Sums of Squares and Matrix Completion

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Basic Question

$X \subset \mathbb{R}^n$ is a compact set defined by quadratic equations $p_1 = 0, \ldots, p_s = 0$. Want to optimize a quadratic Q on X.

P_X convex cone of quadratics nonnegative on X.

"Obviously nonnegative" quadratics on X: Sums of squares of linear polynomials + linear combinations of p_i's.

Σ_X convex cone of sums of squares on X.

Main Question: For what X are all nonnegative quadratics "obviously nonnegative"? Sum of Squares hierarchy converges in one step.

Going Deeper

Quadratic polynomials are symmetric matrices. Work modulo linear subspace L:

$$L = \text{span}(p_1, \ldots, p_s).$$

The cone Σ_X is a projection of the cone of PSD matrices $\mathcal{S}_+^n$.

The dual cone Σ_X^* is the slice of $\mathcal{S}_+^n$ with the orthogonal complement $L^\perp$. Σ_X^* is the *Hankel Spectrahedron* of X.

Observation: $\Sigma_X = P_X$ iff Σ_X^* has only rank 1 extreme rays.

Equivalent Formulation: Describe all spectrahedral cones $C = L \cap \mathcal{S}_+^n$ which have only rank one extreme rays.

Examples

First example: $L = \mathcal{S}^n$ and $C = \mathcal{S}^n_+$.

Second example: L is the subspace of Hankel matrices:

$$\begin{bmatrix} a_1 & a_2 & a_3 & \iddots & \iddots & a_n \\ a_2 & a_3 & \iddots & \iddots & a_n & a_{n+1} \\ a_3 & \iddots & \iddots & a_n & a_{n+1} & \iddots \\ \iddots & \iddots & \iddots & \iddots & \iddots & \iddots \\ \iddots & \iddots & \iddots & \iddots & \iddots & \iddots \\ a_n & a_{n+1} & \iddots & \iddots & \iddots & a_{2n-1} \end{bmatrix}$$

Third Example: L is a hyperplane. S-Lemma implies that C has only rank 1 extreme rays.

One More Example!

Fourth example: L is a subspace of block-Hankel Matrices:

$$C = \left[\begin{array}{c|c} H_1 & H_2 \\ \hline H_2 & H_3 \end{array}\right]$$

where each H_i is Hankel. Can be any size:

$$C = \left[\begin{array}{c|c|c} H_1 & H_2 & H_3 \\ \hline H_2 & H_4 & H_5 \\ \hline H_3 & H_5 & H_6 \end{array}\right]$$

Tensor products: $M \bigotimes H$

Warmup Question

M is a partially filled-in symmetric matrix.

$$M = \begin{bmatrix} 1 & 1 & 1 & ? \\ 1 & 1 & 1 & ? \\ 1 & 1 & 1 & 2 \\ ? & ? & 2 & 1 \end{bmatrix}$$

Can ? entries be chosen so that M is positive semidefinite?

Warmup Question

M is a partially filled-in symmetric matrix.

$$M = \begin{bmatrix} 1 & 1 & 1 & ? \\ 1 & 1 & 1 & ? \\ 1 & 1 & 1 & 1 \\ ? & ? & 1 & 1 \end{bmatrix}$$

Can ? entries be chosen so that M is positive semidefinite?

Warmup Question, Part II

M is a symmetric matrix where $*$ entries are given and ? entries can be freely chosen.

Can M be completed to a positive semidefinite matrix?

$$M = \begin{bmatrix} * & * & * & ? \\ * & * & * & ? \\ * & * & * & * \\ ? & ? & * & * \end{bmatrix}$$

Obvious Necessary Condition

M is a symmetric matrix where $*$ entries are given and ? entries can be freely chosen.

Can M be completed to a positive semidefinite matrix?

$$\begin{bmatrix} * & * & * & ? \\ * & * & * & ? \\ * & * & * & * \\ ? & ? & * & * \end{bmatrix}$$

Necessary Condition: Filled in principal minors of M must be PSD.

Question: When is this necessary condition also sufficient?

Sum of Squares Reformulation

$$\begin{bmatrix} * & * & * & ? \\ * & * & * & ? \\ * & * & * & * \\ ? & ? & * & * \end{bmatrix}$$

A symmetric matrix is also a quadratic form!

Let

$$X = \mathrm{Span}(e_1, e_2, e_3) \cup \mathrm{Span}(e_3, e_4).$$

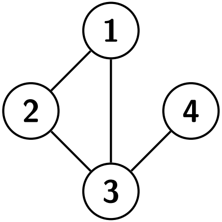
Observe that quadrics vanishing on X are

$$L(X) = \mathrm{span}(x_1 x_4, x_2 x_4).$$

Reformulation: Given a quadratic form nonnegative on X when can we write it as a sum of squares on X (modulo $L(X)$)?

Enter Graphs

$$\begin{bmatrix} * & * & * & ? \\ * & * & * & ? \\ * & * & * & * \\ ? & ? & * & * \end{bmatrix}$$



Draw a graph G where edges correspond to $*$ (no self-edges).

Variety X corresponds to maximal cliques (clique complex) of G.

Theorem: (Grone, Johnson, Sa, Wolkowicz, 1984) The obvious necessary condition is also sufficient if and only if G is chordal.

G is chordal if any cycle of length ≥ 4 has a chord dividing it.

The Theorem

Theorem:(B., R. Sinn, M. Velasco) The following are equivalent:

- $P_X = \Sigma_X$.
- The irreducible components $X_1, \ldots, X_k$ of X are varieties of minimal degree which are linearly joined:

$$(X_1 \cup \cdots \cup X_i) \cap X_{i+1} = \mathrm{Span}(X_1 \cup \cdots \cup X_i) \cap \mathrm{Span}\, X_{i+1}.$$

L is the vector space of all quadratic vanishing on X.

Remarks:

- Variety of minimal degree means: $\deg X = \operatorname{codim} X + 1$. These have been classified classically by Del Pezzo and Bertini.
- From Eisenbud, Green, Hulek, Popescu (2004) the above X are exactly varieties of Castelnuovo-Mumford regularity 2.

What If Not Equal?

L: Coordinate linear subspace of $\mathcal{S}^n$ with corresponding graph G.

$C = L \cap \mathcal{S}_n^+$ spectrahedral cone.

Theorem:(B., R. Sinn, M. Velasco) A number p is the smallest rank of an extreme ray of C greater than 1 if and only if $p+2$ is the length of the smallest chordless cycle in G.

Remark: This also generalizes to arbitrary varieties X, via so-called property $N_{2,p}$.

THANK YOU!

Do Sums of Squares Dream of Free Resolutions?

G.B., R. Sinn, M. Velasco