Provenance Analysis and Games

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joint work with Val Tannen

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Oxford English Dictionary: provenance, n

The fact of coming from some particular source or quarter; origin, derivation.

Provenance analysis for first-order logic

We have seen in Val Tannen's talk:

- Provenance analysis can be generalized from positive query languages to logics with full negation, especially full first-order logic.
- Negation is handeled via transformation to negation normal form.
- In the presence of negation, the semirings "to rule them all" are $\mathbb{N}[X,\overline{X}] := \mathbb{N}[X \cup \overline{X}]/(X\overline{X})$ based on a self-inverse bijection $X \leftrightarrow \overline{X}$
- Applications to model updates, to explanations for missing or wrong query answers, and to repairs for failing integrity constraints.

Provenance in other settings than first-order logic

In this talk, we shall discuss further aspects:

- Provenance for FO can also be understood as a provenance for the associated model-checking games
- Provenance for games is of independent interest, and provides relevant insights into games beyond the question who wins.
- Games for first-order logic are acyclic and have only finite plays. With the appropriate choice of semirings, provenance analysis can be generalized to games that admit infinite plays.
- Provenance analysis for LFP.

Provenance for finite games

Acyclic two player-game $\mathcal{G} = (V, V_0, V_1, T, E)$ with $V = V_0 \cup V_1 \cup T$

V_σ: positions of Player σ, T: terminal positions, $E \subseteq V \times V$: moves

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Valuations $f_\sigma : T \to K$ of terminal positions and $h_\sigma : E \to K \setminus \{0\}$ of moves in a semiring K.

- $f_\sigma(t)$ describes the value of the terminal position v for Player σ.
  $f_\sigma(t) = 0$ means that t is a losing position
- $h_\sigma(vw)$ describes the value (or cost) for Player σ of a move from v to w.
  (Values of moves may be irrelevant. In that case, set $h_\sigma(vw) = 1$.)

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Extension to valuations $f_\sigma : V \to K$ for all positions. A move from v to w contributes to $f_\sigma(v)$ the value $h_\sigma(vw) \cdot f_\sigma(w)$.

$$f_\sigma(v) = \begin{cases} \sum_{w \in vE} h_\sigma(vw) \cdot f_\sigma(w) & \text{if } v \in V_\sigma \\ \prod_{w \in vE} h_\sigma(vw) \cdot f_\sigma(w) & \text{if } v \in V_{1-\sigma} \end{cases}$$

Reachability games and contradictory valuations

For acyclic game graphs $\mathcal{G} = (V, V_0, V_1, T, E)$, and semiring valuations $f_\sigma : V \to K$, Player σ has a winning strategy for the reachability objective $T \setminus f_\sigma^{-1}(0)$ from all positions v with $f_\sigma(v) \neq 0$.

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On a set $U \subseteq V$ the valuations f_0, f_1 are
- contradictory if either $f_0(u) = 0$ or $f_1(u) = 0$ for all $u \in U$,
- weakly contradictory if just $f_0(u) \cdot f_1(u) = 0$,
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If f_0 and f_1 are (weakly) contradictory on the the terminal positions of $\mathcal{G}$, then they are (weakly) contradictory on all positions of $\mathcal{G}$.
For positive semirings, also strongly contradictory valuations on the terminal positions extend to strongly contradictory ones on all positions.

For the Boolean semiring $\mathbb{B} = (\{0,1\}, \vee, \wedge, 0, 1)$ this is just the determinacy of reachability games on well-founded game graphs.

Applications for different semirings

(1) The tropical semiring and the cost of strategies. On $\mathcal{G}$, let $f_0 : T \to \mathbb{R}_+$ and $h_0 : E \to \mathbb{R}_+$ be cost functions for Player 0 on the terminal positions and the moves.

The cost of a play $\pi = v_0 v_1 \dots v_m$ for Player 0 is defined as
$c(\pi) := \sum_{i=0}^{m-1} h_0(v_i v_{i+1}) + f_0(v_m)$.
The cost of a strategy from v is the sum of the costs of all plays from v that are admitted by the strategy.

Proposition. The cost of an optimal strategy from v in a game $\mathcal{G}$ with basic cost functions $f_0 : T \to \mathbb{R}_+$ and $h_0 : E \to \mathbb{R}_+$ is given by the valuation $f_0(v)$ computed in the tropical semiring $(\mathbb{R}_+^\infty, \min, +, \infty, 0)$.

Applications for different semirings

(2) The access control semiring $\mathbb{A} = (\{\mathsf{P} < \mathsf{C} < \mathsf{S} < \mathsf{T} < 0\}, \min, \max, 0, \mathsf{P})$. Let $f_0 : T \to \mathbb{A}$ and $h_0 : E \to \mathbb{A} \setminus \{0\}$ define access levels for the terminal positions and the moves.

The valuation $f_0(v) \in \mathbb{A}$ then describes the minimal clearance level that Player 0 needs to win from position v.

(3) Confidence scores. Based on confidences $f_\sigma : T \to [0,1]$ that Player σ puts into t being a winning position for her, compute confidence scores $f_\sigma(v)$ to describe the confidence of Player σ that she can win from v, as semiring valuations in the Viterbi semiring $\mathbb{V} = ([0,1], \max, \cdot, 0, 1)$.

Counting winning strategies

Let $\mathbb{N}[T]$ be the semiring of polynomials over indeterminates $t \in T$.

For a game $\mathscr{G}$, let $f_\sigma : V \to \mathbb{N}[T]$ be the valuation induced by $f_\sigma(t) = t$.

We can write $f_\sigma(v)$ as a sum of monomials $t_1^{j_1} \cdots t_k^{j_k}$.

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Each monomial $t_1^{j_1} \cdots t_k^{j_k}$ in $f_\sigma(v)$ indicates a strategy of Player σ from v whose set of possible outcomes is precisely $\{t_1, \ldots, t_k\}$, and precisely j_i plays that are compatible with that strategy have the outcome t_i.

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Fix any reachability objective $W \subseteq T$. Let $f_\sigma(v) = f_\sigma^W(v) + g_\sigma^W(v)$ where $f_\sigma^W(v)$ is the sum of those monomials that only contain indeterminates in W.

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Theorem. Player σ has a strategy to reach W from v if, and only if, $f_\sigma^W(v) \neq 0$. Moreover, if $f_\sigma^W(v) = \sum_{j \in J} c_j M_j$ (where M_j are monomials with indeterminates in W), then $\sum_{j \in J} c_j$ is the number of distinct strategies from v that Player σ has for the reachability objective W.

Provenance analysis for first-order logic

Let A be a finite universe and τ a finite relational vocabulary.

$\text{Lit}_A(\tau) := \text{Atoms}_A(\tau) \cup \text{NegAtoms}_A(\tau) \cup \{a \overset{\neq}{=} b : a,b \in A\}$

A K-interpretation for A and τ is a function $\pi : \text{Lit}_A(\tau) \to K$ that maps equalities and inequalities to their truth values.

If, for all atoms $R\overline{a}$, either $\pi(R\overline{a}) = 0$ or $\pi(\neg R\overline{a}) = 0$, ("consistency"), and, moreover, $\pi(R\overline{a}) + \pi(\neg R\overline{a}) \neq 0$ ("completeness"), then π specifies (provenance information for) a unique structure $\mathfrak{A}_\pi$.

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In Val's talk, we have seen how to extend π to a K-interpretation $\pi : \mathrm{FO}(\tau) \to K$ giving provenance values $\pi[\![\varphi]\!] \in K$ to all $\varphi \in \mathrm{FO}(\tau)$.

This extension can also be understood in game-theoretic terms.

Provenance analysis via model-checking games

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Proposition. For all positions φ of the game $\mathcal{G}(A,\psi)$,

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In particular, if the K-interpretation π defines a unique structure $\mathfrak{A}_\pi$, then $\mathfrak{A}_\pi \models \psi \iff f_0(\psi) \neq 0$, and the provenance information $f_0(\psi)$ reveals information about the number and properties of the strategies of Verifier to establish the the truth of ψ in $\mathfrak{A}_\pi$.

Modal logic (ML)

$$\varphi ::= P_i \mid \varphi \vee \varphi \mid \varphi \wedge \varphi \mid \neg\varphi \mid \Diamond\varphi \mid \Box\varphi$$

evaluated on transition systems $\mathfrak{A} = (V, E, (P_i)_{i\in I})$ with $E \subseteq V \times V$ and $P_i \subseteq V$.

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Lit_V, the set of modal literals for V, contains the atoms $P_i v$ and Evw, for $v, w \in V$, and their negations $\neg P_i v$ and $\neg Evw$.

A modal K-interpretation for V is a function $\pi : \mathrm{Lit}_V \to K$. Similar to the case of FO, it extends to a K-valuation $\pi : \mathrm{ML} \times V \to K$:

$$\pi[\![\Diamond\varphi, v]\!] := \sum_{w\in vE} \pi(Evw) \cdot \pi[\![\varphi, w)]\!] \qquad \pi[\![\Box\varphi, v]\!] := \prod_{w\in vE} \pi(Evw) \cdot \pi[\![\varphi, w]\!]$$

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On the other side, modal K-valuations do in general not coincide with K-interpretations for the standard translation of ML into (the modal fragment of) FO, taking $\psi \in \mathrm{ML}$ to $\psi^*(x) \in \mathrm{FO}$ such that $\mathfrak{A}, v \models \psi \iff \mathfrak{A} \models \psi^*(v)$.

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Indeed, this translation maps $\Box\varphi$ to $(\Box\varphi)^*(x) = \forall y(\neg Exy \vee \varphi^*(y))$. But

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These values coincide only in special cases, for instance if $\pi(Evw)$ and $\pi(\neg Evw)$ only take values 0,1, and 1 is an absorbing element in the semiring K, i.e. if $1 + a = 1$, for all $a \in K$.

The guarded fragment of first-order logic

GF $\subseteq$ FO: fragment with interesting algorithmic and model-theoretic properties. It permits only guarded quantification

$$(\exists \bar{y} \,.\, \alpha)\varphi \qquad \text{and} \qquad (\forall \bar{y} \,.\, \alpha)\varphi$$

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A K-interpretation $\pi : \mathrm{Lit}_A(\tau) \to K$ provides valuations for terminal positions and guarded quantification moves of a GF-game. This induces a valuation $f_0(\varphi) \in K$ for every position ψ in the game. Set $\pi[\![\psi]\!] := f_0(\psi)$.

As in the case of modal logic, the standard translation of GF into usual first-order syntax taking $(\forall \bar{y} \,.\, \alpha)\varphi$ to $\forall \bar{y}(\neg\alpha \vee \varphi)$ produces formulae that may have different provenance values in K.

Provenance for reachability games with cycles

Let $\mathcal{G} = (V, V_0, V_1, T, E)$ be a finite, not necessarily acyclic, game graph.

Given a valuation $f_\sigma : T \to K$ in a semiring K for the terminal nodes, the rules defining valuations for the other nodes have now to be read as an equation system in indeterminates X_v (for $v \in V$):

$$X_v = f_\sigma(v) \quad \text{for } v \in T$$

$$X_v = \sum_{w \in vE} h_\sigma(vw) \cdot X_w \quad \text{if } v \in V_\sigma$$

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To make sure that a solution of such a system exists, we assume that the semiring K is naturally ordered and ω-continuous.

ω-continuous semirings

A semiring is naturally ordered if $a \leq b :\Leftrightarrow \exists x(a + x = b)$ is a partial order.

A semiring K is ω-continuous if it is naturally ordered and every ω-chain $a_0 < a_1 < \dots$ has a supremum $\sup_{i<\omega} a_i$, such that the associated countable summation operator $\sum_{i<\omega} b_i := \sup_{i<\omega}(b_0 + \cdots + b_i)$ is compatible with the operations of K.

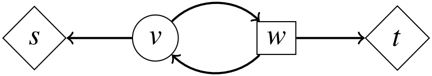
A formal power series $f \in K[\![X]\!]$ in variables $X = (X_1, \dots, X_n)$ is a possibly infinite sum of monomials $c \cdot X_1^{e_1} \dots X_n^{e_n}$.

Let $F = (f_1 \dots f_n)$ be a system of formal power series $f_i \in K[\![X]\!]$. If K is ω-continuous, then by Kleene's Fixed-Point Theorem, the equation system $F(X) = X$ has a least fixed-point solution $\text{lfp}(F)$ which is the supremum of the Kleene approximants F^k, defined by $F^0 = 0$, $F^{k+1} = F(F^k)$.

Semirings of power series

Notice that $(\mathbb{N}, +, \cdot, 0, 1)$ is not ω-continuous, but its completion $\mathbb{N}^\infty$ is.
The completion of $\mathbb{N}[X]$ is not $\mathbb{N}^\infty[X]$ but the semiring of (possibly infinite) formal power series, denoted $\mathbb{N}^\infty[\![X]\!]$.

Example.



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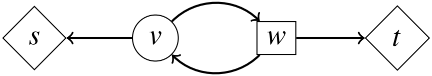

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Solution in $\mathbb{N}^\infty[\![s,t]\!]$: $f(v) = s \cdot (1 + t + t^2 + \cdots)$ and $f(w) = s \cdot (t + t^2 + \cdots)$

Evaluation.

- $f(v)(0,t) = f(w)(0,t) = 0$

  Neither from v nor from w, Player 0 has a strategy to reach t.
- $f(v)(s,0) = s$ but $f(w)(s,0) = 0$:

  Player 0 has a strategy to reach s from v, but not from w.

Counting strategies

Again, valuations in $\mathbb{N}^\infty[\![X]\!]$ give more information than just who wins.

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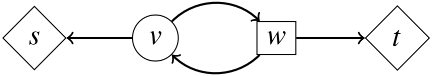

For every $n < \omega$, the monomial $s \cdot t^n$ in $f(v) = s \cdot (1 + t + t^2 + \cdots)$ tells us that Player 0 has precisely one strategy from v that admits $n+1$ consistent plays, one of which has outcome s, and the other n have outcome t.

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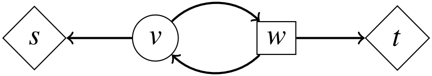

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By evaluating these formal power series in the tropical semiring, the Viterbi semiring, or the access control semiring, we obtain information about the cost of optimal strategies, and the confidence of winning or the required clearance levels for winning reachability games.

Least fixed-point logic

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posLFP is the fragment of LFP that makes use of least fixed points only (which may appear only positively). posLFP is at the bottom level of the alternation hierarchy of LFP. In general, and for instance on $(\mathbb{N}, +, \cdot)$, this hierarchy is strict.

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LFP extends FO by least and greatest fixed-points for monotone definable operators. It is a logic of great importance in finite model theory.

On ordered finite structures, LFP captures PTIME. (Immerman, Vardi)

posLFP is the fragment of LFP that makes use of least fixed points only (which may appear only positively). posLFP is at the bottom level of the alternation hierarchy of LFP. In general, and for instance on $(\mathbb{N}, +, \cdot)$, this hierarchy is strict.

Theorem (Immerman) On finite structures, LFP $\equiv$ posLFP.

The model checking games for general LFP-formulae are parity games, which are not known to solvable in polynomial time. However, the model-checking games for posLFP are reachability games.

Provenance for positive least fixed-point logic

For a finite universe A and a finite relational vocabulary, consider a K-interpretation $\pi : \mathrm{Lit}_A(\tau) \to K$ into an ω-continuous semiring K.

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For a general form of provenance for posLFP, use the semirings $\mathbb{N}^\infty[\![X, \overline{X}]\!]$.

Beyond pos LFP

How to deal with full LFP? By moving to formulae in negation normal form, we have to take care of greatest fixed points. However, their existence (and meaning) is unclear in arbitrary ω-continuous semirings. (At least to me!)

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Consider monomials over a finite set X of provenance tokens, with exponents in $\mathbb{N}^\infty$. Absorption ordering: $x_1^{i_1} \cdots x_m^{i_m} \leq x_1^{j_1} \cdots x_m^{j_m} \iff i_k \geq j_k$ for all k.

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Proposition. $\mathbb{S}^\infty[X]$ is a complete lattice with respect to the natural order.

Hence the Tarski-Knaster fixed-point theory applies to $\mathbb{S}^\infty[X]$, and we can inductively define provenance values in $\mathbb{S}^\infty[X]$ for arbitrary LFP-formulae.

Absorptive strategies

We have seen that with any strategy $\mathscr{S}$, we can associate a monomial $M_{\mathscr{S}}$ over the set of terminal positions. The value of a strategy is the product over the values of the plays it admits. Nonterminating plays have value 0.

Absorption: $\mathscr{S} \succeq \mathscr{S}'$ if $M_{\mathscr{S}} \geq M_{\mathscr{S}'}$
This means: for any outcome t, $\mathscr{S}$ admits less plays with outcome t than $\mathscr{S}'$.

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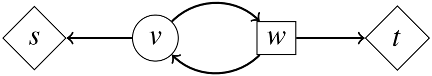
Theorem. Let $\mathscr{G}$ be a reachability game. The provenance values in $\mathbb{S}^{\infty}[T]$ at v give the values of all absorption-dominant strategies from v.

Reachability versus safety games

Example.



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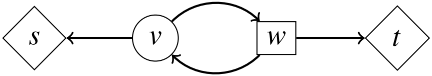

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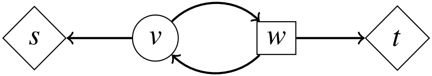

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But what if we analyse this game as a safety game:

- The value of a non-terminating play is 1, not 0.
- We have to compute greatest fixed-point solutions of the equation system.

Greatest fixed point solution in $\mathbb{S}^\infty[s,t]$: $f(v) = s + t^\infty$ and $f(w) = st + t^\infty$

For safety, Player 0 has two absorptive strategies: move to s, or move to w. From v the first one admits a unique play with outcome s, the second one admits infinitely many plays with outcome t (and one non-terminating play).

Work in Progress

Provenance analysis for more general infinite games, in particular for parity games.

For such games, it does not suffice to track terminal positions.
Instead track the moves, to get provenance values that tell you which moves are used, and how often, by a strategy.

Hierarchical equations systems, with interleaving least and greatest fixed points, are used to compute provenance values for parity games.

Where are the limits of this approach?

Algorithmic questions