Stochastic Domain Theory

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Stochastic Processes and Domains

Scott's Stochastic Lambda Calculus

- Untyped lambda calculus with probabilistic choice
- Probabilistic choice implemented via *random variables* $X\colon ([0,1],\lambda) \to \mathcal{P}(\mathbb{N})$ using Borel sets generated by *Scott topology*.

Barker's Randomized PCF

- PCF (simply typed lambda calculus $+\ \mathbb{N}\ +\ \mathbb{B}\ +$ `rec`) with *randomized choice*
- Models randomized algorithms – reveals speedup in Miller-Rabin Prime Testing Algorithm

Stochastic Lambda Calculus for Probabilistic Programming

This talk: *Applying domain theory to stochastic processes.*

Stochastic Processes and Skorohod's Theorem

A *stochastic process* is a time-indexed family $\{X_t \mid t \in T \subseteq \mathbb{R}_+\}$ of random variables $X_t : \Omega \to S$, where $(\Omega, \Sigma_\Omega, \mu)$ is a probability space, and S is a Polish space.

Note: If S is Polish, then so is $(Prob(S), d_p)$, where d_p is the *Prokhorov metric*. d_P generates the *weak topology* on $Prob(S)$.

Examples:

- Brownian motion, Lévy processes, Markov chains

  MCMC – Markov chain Monte Carlo
  Theme in *Probabilistic Programming Semantics*

Stochastic Processes and Skorohod's Theorem

Let λ denote Lebesgue measure on $[0,1]$.

Skorohod's Theorem

If S is a Polish space, and $\nu \in \mathrm{Prob}\, S$, then there is a random variable $X\colon [0,1] \to S$ with $X_* \lambda = \nu$; i.e., $\nu(A) = \lambda(X^{-1}(A))$ $\forall A$ measurable.

Moreover, if $\nu_n, \nu \in \mathrm{Prob}\, S$ satisfy $\nu_n \to_w \nu$, then there are random variables $X_n, X\colon [0,1] \to S$ with $X_* \lambda = \nu, X_{n*} \lambda = \nu_n$ and $X_n \to X$ λ-a.e.

So: 1) *Every* stochastic process arises from some $X_t\colon [0,1] \to S$.

2) Convergence in $(\mathrm{Prob}\, S, \mathit{weak})$ is equivalent to pointwise convergence a.e. of measurable maps $X\colon [0,1] \to S$.

Goal: Obtain domain-theoretic version of Skorohod's Theorem with Skorohod's Theorem as a Corollary.

Interlude: Some Domain Theory

Domains are partially ordered sets with additional properties:

Directed completeness

$\emptyset \neq A \subseteq D$ *directed* if $x, y \in A \ \Rightarrow \ (\exists z \in A)\, x, y \leq z$.
D *directed complete:* A directed $\ \Rightarrow\ \sup A$ exists.

Approximation

$x \ll y$ iff $y \leq \sup A$ directed $\ \Rightarrow\ (\exists a \in A)\, x \leq a$.

Domain: $\twoheaddownarrow y = \{x \mid x \ll y\}$ directed and $y = \sup \twoheaddownarrow y$

Basis: $B_D \subseteq D$ satisfying $\downarrow y \cap B_D \subseteq \twoheaddownarrow y$ & $y = \sup \downarrow y \cap B_D$ $(\forall y \in D)$

Scott Topology

U *Scott open* if:

- $U = \uparrow U = \{x \in D \mid (\exists u \in U)\, u \leq x\}$ and
- A directed, $\sup A \in U \ \Rightarrow\ A \cap U \neq \emptyset$.

Example: $\twoheaduparrow x = \{y \mid x \ll y\}$ is Scott open $(\forall x \in D)$.

Interlude: Some Domain Theory

Morphisms

$f : D \to E$ is *Scott continuous* if:

- f is monotone, and
- A directed $\Rightarrow$ $f(\sup A) = \sup f(A)$.

Lawson Topology

Basis: $\{\twoheaduparrow x \setminus \uparrow F \mid x \in D,\ F \in \mathcal{P}_{<\omega} D\}$

Hausdorff refinement of Scott topology.

All the domains we discuss are Lawson compact.

Towards a Domain-theoretic Skorohod Theorem

First step: We can use any standard probability space for $([0,1], \lambda)$:

Let $\mathcal{C} = 2^\omega$ denote a countable product of 2-point groups, and let $\mu_\mathcal{C}$ denote Haar measure on $\mathcal{C}$.

Theorem:
If S is a Polish space, and $\nu \in \mathrm{Prob}\, S$, then there is a random variable $X \colon \mathcal{C} \to S$ with $X_* \mu_\mathcal{C} = \nu$.

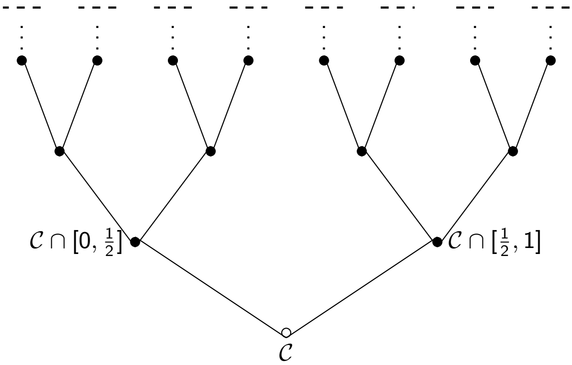
Moreover, if $\nu_n, \nu \in \mathrm{Prob}\, S$ satisfy $\nu_n \to_w \nu$, then there are random variables $X_n, X \colon \mathcal{C} \to S$ with $X_* \mu_\mathcal{C} = \nu, X_{n*} \mu_\mathcal{C} = \nu_n$ and $X_n \to X$ $\mu_\mathcal{C}$-a.e.

Proof: Use $\varphi \colon \mathcal{C} \twoheadrightarrow [0,1]$. $\square$

Towards a Domain-theoretic Skorohod Theorem

Second Step: Embed $\mathcal{C}$ in an appropriate domain:

$\mathbb{CT} = \{0,1\}^\infty$ is a domain in the prefix order.



$$\mathcal{C} \simeq (\{0,1\}^\omega, \Sigma(\mathbb{CT})|_{\{0,1\}^\omega}) = (\mathsf{Max}\,\mathbb{CT}, \Lambda(\mathbb{CT})|_{\mathsf{Max}\,\mathbb{CT}})$$

Third Step: Which domains represent Polish spaces?

BCD_ω – countably based bounded complete domains and Scott continuous maps.

- $D^\infty \simeq [D^\infty \to D^\infty]$ is in BCD_ω.
- $\mathbb{CT} = \{0,1\}^\infty$ is a bounded complete domain.

Theorem: (Lawson; Ciesielski, Flagg & Kopperman)

Each countably-based bounded complete domain D satisfies $\mathrm{Max}\, D$ is a Polish space in the inherited Scott topology. Moreover, $\mathrm{Max}\, D$ is a G_δ in D.

Conversely, every Polish space can be embedded as $\mathrm{Max}\, D$ for a countably based bounded complete domain D.

Examples:

1) $\mathcal{C} \simeq \mathrm{Max}\, \mathbb{CT} \hookrightarrow \mathbb{CT}$.

2) $\mathbb{R} \simeq \mathrm{Max}\, \mathbb{IR} \hookrightarrow \mathbb{IR} = (\{[a,b] \mid a \leq b \in \mathbb{R}\} \cup \{\mathbb{R}\}, \supseteq)$.

Domain-theoretic Skorohod Theorem (cont'd)

Skorohod's Theorem for Domains

If D is a countably based bounded complete domain and $\nu \in \mathrm{Prob}\, D$, then there is a Scott-continuous map $X \colon \mathbb{CT} \to D$ with $X_* \, \mu_{\mathcal{C}} = \nu$.

Moreover, if $\nu_n, \nu \in \mathrm{Prob}\, D$ satisfy $\nu_n \to_w \nu$, then there are Scott-continuous maps $X_n, X \colon \mathbb{CT} \to D$ with $X_* \, \mu_{\mathcal{C}} = \nu$, $X_{n*} \, \mu_{\mathcal{C}} = \nu_n$ and $X_n \to X$ in $[\mathbb{CT} \to D]$.

BCD_ω is Cartesian closed:

- $[D \to E] = \{f \colon D \to E \mid f \text{ Scott continuous}\}$
- $f \leq g$ iff $f(x) \leq g(x)$ $(\forall x \in D)$.

So: $X \mapsto X_* \mu_{\mathcal{C}} \colon [\mathbb{CT} \to D] \twoheadrightarrow (\mathrm{Prob}\, D, \mathit{weak})$ is continuous surjection

Domain-theoretic Skorohod Theorem (cont'd)

Skorohod's Theorem for Domains

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Moreover, if $\nu_n, \nu \in \mathrm{Prob}\, D$ satisfy $\nu_n \to_w \nu$, then there are Scott-continuous maps $X_n, X \colon \mathbb{CT} \to D$ with $X_* \, \mu_{\mathcal{C}} = \nu$, $X_{n*} \, \mu_{\mathcal{C}} = \nu_n$ and $X_n \to X$ in $[\mathbb{CT} \to D]$.

Corollary: Skorohod's Theorem

Proof: If S is Polish, then $(\mathrm{Prob}\, S, \mathit{weak}) \hookrightarrow (\mathrm{Max}\, \mathrm{Prob}\, D, \mathit{weak})$ for some BCD_ω D. Then $\nu \in \mathrm{Prob}\, S \;\Rightarrow\; (\exists X \colon \mathbb{CT} \to D)\, X_* \mu_{\mathcal{C}} = \nu$.

$X|_{\mathcal{C}} \colon \mathcal{C} \to D$ is measurable is easy argument. □

Note: $(\mathbb{CT}, \mu_{\mathcal{C}})$ is a standard probability space (mod 0), so we get more information about X, X_n: they're all Scott continuous.

Deflations

$\phi\colon D \to D$ is a *deflation* if ϕ is Scott continuous and $\phi(D)$ is finite.

$D \in \mathsf{BCD}_\omega \implies 1_D = \sup_n \phi_n,\ \phi_n \le \phi_{n+1}$, deflations

Example: $\pi_n\colon \mathbb{CT} \to \downarrow\mathcal{C}_n,$ where $\mathcal{C}_n \simeq 2^n$

Prob functorial $\implies 1_{\mathsf{Prob}\,D} = \sup_n \phi_{n*}$

Example: $\mu_{\mathcal{C}} = \sup_n \pi_{n*}\,\mu_{\mathcal{C}} = \sup_n \mu_{\mathcal{C}_n}$

So: If $D \in \mathsf{BCD}_\omega$ and $\mu \in \mathsf{Prob}\,D$, then $\mu = \sup_n \phi_{n*}\,\mu$
with $\phi_{n*}\,\mu = \sum_{x\in F_n} r_x\delta_x$, where F_n finite for all n.

Interlude: Valuations and Probability Measures

The Domain of Valuations

$\mathbb{V}(D) = \{v \colon \Sigma(D) \to [0,1] \mid v \text{ a valuation}\}$:

- $v(\emptyset) = 0$ & $v(D) \leq 1$
- $v(U \cup V) + v(U \cap V) = v(U) + v(V)$ $(\forall U, V \in \Sigma(D))$
- v Scott continuous: $v(\bigcup_i U_i) = \sup_i v(U_i)$ $(\forall \{U_i\}_i$ directed).

Define: $v \leq v'$ iff $v(U) \leq v'(U)$ $(\forall U \in \Sigma(D))$.

Fact: D (Lawson compact) domain $\Longrightarrow$
$\mathbb{V}(D)$ (Lawson compact) domain

Interlude: Valuations and Probability Measures

Prob D is a Domain

1° $\mathbb{V}_1(D) = \{v \in \mathbb{V}(D) \mid v(D) = 1\}$ also a domain in pointwise order.

2° $\mathrm{Prob}\, D \simeq \mathbb{V}(D)$

$\mu \leq \nu$ iff $\int f d\mu \leq \int f d\nu$ ($\forall f : D \to \mathbb{R}_+$ Scott continuous)

3° D Lawson compact $\Rightarrow$ $(\mathrm{Prob}(D), \mathit{weak}) = (\mathrm{Prob}(D), \mathit{Lawson})$.

(Edalat; van Breugel, M., Ouaknine and Worrell
- plays off Portmanteau Theorem).

Finitary Mappings

Fix $\mu \in \mathrm{Prob}\, D$, and fix $\phi_{n*}\, \mu = \sum_{x \in F_n} r_x \delta_x$.

Want: $f_n \colon \mathcal{C}_{m_n} \to D$ with $f_{n*}\, \mu_{\mathcal{C}_{m_n}} = \phi_{n*}\, \mu$ for some $m_n > n$.

Requires: r_x must be dyadic for $x \in F_n$. ↯

So, we *approximate* $\phi_{n*}\, \mu$:

Choose $m_n > n, |F_n|$ with $r_x - s_x < \frac{1}{2^{m_n}}$ $(\forall x \in F_n)$, where

$$s_x = \max \downarrow (r_x \cap Dyad_{m_n}), \text{ with } Dyad_{m_n} = \{\tfrac{k}{2^{m_n}} \mid k \leq 2^{m_n}\}.$$

Choose $y_x \ll x$ for each $x \in F_n$.

Let $\nu_n = \sum_{x \in F_n} s_x \delta_{y_x} + (1 - \sum s_x)\delta_\perp$.

Define $f_n \colon \mathcal{C}_{m_n} \to F_n \cup \{\perp\} \subseteq D$ by

$f_n^{-1}(x) = s_x$ $(\forall x \in F_n)$, and $f_n^{-1}(\perp) = 1 - \sum_{x \in F_n} s_x$.

Then $f_{n*}\, \mu_{\mathcal{C}_{m_n}} = \nu_n \ll \phi_{n*}\, \mu$

Outline of Proof

We can extend f_n to $\widetilde{f}_n\colon {\downarrow}\mathcal{C}_{m_n} \to D$ with $\widetilde{f}_{n*}\,\mu_{\mathcal{C}_{m_n}} = \phi_{n*}\,\mu$.

Problem: $\{\widetilde{f}_n\}_n$ is not a chain.

Proposition: Let $\nu = \sum_{x\in F} r_x\delta_x \le \sum_{y\in G} s_y\delta_y = \nu' \in \mathrm{Prob}\, D$.

Assume r_x, s_y are dyadic rationals for each $x \in F, y \in G$.

Suppose $f_m\colon \mathcal{C}_{k_m} \to D$ satisfies $f_{m*}\,\mu_{\mathcal{C}_{k_m}} = \nu$.

Then there are $n > m$, $k_n > k_m$, and $f_n\colon \mathcal{C}_{k_n} \to D$ satisfying:

- $f_{n*}\,\mu_{\mathcal{C}_{k_n}} = \nu'$, and
- $f_m \circ \pi_{k_m k_n} \le f_n$, where $\pi_{k_m k_n}\colon \mathcal{C}_{k_n} \to \mathcal{C}_{k_m}$ is the canonical projection, which implies $\widetilde{f}_m \le \widetilde{f}_n$.

The proof uses the Splitting Lemma, the fact that if r_x, s_y are dyadic, then the transport numbers $t_{x,y}$ are, too, and a generalization of Hall's Marriage Problem.

The proof of the first part of the Theorem follows by recursively defining the mappings $\widetilde{f}_n$, starting with $\widetilde{f}_0\colon \mathcal{C}_0 \to D$ by $f_0(\mathbf{1}) = \perp_D$, and then letting $X = \sup_n \widetilde{f}_n$.

Questions?