Structure vs. Power: game comonads in finite model theory

Samson Abramsky

Department of Computer Science, University of Oxford
Structure vs. Power

Foundations of CS: the Great Divide

- **Structure**: structure and compositionality. E.g. semantics, type theory, ... 
- **Power**: expressiveness and efficiency. Track A, but also “Track B/A”!

We understand very little about how **Structure** can talk to **Power**, and *vice versa*.

The Logical Structures in Computation program aimed to address this.

We shall talk about one direct off-shoot of the program:


Model theory and deception

- In model theory, we see a structure, not “as it really is” (up to isomorphism) but only up to **definable properties**.

- The crucial notion is equivalence of structures up to the equivalence $\equiv^L$ induced by the logic $L$:

  $$\mathcal{A} \equiv^L \mathcal{B} := \forall \phi \in L. \mathcal{A} \models \phi \iff \mathcal{B} \models \phi$$

- It is always true that if a class of structures $\mathcal{K}$ is definable in $L$, then $\mathcal{K}$ must be saturated under $\equiv^L$.

- In most cases of interest in FMT, the converse is true too.

- In descriptive complexity, we seek to characterize a complexity class $\mathcal{C}$ (for decision problems) as those classes of structures $\mathcal{K}$ (e.g. graphs) definable in $L$. 
Homomorphisms and pebble games

Homomorphisms play a fundamental role in FMT, CSP, DB:

- existence of homomorphisms $\equiv$ CSP
- preservation of conjunctive queries, fundamental in DB

Existential $k$-pebble games (Kolaitis and Vardi 90): Spoiler moves pebbles in $A$, Duplicator responds in $B$.

Proposition (KV90)

The following are equivalent:

- Duplicator has a winning strategy in the existential $k$-pebble game.
- Every sentence of the existential positive $k$-variable fragment of first-order logic satisfied by $A$ is also satisfied by $B$. 
A novel perspective

We shall study $k$-pebble games, not as an external artefact, but as a semantic construction on relation structures.

Given a structure $A$ over a relational signature $\sigma$, we shall introduce a new structure $T_k A$ corresponding to Spoiler playing his part of an existential $k$-pebble game on $A$, with the potential codomain $B$ left unspecified.

The idea is that we can exactly recover the content of a Duplicator strategy in $B$ by giving a homomorphism from $T_k A$ to $B$.

Thus the notion of **local approximation** built into the $k$-pebble game is internalised into the category of $\sigma$-structures and homomorphisms.

Formally, this construction will be shown to give a **comonad** on this category.

This leads to comonadic characterisations of a number of central concepts in Finite Model Theory.
Pebbling as a semantic construction

Given a structure $A$, the set of plays in $A$ by the Spoiler is represented by the set of finite non-empty sequences of moves $(p, a)$, where $p \in [k]$ is a pebble index, and $a \in A$.

Notation: $s = [(p_1, a_1), \ldots, (p_n, a_n)]$.

This forms the universe of $T_k A$.

How do we lift the relations on $A$ to $T_k A$?

Given e.g. a binary relation $E$, we define $E^{T_k A}$ to be the set of pairs of plays $s, t \in T_k A$ such that

- $s$ and $t$ are comparable in the prefix ordering, so $s \sqsubseteq t$ or $t \sqsubseteq s$.

- If $s \sqsubseteq t$, then the pebble index of the last move in $s$ does not appear in the suffix of $s$ in $t$; and symmetrically if $t \sqsubseteq s$.

- $E^A(\varepsilon_A(s), \varepsilon_A(t))$, where $\varepsilon_A : T_k A \longrightarrow A$ sends a play $[(p_1, a_1), \ldots, (p_n, a_n)]$ to $a_n$, the $A$-component of its last move.
The pebbling functor

We now extend $\mathbb{T}_k$ to a functor $\mathbb{T}_k : \mathcal{R}(\sigma) \rightarrow \mathcal{R}(\sigma)$. If $f : A \rightarrow B$ is a homomorphism, we define $\mathbb{T}_k f : \mathbb{T}_k A \rightarrow \mathbb{T}_k B$ to be the map

$$[(p_1, a_1), \ldots, (p_n, a_n)] \mapsto [(p_1, f(a_1)), \ldots, (p_n, f(a_n))] .$$

It is clear that this is a homomorphism from $\mathbb{T}_k A$ to $\mathbb{T}_k B$. Moreover, it is easily verified that $\mathbb{T}_k (g \circ f) = \mathbb{T}_k (g) \circ \mathbb{T}_k (f)$, and $\mathbb{T}_k (\text{id}_A) = \text{id}_{\mathbb{T}_k A}$, so $\mathbb{T}_k$ is a functor.

We have already defined the map $\varepsilon_A : \mathbb{T}_k A \rightarrow A$ for each structure $A$. It is easy to see that it is a homomorphism. We also note that this defines a natural transformation. That is, for each homomorphism $f : A \rightarrow B$, the following diagram commutes.
The Co-Kleisli Category

The objects are the same as those of \( \mathcal{R}(\sigma) \), while a morphism from \( A \) to \( B \) in \( \mathcal{K}(T_k) \) is a homomorphism \( f : T_k A \rightarrow B \).

To compose Kleisli maps, we use the **Kleisli coextension**, which takes \( f : T_k A \rightarrow B \) to \( f^* : T_k A \rightarrow T_k B \).

\[
f^* : [(p_1, a_1), \ldots, (p_n, a_n)] \mapsto [(p_1, f(s_1)), \ldots, (p_n, f(s_n))]
\]

where \( s_i = [(p_1, a_1), \ldots, (p_i, a_i)], i = 1, \ldots n \).

Then the composition of \( f : T_k A \rightarrow B \) and \( g : T_k B \rightarrow C \) is given by \( g \circ f^* : T_k A \rightarrow C \).

We write \( A \rightarrow_k B \) if there exists a morphism from \( A \) to \( B \) in \( \mathcal{K}(T_k) \).

**Theorem**

The following are equivalent:

1. There is a winning strategy for Duplicator in the existential \( k \)-pebble game from \( A \) to \( B \).
2. \( A \rightarrow_k B \).
Grading

Conceptually, we can think of the morphisms $f : A \rightarrow_k B$ in the co-Kleisli category for $T_k$ as those which only have to respect the $k$-local structure of $A$.

The lower the value of $k$, the less information available to Spoiler, and the easier it is for Duplicator to have a winning strategy.

Equivalently, the easier it is to have a morphism $A \rightarrow_k B$, i.e. a morphism from $A$ to $B$ in the co-Kleisli category.

This leads to a natural \textbf{weakening principle}: if we have a morphism from $T_kA$ to $B$, then this should yield a morphism from $T_lA$ to $B$ when $l < k$.

Note that there is an inclusion $T_lA \subseteq T_kA$ when $l < k$.

\textbf{Proposition}

\textit{The inclusion maps form a natural transformation $i_{l,k} : T_l \rightarrow T_k$ which is a morphism of comonads, i.e. it preserves the counit and comultiplication.}
Game comonads as syntax-free model theory

We can use the pebbling comonad to characterize several key logical equivalences:

- We have the logic $C^k$, the $k$-variable logic with counting quantifiers, which plays a central rôle in finite model theory.

**Theorem**

For all finite $A, B$: $A \cong_K B \iff A \equiv_{C^k} B$.

- We can also capture equivalence in $k$-variable logic, and in existential $k$-variable logic.
Tree-width

A beautiful feature of these comonads is that they let us capture crucial combinatorial invariants of structures using the indexed comonadic structure.

Conceptually, we can think of the morphisms $f : A \to_k B$ in the co-Kleisli category for $\mathcal{T}_k$ as those which only have to respect the $k$-local structure of $A$.

The lower the value of $k$, the less information available to Spoiler, and the easier it is for Duplicator to have a winning strategy.

Equivalently, the easier it is to have a morphism $A \to_k B$, i.e. a morphism from $A$ to $B$ in the co-Kleisli category.

What about morphisms $A \to \mathcal{T}_kB$? Pebbling $B$ makes it harder for Duplicator to win the homomorphism game.
Coalgebras

Another fundamental aspect of comonads is that they have an associated notion of **coalgebra**. A coalgebra for \( T_k \) is a morphism \( \alpha : A \to T_k A \) such that the following diagrams commute:

\[
\begin{array}{ccc}
A & \xrightarrow{\alpha} & T_k A \\
\downarrow{\alpha} & & \downarrow{\delta_A} \\
T_k A & \xrightarrow{T_k \alpha} & T_k T_k A \\
\end{array}
\]

N.B. \( \delta : T_k A \to T_k T_k A \) is the comultiplication of the comonad, which is the coextension of \( \text{id}_{T_k A} \).

Note that a coalgebra structure \( \alpha \) on \( A \) implies that a homomorphism exists from \( A \) to \( B \) whenever a homomorphism exists from \( T_k A \) to \( B \). Given \( h : T_k A \to B \), we can form \( h \circ \alpha : A \to B \).

Thus we should only expect a coalgebra structure to exist when the \( k \)-local information on \( A \) is sufficient to determine the structure of \( A \).
Coalgebra number

We now consider how the comonadic structure of $k$-pebbling can be used to characterize treewidth.

**Theorem**

For all structures $A$, $\text{tw}(\mathbb{T}_k A) < k$.

Thus although $\mathbb{T}_k A$ is always infinite, it has treewidth bounded by $k$.

We define the **coalgebra number** $\kappa(A)$ of a finite structure $A$ to be the least $k$ such that there is a coalgebra $\alpha : A \rightarrow \mathbb{T}_k A$.

**Theorem**

For all finite structures $A$:

$$\kappa(A) = \text{tw}(A) + 1.$$ 

In fact, coalgebras on a structure correspond bijectively to certain “nice” tree decompositions.
Further work has shown that the same ideas apply to other important logic comparison games:

- Ehrenfeucht-Fraissé games, comparison by quantifier rank
- Bisimulation games for the modal fragment

In each case, we can define corresponding comonads which characterize the logical equivalences.

Moreover, the coalgebra numbers pick out important invariants.

There is clearly a general paradigm here . . .
The Ehrenfeucht-Fraissé Comonad $E_k$

- $E_k A$ – sequences of elements of $A$ of length $\leq k$.

- Relations defined in similar fashion to $T_k A$ (without need for pebble condition).

This gives syntax-free characterizations of 3 important equivalences:

- quantifier-rank fragments of existential-positive logic
- quantifier-rank fragments of FO logic
- quantifier-rank fragments of FO logic with counting

The coalgebra number for these comonads characterizes **tree-depth** (Nešetřil and Ossona de Mendez) in exactly the same way as for tree-width in the case of the pebbling comonad.

This invariant plays an important rôle in Rossman’s proof of the Homomorphism Preservation Theorem.
The Unravelling Comonad $\mathbb{U}_k$

Given a Kripke structure (labelled transition system with unary predicates):

- $\mathbb{U}_k A$ is the set of all paths $w_0 i_1 w_1 \cdots i_{k-1} w_n$ where $R^A_{i_j}(w_{j-1}, w_j)$, $1 \leq j \leq n$, and $0 \leq n \leq k$.
- $R^A_{\mathbb{U}_k}(s, t)$ iff $t = siw$; $P^A_{\mathbb{U}_k}(siw)$ iff $P^A(w)$ iff $P^A_{\mathbb{U}_k}(w)$.

This yields syntax-free characterizations of equivalences modulo the modal fragment up to modal depth $k$.

The coalgebra number characterizes the following property:

**Theorem**

*There is a $k$-coalgebra on $A$ if and only if the multigraph $G(A)$ is a rooted forest of height $\leq k$.***

It should be possible to generalize this comonad to the Guarded fragment.
No Mo’ Comonads?

Many further directions to pursue:

- Other cases, e.g. games for branching quantifiers or Dependence Logic. May need richer forms of indexing or grading.

- What is the general pattern?

- There is a colimit structure

  \[ T_\omega A = \lim_{\to} T_0 A \to T_1 A \to T_2 A \to \cdots \]

  Similarly for \( E_k \) and \( U_k \). What are their properties?

- No-go theorem for finite representation of \( T_k A \).

- Combining these comonads with **quantum monads**.

- Analyze, and give structural, and hopefully more general forms of major results such as Rossman’s Homomorphism Preservation Theorem.

- Combine with categorical treatments of the Logic-Automata nexus (Bojanczyk, Gehrke-Petrisan-Colcombet, Adamek-Milius) to look at results such as decidability of the Guarded fragment.