Tensor topology

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After a few months, though, I realized something. I hadn't gotten any better at understanding tensor products, but I was getting used to not understanding them. It was pretty amazing. I no longer felt anguished when tensor products came up; I was instead almost amused by their cunning ways.

Reflexive objects

linear λ-calculus in monoidal closed category

$$U \simeq [U \to U]$$

Idempotent subunits

Categorify central idempotents in ring

$$\begin{aligned}\mathrm{ISub}(\mathbf{C}) = \{s \colon S \rightarrowtail I \mid \mathrm{id}_S \otimes s \colon S \otimes S \to S \otimes I \text{ iso} \\ \exists S \otimes (-) \Rightarrow (-) \otimes S \text{ iso } \}/\simeq\end{aligned}$$

Example: order theory

Frame: complete lattice, $\wedge$ distributes over $\bigvee$

e.g. open subsets of topological space

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e.g. $[0, \infty]$, $\mathrm{Pow}(M)$

$$\begin{tikzcd}
\mathbf{Frame} \arrow[r, hook, shift left=2] \arrow[r, phantom, "\perp"] & \mathbf{Quantale} \arrow[l, shift left=2, "\mathrm{ISub}"] \\
\{x \in Q \mid x^2 = x \leq 1\} & Q \arrow[l, mapsto]
\end{tikzcd}$$

'idempotent subunits are side-effect-free observations'

Example: logic

$$\begin{aligned}\mathrm{ISub}(\mathrm{Sh}(X)) &= \{S \rightarrowtail 1\} \\ &= \{S \subseteq X \mid S \text{ open}\} \in \textbf{Frame}\end{aligned}$$

‘idempotent subunits are truth values’

Example: algebra

$$\mathrm{ISub}(\mathbf{Mod}_R) =$$
$$\left\{ S \subseteq R \text{ ideal} \;\middle|\; S = S^2 = \{x_1y_1 + \cdots x_ny_n \mid x_i, y_i \in S\} \right\}$$

'idempotent subunits are idempotent ideals'

Example: analysis

Hilbert module is $C_0(X)$-module with $C_0(X)$-valued inner product

$$C_0(X) = \{f \colon X \to \mathbb{C} \mid \forall \varepsilon > 0 \; \exists K \subseteq X \colon |f(X \setminus K)| < \varepsilon\}$$

$$\mathrm{ISub}(\mathbf{Hilb}_{C_0(X)}) = \{S \subseteq X \text{ open}\}$$

'idempotent subunits are open subsets of base space'

Example: geometry

Hilbert bundle is bundle $E \twoheadrightarrow X$ with Hilbert spaces for fibres

$$\mathrm{ISub}(\mathbf{Hilb}_X) = \{S \subseteq X \text{ open}\}$$

‘idempotent subunits are open subsets of base space’

It is the things you can prove that tell you how to think about tensor products. In other words, you let elementary lemmas and examples shape your intuition of the mathematical object in question. There's nothing else, no magical intuition will magically appear to help you "understand" it.

Semilattice

Proposition: $\mathrm{ISub}(\mathbf{C})$ is a semilattice, $\wedge = \otimes$, $1 = \mathrm{id}_I$

$$\begin{tikzcd}
T \arrow[dr, tail] & \\
& I \\
S \arrow[ur, tail] \arrow[uu, dashed] &
\end{tikzcd}$$

Caveat: $\mathbf{C}$ must be firm, i.e. $s \otimes \mathrm{id}_T$ monic, and size issue

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$$\begin{tikzcd}
\mathbf{SemiLat} \arrow[loop left, "\mathrm{id}"] \arrow[r, dashed, shift left] \arrow[r, phantom, "\perp"] & \mathbf{FirmCat} \arrow[l, shift left, "\mathrm{ISub}"]
\end{tikzcd}$$

Spatial categories

Call **C** spatial when ISub(**C**) is frame

$$\begin{tikzcd}
\mathbf{SemiLat} \arrow[rr, shift left=2] \arrow[rr, phantom, "\perp"] \arrow[d, shift right=2] \arrow[d, phantom, "\dashv"] & & \mathbf{Frame} \arrow[ll, hook', shift left=2] \arrow[d, shift right=2] \arrow[d, phantom, "\dashv"] \\
\mathbf{FirmCat} \arrow[u, shift right=2, "\mathrm{ISub}"'] \arrow[rr, dashed, shift left=2, "?"] \arrow[rr, phantom, "\perp"] & & \mathbf{SpatCat} \arrow[u, shift right=2, "\mathrm{ISub}"'] \arrow[ll, hook', shift left=2]
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Idea: $\widehat{\mathbf{C}} = [\mathbf{C}^{\mathrm{op}}, \mathbf{Set}]$ is cocomplete

$$F \widehat{\otimes} G(A) = \int^{B,C} \mathbf{C}(A, B \otimes C) \times F(B) \times G(C)$$

Lemma: $\mathrm{ISub}(\widehat{\mathbf{C}}, \widehat{\otimes})$ is frame

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Lemma: $\mathrm{ISub}(\widehat{\mathbf{C}}, \widehat{\otimes})$ is frame, but $\mathrm{ISub}(\widehat{\mathbf{C}}) \neq \widehat{\mathrm{ISub}(\mathbf{C})}$

Support

Say $s \in \mathrm{ISub}(\mathbf{C})$ supports $f \colon A \to B$ when

$$\begin{tikzcd}
A \arrow[r, "f"'] \arrow[d, dashed] & B \\
B \otimes S \arrow[r, "\mathrm{id} \otimes s"'] & B \otimes I \arrow[u, "\simeq"']
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$$\begin{tikzcd}
f \arrow[r, mapsto] & \{s \mid s \text{ supports } f\} \\
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\mathbf{C^2} \arrow[r, "\mathrm{supp}"] \arrow[dr, "F"'] & \mathrm{Pow}(\mathrm{ISub}(\mathbf{C})) \arrow[d, dashed, "\widehat{F}"] \\
 & Q \in \mathbf{Frame}
\end{tikzcd}$$

universal with $F(f) = \bigvee \{F(s) \mid s \in \mathrm{ISub}(\mathbf{C}) \text{ supports } f\}$

Spatial categories

Call $F\colon \mathbf{C}^{\mathrm{op}} \to \mathbf{Set}$ supported when

$$F(A) \simeq \{f\colon A \to B \mid \mathrm{supp}(f) \cap U \neq \emptyset\}$$

for some $B \in \mathbf{C}$ and $U \subseteq \mathrm{ISub}(\mathbf{C})$.

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\mathbf{C} \arrow[rr, mapsto] & & {[\mathbf{C}^{\mathrm{op}}, \mathbf{Set}]_{\mathrm{supp}}} \\
& & = \mathrm{Sh}(\mathbf{C}, J)?
\end{tikzcd}$$

Complements

Subunit is split when $$\begin{tikzcd} S \arrow[loop left, "\mathrm{id}"] \arrow[r, tail, shift left, "s"] & I \arrow[l, dashed, two heads, shift left] \end{tikzcd}$$

SISub($\mathbf{C}$) is a sub-semilattice of ISub($\mathbf{C}$)

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If $\mathbf{C}$ has zero object, ISub($\mathbf{C}$) has least element 0

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Proposition: when $\mathbf{C}$ has finite biproducts,
then $s, s^{\perp} \in$ SISub($\mathbf{C}$) are complements
if and only if they are biproduct injections

Corollary: if $\oplus$ distributes over $\otimes$,
then SISub($\mathbf{C}$) is a Boolean algebra
(universal property?)

Linear logic

if $T\colon \mathbf{C} \to \mathbf{C}$ monoidal monad, $\mathrm{Kl}(T)$ is monoidal
semilattice morphism
$\{\eta_I \circ s \mid s \in \mathrm{ISub}(\mathbf{C}),\ T(s) \text{ is monic in } \mathbf{C}\} \to \mathrm{ISub}(\mathrm{Kl}(T))$
is not injective, nor surjective

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model for linear logic: $*$-autonomous category $\mathbf{C}$ with finite
products, monoidal comonad $!\colon (\mathbf{C}, \otimes) \to (\mathbf{C}, \times)$
(then $\mathrm{Kl}(!)$ cartesian closed)
if ε epi, then $\mathrm{ISub}(\mathbf{C}, \times) \simeq \mathrm{ISub}(\mathrm{Kl}(!), \times)$
(but hard to compare to $\mathrm{ISub}(\mathbf{C}, \otimes)$)

Further

Do you work with morphisms into a tensor unit?

- causality
- proof analysis
- control flow
- data flow
- concurrency
- graphical calculus

Restriction

The full subcategory $\mathbf{C}\big|_s$ of A with $\mathrm{id}_A \otimes s$ invertible is:

- monoidal with tensor unit S
- coreflective: $$\begin{tikzcd} \mathbf{C}\big|_s \arrow[r, hook, shift left=2] \arrow[r, phantom, "\perp"] & \mathbf{C} \arrow[l, dashed, shift left=2] \end{tikzcd}$$
- tensor ideal: if $A \in \mathbf{C}$ and $B \in \mathbf{C}\big|_s$, then $A \otimes B \in \mathbf{C}\big|_s$
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Proposition: $\mathrm{ISub}(\mathbf{C}) \simeq \{$monocoreflective tensor ideals in $\mathbf{C}\}$

Localisation

A graded monad is a monoidal functor $\mathbf{E} \to [\mathbf{C}, \mathbf{C}]$

$(\eta \colon A \to T(1),\ \mu \colon T(t) \circ T(s) \to T(s \otimes t))$

Lemma: $s \mapsto \mathbf{C}\big|_s$ is an ISub($\mathbf{C}$)-graded monad

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universal property of localisation for $\Sigma = \{\mathrm{id}_E \otimes s \mid E \in \mathbf{C}\}$

$$\begin{tikzcd}
\mathbf{C} \arrow[r, "(-) \otimes S"] \arrow[dr, "F \text{ inverting } \Sigma"'] & \mathbf{C}\big|_s = \mathbf{C}[\Sigma^{-1}] \arrow[d, dashed, "\simeq"'] \\
 & \mathbf{D}
\end{tikzcd}$$