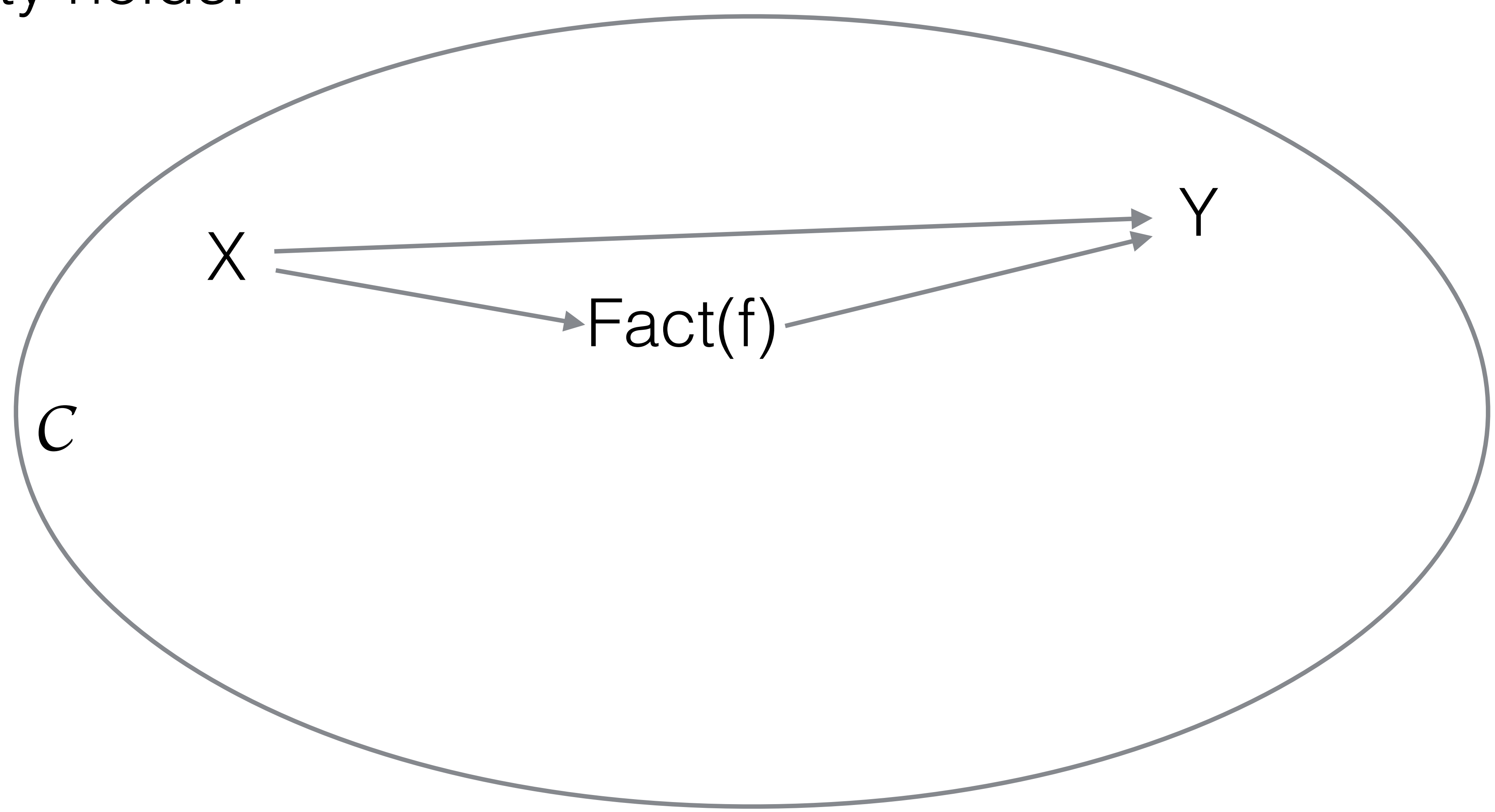

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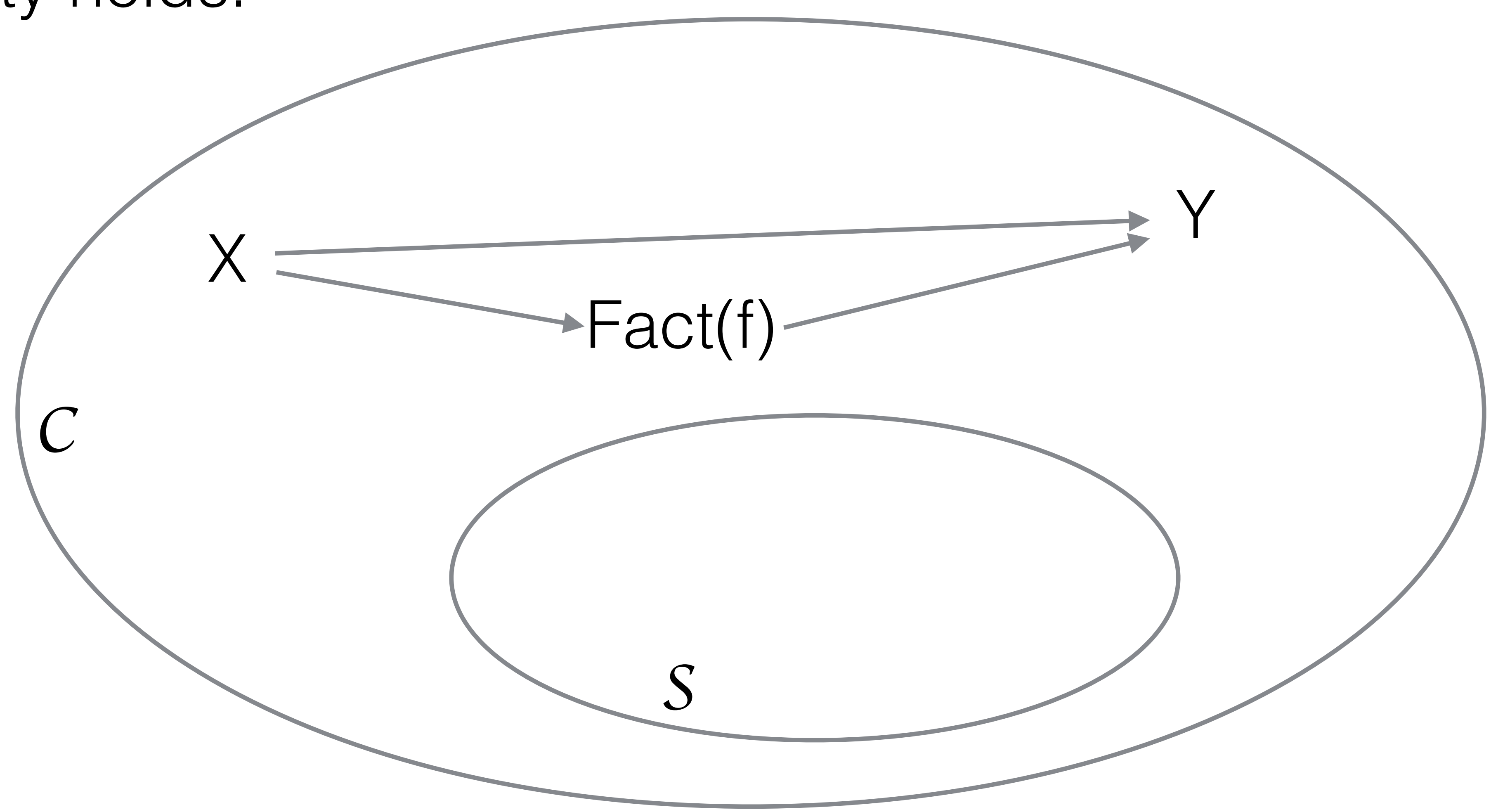

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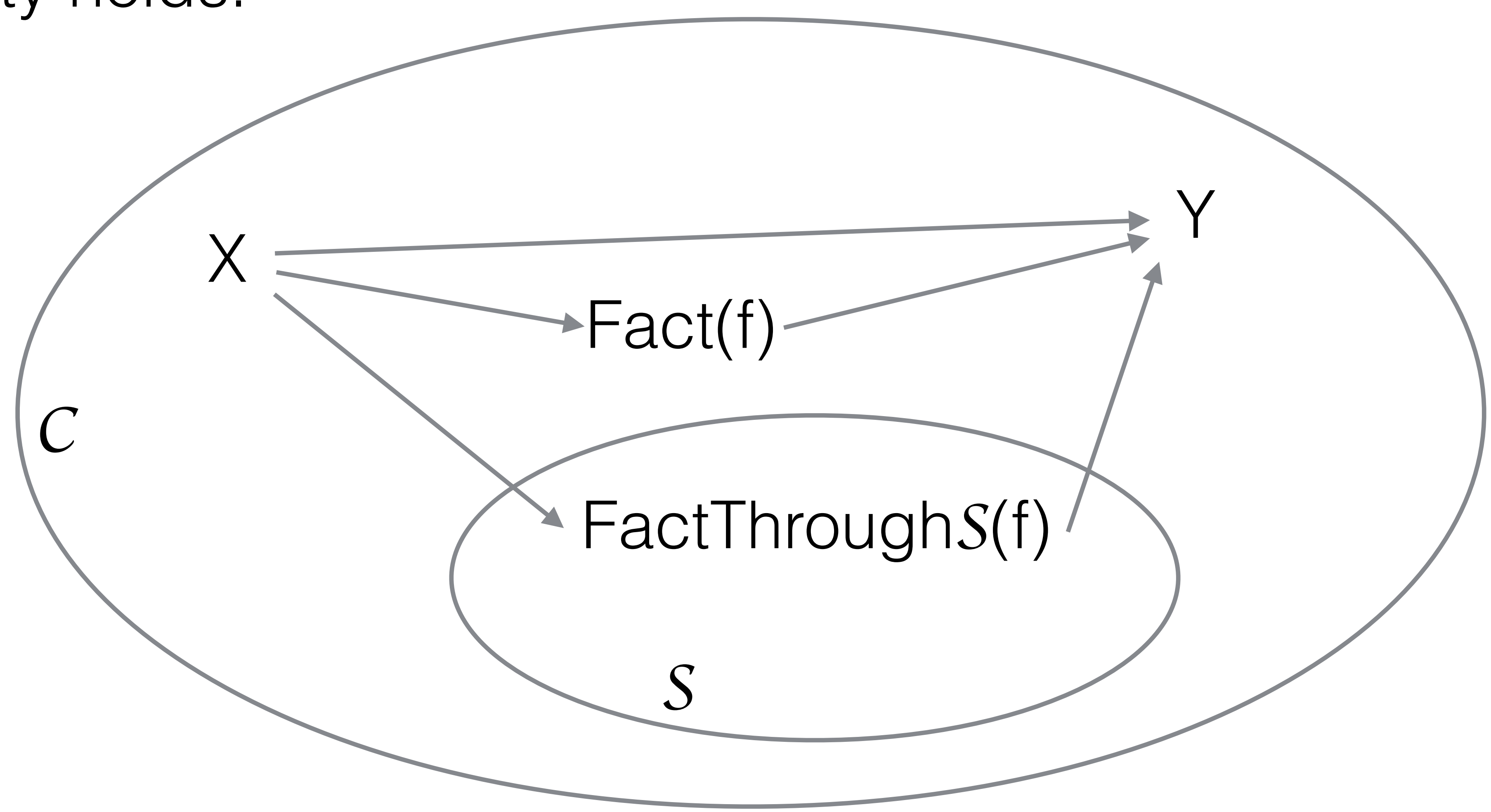

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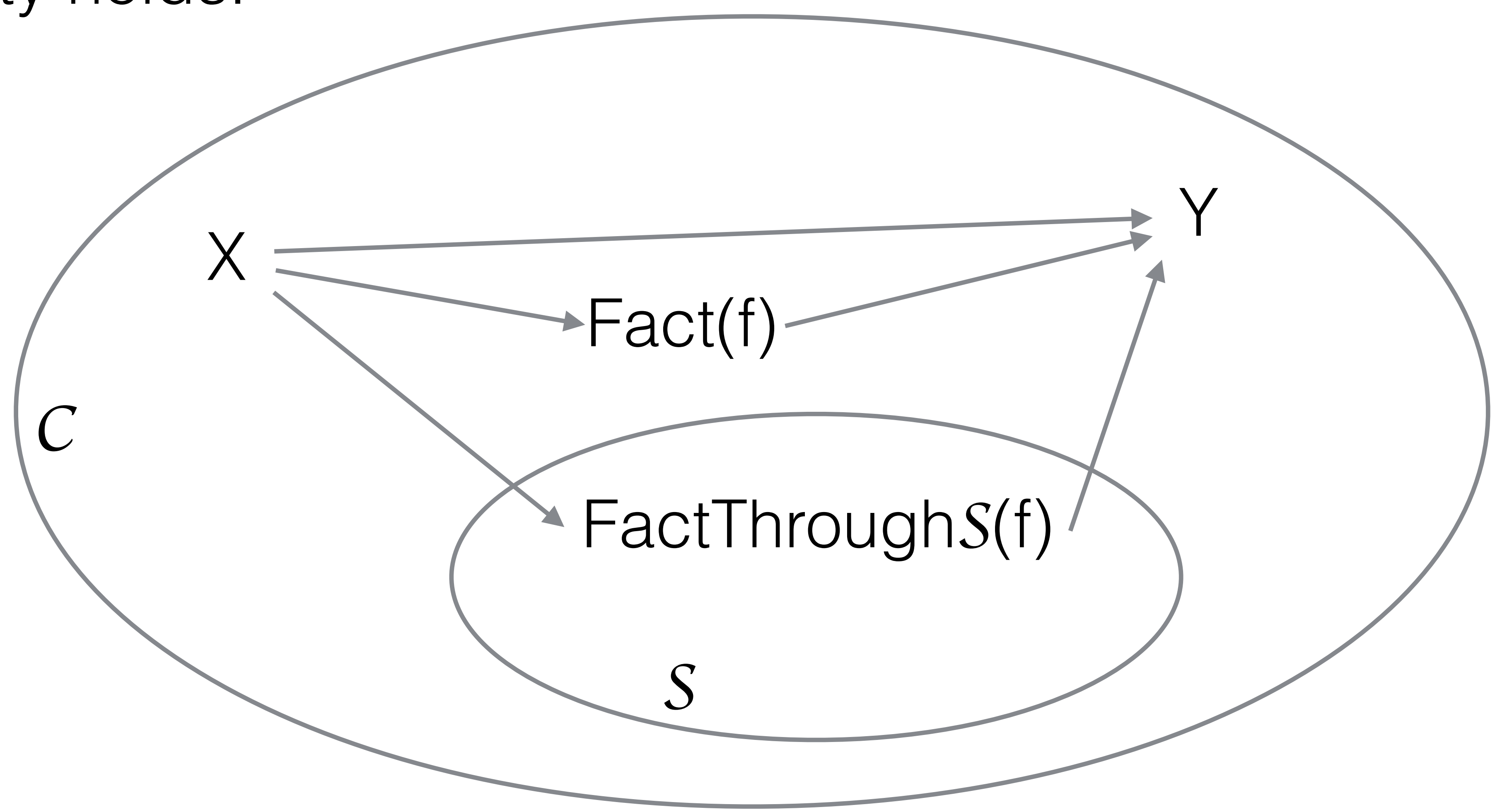

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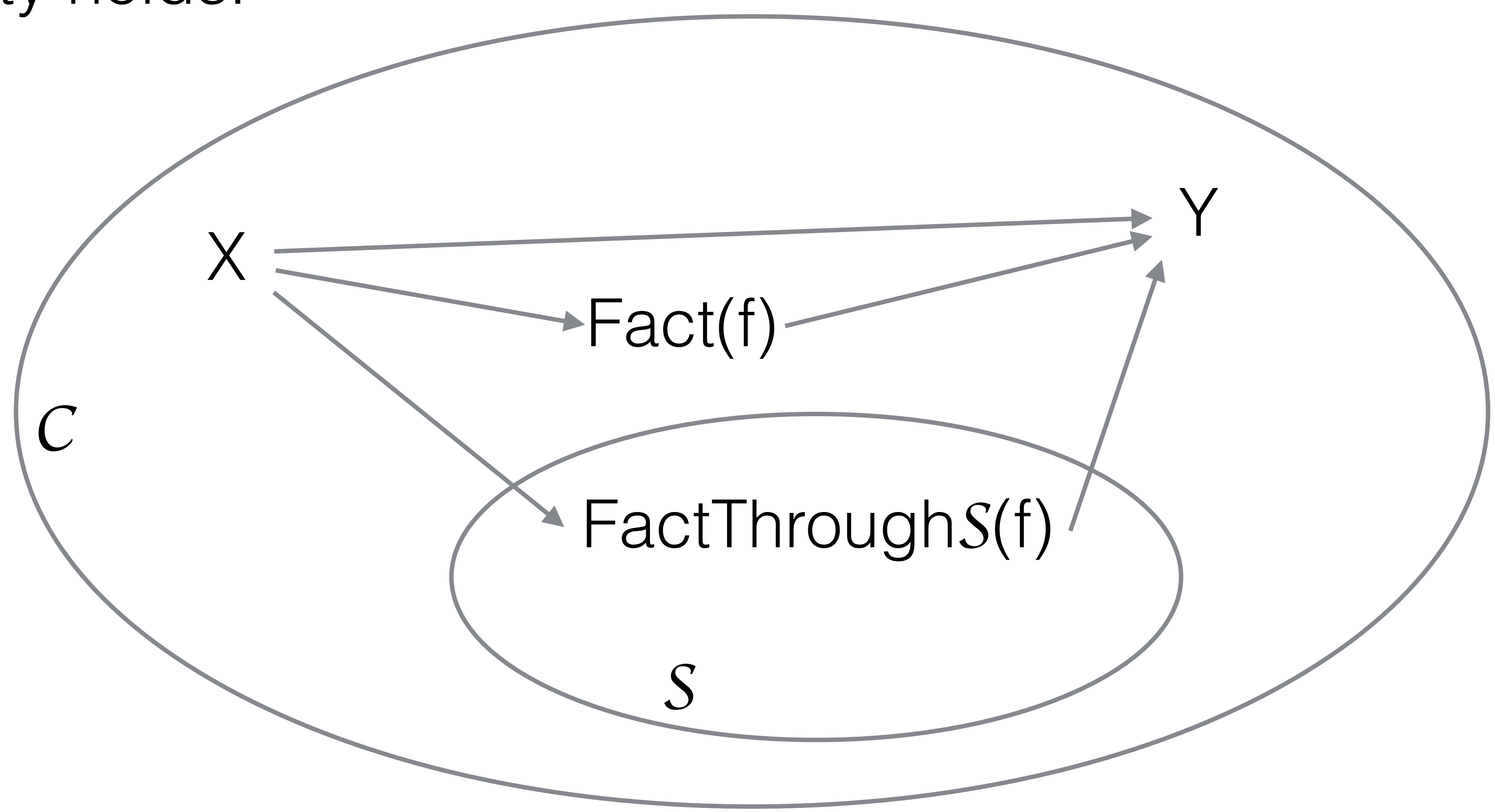

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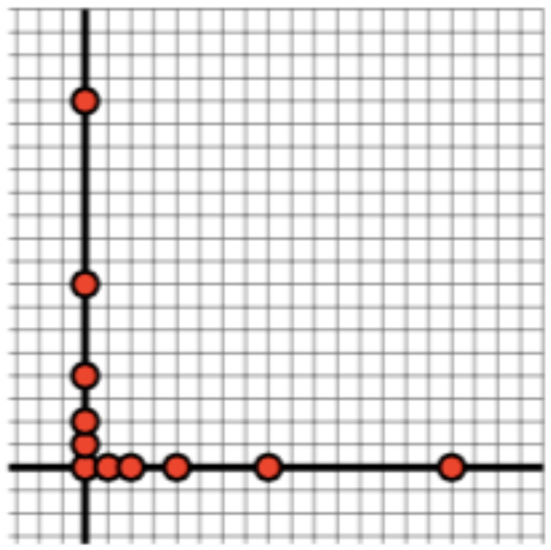

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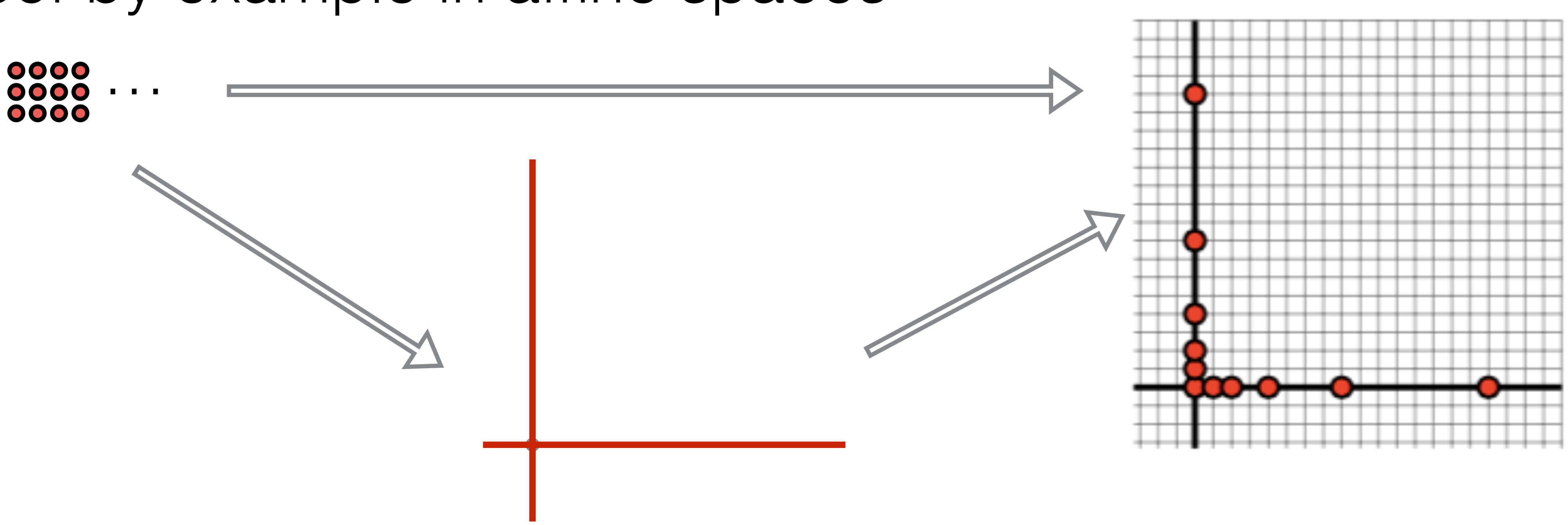

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But, we do not know how to compute it…

The core algorithmic problem

Open problem: Given n×n matrices $A_1,\ldots,A_k$, compute the least finite union of subspaces of matrices that covers the generated semigroup.

For instance consider the matrice **Rot(α)** for some rational number α.

<u>If α is a rational multiple</u> of π, it should output the (finite union) of the dimension one spaces **Vec(Rot(nα))** for integer n.

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Conclusion

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- new categorical concepts on the way,
- new ways to construct categories that yield natural classes of minimizable automata using « glueings ».

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And then ?

- Make this construction effective… (generalization of sequencialization)
- tree automata
- algebras (monoids,…)
- infinite objects (ω-semigroup, o-semigroup, monads…).

Questions ?