

Tensor topology

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After a few months, though, I realized something. I hadn't gotten any better at understanding tensor products, but I was getting used to not understanding them. It was pretty amazing. I no longer felt anguished when tensor products came up; I was instead almost amused by their cunning ways.

Reflexive objects

linear λ -calculus in monoidal closed category

$$U \simeq [U \rightarrow U]$$

Idempotent subunits

Categorify central idempotents in ring

$$\text{ISub}(\mathbf{C}) = \{s: S \rightarrow I \mid \text{id}_S \otimes s: S \otimes S \rightarrow S \otimes I \text{ iso} \\ \exists S \otimes (-) \Rightarrow (-) \otimes S \text{ iso}\} / \simeq$$

Example: order theory

Frame: complete lattice, \wedge distributes over \vee
e.g. open subsets of topological space

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$$\begin{array}{ccc} \mathbf{Frame} & \begin{array}{c} \xrightarrow{\quad} \\ \perp \\ \xleftarrow{\quad} \\ \text{ISub} \end{array} & \mathbf{Quantale} \\ \{x \in Q \mid x^2 = x \leq 1\} & \xleftarrow{\quad} & Q \end{array}$$

'idempotent subunits are side-effect-free observations'

Example: logic

$$\begin{aligned} \text{ISub}(\text{Sh}(X)) &= \{S \multimap 1\} \\ &= \{S \subseteq X \mid S \text{ open}\} \in \mathbf{Frame} \end{aligned}$$

‘idempotent subunits are truth values’

Example: algebra

$$\text{ISub}(\mathbf{Mod}_R) = \{S \subseteq R \text{ ideal} \mid S = S^2 = \{x_1y_1 + \cdots + x_ny_n \mid x_i, y_i \in S\}\}$$

'idempotent subunits are idempotent ideals'

Example: analysis

Hilbert module is $C_0(X)$ -module with $C_0(X)$ -valued inner product

$$C_0(X) = \{f: X \rightarrow \mathbb{C} \mid \forall \varepsilon > 0 \exists K \subseteq X: |f(X \setminus K)| < \varepsilon\}$$

$$\text{ISub}(\mathbf{Hilb}_{C_0(X)}) = \{S \subseteq X \text{ open}\}$$

‘idempotent subunits are open subsets of base space’

Example: geometry

Hilbert bundle is bundle $E \rightarrow X$ with Hilbert spaces for fibres

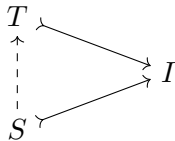
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It is the things you can prove that tell you how to think about tensor products. In other words, you let elementary lemmas and examples shape your intuition of the mathematical object in question. There's nothing else, no magical intuition will magically appear to help you "understand" it.

Semilattice

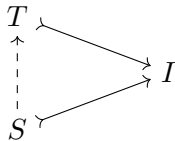
Proposition: $\text{ISub}(\mathbf{C})$ is a semilattice, $\wedge = \otimes$, $1 = \text{id}_I$



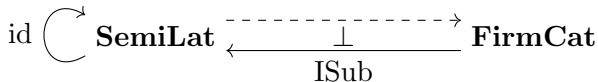
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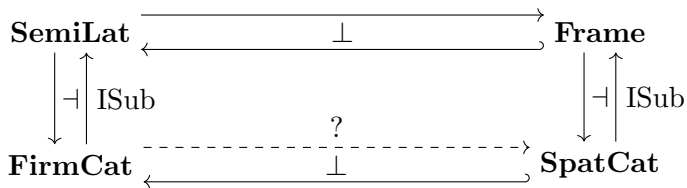


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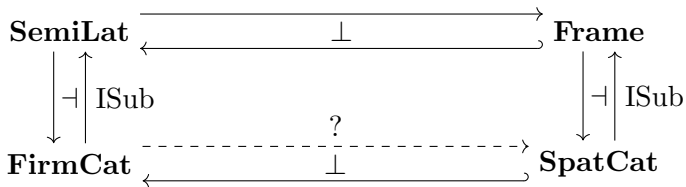
Spatial categories

Call \mathbf{C} *spatial* when $\text{ISub}(\mathbf{C})$ is frame



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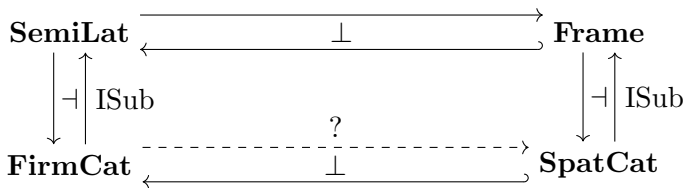
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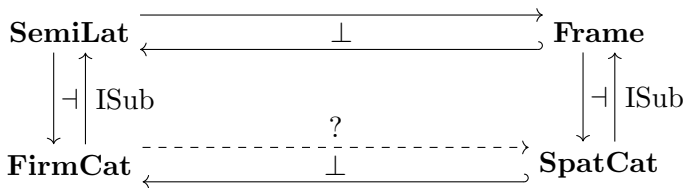
Idea: $\widehat{\mathbf{C}} = [\mathbf{C}^{\text{op}}, \mathbf{Set}]$ is cocomplete

$$F \widehat{\otimes} G(A) = \int^{B, C} \mathbf{C}(A, B \otimes C) \times F(B) \times G(C)$$

Lemma: $\text{ISub}(\widehat{\mathbf{C}}, \widehat{\otimes})$ is frame

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Lemma: $\text{ISub}(\widehat{\mathbf{C}}, \widehat{\otimes})$ is frame, but $\text{ISub}(\widehat{\mathbf{C}}) \neq \widehat{\text{ISub}(\mathbf{C})}$

Support

Say $s \in \text{ISub}(\mathbf{C})$ **supports** $f: A \rightarrow B$ when

$$\begin{array}{ccc} A & \xrightarrow{\quad} & B \\ \vdots & & \uparrow \simeq \\ B \otimes S & \xrightarrow{\quad \text{id} \otimes s \quad} & B \otimes I \end{array}$$

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 & \searrow F & \downarrow \hat{F} \\
 & & Q \in \mathbf{Frame}
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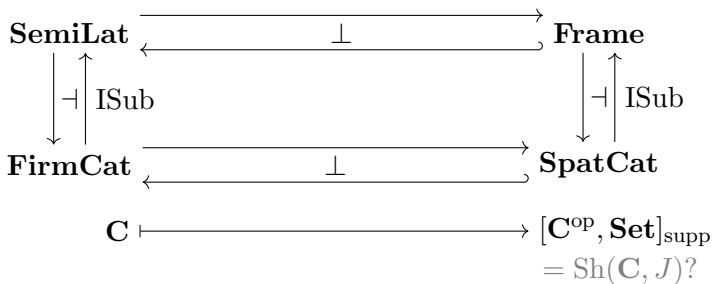
universal with $F(f) = \bigvee \{F(s) \mid s \in \text{ISub}(\mathbf{C}) \text{ supports } f\}$

Spatial categories

Call $F: \mathbf{C}^{\text{op}} \rightarrow \mathbf{Set}$ **supported** when

$$F(A) \simeq \{f: A \rightarrow B \mid \text{supp}(f) \cap U \neq \emptyset\}$$

for some $B \in \mathbf{C}$ and $U \subseteq \text{ISub}(\mathbf{C})$.



Complements

Subunit is **split** when $\text{id} \circlearrowleft S \overset{s}{\overset{\leftarrow}{\dashrightarrow}} I$
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Proposition: when \mathbf{C} has finite biproducts,
then $s, s^\perp \in \text{SISub}(\mathbf{C})$ are complements
if and only if they are biproduct injections

Corollary: if \oplus distributes over \otimes ,
then SISub(\mathbf{C}) is a **Boolean** algebra
(universal property?)

Linear logic

if $T: \mathbf{C} \rightarrow \mathbf{C}$ monoidal monad, $\text{Kl}(T)$ is monoidal
semilattice morphism

$\{\eta_I \circ s \mid s \in \text{ISub}(\mathbf{C}), T(s) \text{ is monic in } \mathbf{C}\} \rightarrow \text{ISub}(\text{Kl}(T))$
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model for linear logic: $*$ -autonomous category \mathbf{C} with finite
products, monoidal comonad $!: (\mathbf{C}, \otimes) \rightarrow (\mathbf{C}, \times)$
(then $\text{Kl}(!)$ cartesian closed)

if ε epi, then $\text{ISub}(\mathbf{C}, \times) \simeq \text{ISub}(\text{Kl}(!), \times)$
(but hard to compare to $\text{ISub}(\mathbf{C}, \otimes)$)

Further

Do you work with morphisms into a tensor unit?

- ▶ causality
- ▶ proof analysis
- ▶ control flow
- ▶ data flow
- ▶ concurrency
- ▶ graphical calculus

Restriction

The full subcategory $\mathbf{C}|_s$ of \mathbf{C} with $\text{id}_A \otimes s$ invertible is:

- ▶ monoidal with tensor unit S
- ▶ coreflective: $\mathbf{C}|_s \begin{array}{c} \xrightarrow{\quad} \\ \dashleftarrow{\perp} \\ \xrightarrow{\quad} \end{array} \mathbf{C}$
- ▶ tensor ideal: if $A \in \mathbf{C}$ and $B \in \mathbf{C}|_s$, then $A \otimes B \in \mathbf{C}|_s$
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Proposition: $\text{ISub}(\mathbf{C}) \simeq \{\text{monocoreflective tensor ideals in } \mathbf{C}\}$

Localisation

A **graded monad** is a monoidal functor $\mathbf{E} \rightarrow [\mathbf{C}, \mathbf{C}]$
($\eta: A \rightarrow T(1)$, $\mu: T(t) \circ T(s) \rightarrow T(s \otimes t)$)

Lemma: $s \mapsto \mathbf{C}|_s$ is an $\text{ISub}(\mathbf{C})$ -graded monad

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universal property of **localisation** for $\Sigma = \{\text{id}_E \otimes s \mid E \in \mathbf{C}\}$

$$\begin{array}{ccc} \mathbf{C} & \xrightarrow{(-) \otimes S} & \mathbf{C}|_s = \mathbf{C}[\Sigma^{-1}] \\ & \searrow F \text{ inverting } \Sigma & \downarrow \text{dashed} \\ & & \mathbf{D} \end{array}$$

\simeq