Small (Explicit) Extended Formulation for Knapsack Cover Inequalities from Monotone Circuits

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&
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Extended formulations vs circuit complexity

[HRUBEŠ '12, GJW '16]

∃ connection between circuit complexity and extended formulations

What is the smallest LP for solving problem A?

What is the smallest depth circuit for solving problem B?
Extended formulations vs circuit complexity

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What is the smallest LP for solving problem \( A \)?

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Extended formulations vs circuit complexity

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exists a connection between circuit complexity and extended formulations

What is the smallest LP for solving problem $A$?

$Q$

$P$

What is the smallest depth circuit for solving problem $B$?

$2^{\Omega(n)}$ size LP lower bound for matchings $\implies$ matching requires monotone circuits of $\Omega(n)$ depth
Extended formulations vs circuit complexity

[Hrubeš ’12, GJW ’16]

∃ connection between circuit complexity and extended formulations

What is the smallest LP for solving problem $A$?

What is the smallest depth circuit for solving problem $B$?

We use that certain functions have small depth monotone circuits to give small explicit LPs for covering problems.
Our problem

1. “Simplest” binary integer program with single covering constraint:

\[100x_1 + 50x_2 + 50x_3 \geq 101\]
\[x_1, x_2, x_3 \in \{0, 1\}\]

2. Write down an LP relaxation:

\[100x_1 + 50x_2 + 50x_3 \geq 101\]
\[0 \leq x_1, x_2, x_3 \leq 1\]

\# inequalities = 7

3. Adversary finds worst objective function (= evaluates integrality gap)

If \(\min x_2 + x_3\), then
\nIP optimum = 1 \quad \Longrightarrow \quad \text{Integrality gap} = 50
\nLP optimum = 1/50
Our problem

1. “Simplest” binary integer program with single covering constraint:

\[ 100x_1 + 50x_2 + 50x_3 \geq 101 \]
\[ x_1, x_2, x_3 \in \{0, 1\} \]

2. Write down an LP relaxation:

\[ x_1 = 1 \]
\[ x_2 + x_3 \geq 1 \]
\[ 0 \leq x_2, x_3 \leq 1 \]

# inequalities = 5

3. Adversary finds worst objective function (= evaluates integrality gap)

For any objective fn

IP optimum = LP optimum \[\implies\text{Integrality gap} = 1\]
Our problem — formally

“Simplest” 0/1-set defined by single covering constraint

\[ \sum_{i=1}^{n} a_{i}x_{i} \geq \beta \quad \text{where } a \in \mathbb{Z}_+^{n} \text{ and } \beta \in \mathbb{Z}_+ \]

\[ x \in \{0, 1\}^n \]

convex hull of integer solutions = min-knapsack polytope
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convex hull of integer solutions = \textbf{min-knapsack polytope}

2. Write down an LP relaxation = \textbf{extended formulation}

\[ Ax + By = c \quad \text{equality constraints} \]
\[ Dx + Ey \geq f \quad \text{inequality constraints} \]
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1. “Simplest” 0/1-set defined by single covering constraint

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3. Evaluate the relaxation vs all possible objective functions in terms of

- **size** = # inequalities
- **integrality gap** = \( \sup_{\text{IP optimum}} \frac{\text{LP optimum}}{\text{IP optimum}} \)
State of the Art

Knapsack-cover ineqs [Balas ’75, Hammer et al. ’75, Wolsey ’75, Carr et al. ’06]

There are exponentially many (but approximately separable) inequalities that bring the integrality gap down to 2

Many applications:
- Network design
- Facility location
- Scheduling

Question

Is there a poly-size relaxation with any constant integrality gap?

Previously:
- [Bienstock and McClosky ’12] can be done when the objective is sorted
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Previously:
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... our first feeling: maybe there is no such relaxation?
Our results (1/2)

Theorem (Existential – Bazzi, F, Huang, Svensson ’17)

The min-knapsack polytope can be \((2 + \varepsilon)\)-approximated by an LP of size \((n/\varepsilon)^{O(1)} \cdot 2^{O(d)}\) where \(d\) is the minimum depth of a monotone circuit that computes (truncations of) the corresponding weighted threshold function.
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- [Beimel & Weinreb ’08] Weighted threshold functions

\[
f(x_1, \ldots, x_n) = \begin{cases} 
1 & \text{if } \sum_{i=1}^{n} a_i x_i \geq \beta \\
0 & \text{otherwise}
\end{cases}
\]

have monotone circuits of depth \(O(\log^2 n)\)
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Corollary (Existential – Bazzi, F, Huang, Svensson ’17)

The min-knapsack polytope can be \((2 + \varepsilon)\)-approximated by an LP of size \((1/\varepsilon)^{O(1)} \cdot n^{O(\log n)}\).
A galaxy of hierarchies
Strengthening relaxations using formulas

\[ S \subseteq \{0, 1\}^n \] \text{0/1-set}
\[ \phi \text{ Boolean formula defining } S \]
\[ Q \subseteq [0, 1]^n \] \text{convex relaxation of } S

\[ \rightarrow \]

\textbf{New} relaxation \( \phi(Q) \)
with \( S \subseteq \phi(Q) \subseteq Q \)
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\[ \text{New relaxation } \phi(Q) \]
\[ \text{with } S \subseteq \phi(Q) \subseteq Q \]

Starting from \( \phi \), recursively apply rules:

- Replace \( x_i \) by \( Q \cap \{x : x_i = 1\} \)
- Replace \( \neg x_i \) by \( Q \cap \{x : x_i = 0\} \)
- Replace \( \phi_1 \land \phi_2 \) by \( \phi_1(Q) \cap \phi_2(Q) \)
- Replace \( \phi_1 \lor \phi_2 \) by \( \text{conv}(\phi_1(Q) \cup \phi_2(Q)) \)
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0/1-set: \( S = \{x \in \{0, 1\}^3 : x_1 + x_2 + x_3 \geq 2\} \)
Formula: \( \phi = (x_1 \lor x_2) \land (x_2 \lor x_3) \land (x_3 \lor x_1) \)
Relaxation: \( Q := [0, 1]^3 \)
**Strengthening relaxations using formulas**

\[ S \subseteq \{0, 1\}^n \text{ 0/1-set} \]

\[ \phi \text{ Boolean formula defining } S \]

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→

**New relaxation** \[ \phi(Q) \]

with \[ S \subseteq \phi(Q) \subseteq Q \]

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←**monotone**

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Proposition (F, Huynh & Weltge ’17)

If \( Q \subseteq [0, 1]^n \) is a polytope then \( \phi(Q) \) also, and moreover

\[
xc(\phi(Q)) \leq |\phi| \cdot xc(Q)
\]
Our results (2/2)

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**Theorem (F, Huynh & Weltge ’17)**

Assuming $\phi$ monotone,

- $Q$ satisfies all valid pitch $\leq k$ inequalities
- $\phi(Q)$ satisfies all valid pitch $\leq k + 1$ inequalities
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The pitch measures how “complex” ineqs are (Bienstock & Zuckerberg ’04):
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The **pitch** measures how “complex” ineqs are (Bienstock & Zuckerberg ’04):

- $x_1 \geq 1, \quad x_1 + x_3 + x_7 \geq 1$ have pitch 1
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The *pitch* measures how “complex” ineqs are (Bienstock & Zuckerberg ’04):

- $x_1 \geq 1, x_1 + x_3 + x_7 \geq 1$ have pitch 1
- $x_1 + x_5 \geq 2, 2x_3 + x_4 + x_5 \geq 2$ have pitch 2
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Assuming \( \phi \) monotone,

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\begin{align*}
Q & \text{ satisfies all valid } \\
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\implies \quad & \phi(Q) \text{ satisfies all valid } \\
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\]

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- \( x_1 \geq 1, \quad x_1 + x_3 + x_7 \geq 1 \) have pitch 1
- \( x_1 + x_5 \geq 2, \quad 2x_3 + x_4 + x_5 \geq 2 \) have pitch 2
- \( 2x_3 + x_4 + 2x_7 \geq 3 \)
Our results (2/2)

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*If* \( Q \subseteq [0, 1]^n \) *is a polytope then* \( \phi(Q) \) *also, and moreover*

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**Theorem (F, Huynh & Weltge ’17)**

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\[
\begin{align*}
Q \text{ satisfies all valid } & \quad \Rightarrow \\
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- \( 2x_3 + x_4 + 2x_7 \geq 3 \) has pitch 2
BZ’s approximation of the CG-closures

**Theorem (Bienstock & Zuckerberg ’06)**

If $Q = \{ x \in [0, 1]^n : Ax \geq b \}$ for $A, b$ nonnegative, then

$$Q \cap \{ x \mid x \text{ satisfies all valid pitch } \leq k \text{ ineqs} \}$$

is $(1 + \varepsilon)$-approx of the $\ell$-th CG-closure of $Q$ whenever $k = \Omega(\ell/\varepsilon)$.
Comparison to BZ’04

Main theorem from Bienstock & Zuckerberg ’04, where \( g(k) = \Omega(k^2) \):

**Theorem 1.2.** Let \( k \geq 1 \) be a fixed integer. Consider a set-covering problem

\[
\min \{ c^T x : Ax \geq e, \ x \in \{0,1\}^n \},
\]

where \( A \) is an \( m \times n \), 0-1 matrix and \( e \) is the vector of \( m \) 1s. Let \( P_k \) denote the set of all valid inequalities for \( \{ x \in \{0,1\}^n : Ax \geq e \} \) of pitch \( \leq k \). Then there exists a positive integer \( g(k) \), a polytope \( Q_k \subseteq R^n \), and a polytope \( \tilde{Q}_k \subseteq R^{(m+n)g(k)} \) satisfying the following:

(a) \( \{ x \in \{0,1\}^n : Ax \geq e \} \subseteq Q_k \).
(b) \( a^T x \geq a_0 \) for all \( x \in Q_k \) and for all \( (a,a_0) \in P_k \).
(c) \( Q_k \) is the projection to \( R^n \) of \( \tilde{Q}_k \).
(d) \( Q_k \) can be described by a system of at most \( (m+n)g(k) \) linear constraints, with integral coefficients of absolute value at most \( k \). This system can be computed in time polynomial in \( n \) and \( m \) for fixed \( k \).
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(a) $\{x \in \{0,1\}^n \ Ax \geq e\} \subseteq Q_k$.

(b) $a^T x \geq a_0$ for all $x \in Q_k$ and for all $(a,a_0) \in P_k$.

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- If use \( Q \cap \phi^k([0,1]^n) \), get extended formulation of size
  - \( \leq xc(Q) + 2n \cdot (mn)^k \) for obvious CNF formula deciding \( S \)
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- Was simplified (but not improved) earlier by Mastrolili '17
- If use $Q \cap \phi^k([0,1]^n)$, get extended formulation of size
  - $\leq xc(Q) + 2n \cdot (mn)^k$ for obvious CNF formula deciding $S$
  - $\leq xc(Q) + 2n \cdot |\phi|^k$ where $\phi$ is any formula deciding $S$
Theorem (F, Huynh & Weltge ’17)

Assuming $\phi$ monotone,

\[ Q \text{ satisfies all valid } \text{pitch} \leq k \text{ inequalities} \quad \implies \quad \phi(Q) \text{ satisfies all valid } \text{pitch} \leq k + 1 \text{ inequalities} \]

Proof (inspired by Karchmer & Widgerson ’90)

Assume $\sum_{i \in I^+} c_i x_i \geq \delta$ pitch-$(k + 1)$ ineq not valid for $\phi(Q)$

Letting $a \in \{0, 1\}^n$ with $a_i = 0 \iff i \in I^+$, have:

- $\phi(a) = 0$
- $\exists$ violator $\tilde{x} \in \phi(Q)$
If $\phi = \phi_1 \land \phi_2$ then

- $\phi_1(a) = 0$ or $\phi_2(a) = 0$
- $\exists$ violator $\tilde{x}_1 \in \phi_1(Q)$ and $\exists$ violator $\tilde{x}_2 \in \phi_2(Q)$
If $\phi = \phi_1 \land \phi_2$ then

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If $\phi = \phi_1 \lor \phi_2$ then

- $\phi_1(a) = 0$ and $\phi_2(a) = 0$
- $\exists$ violator $\tilde{x}_1 \in \phi_1(Q)$ or $\exists$ violator $\tilde{x}_2 \in \phi_2(Q)$
Final leaf $x_j$ has:

$a_j = 0 \iff j \in I^+$

$\exists$ violator $\bar{x} \in Q \cap \{x : x_j = 1\}$

contradicts hypothesis that $Q$ satisfies pitch $\leq k$ ineq

$$\sum_{i \neq j} c_i x_i \geq \delta - c_j$$
Final leaf $x_j$ has:

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contradicts hypothesis that $Q$ satisfies pitch $\leq k_{\text{ineq}}$

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**contradicts** hypothesis that $Q$ satisfies pitch $\leq k$ ineq

$$\sum_{i \neq j} c_i x_i \geq \delta - c_j$$
Knapsack-cover inequalities

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**Intuition**: KC ineq is pitch-1 w.r.t. large items ← items \( s_i \) such that \( s_i \geq D(a) \)
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The relaxation

1. **Sort** item sizes: \( s_1 \geq s_2 \geq \cdots \geq s_n \)

2. **Parametrize** the KC inequalities by:
   - \( \alpha := \) index of last large item
   - \( \beta := \sum_{i \leq \alpha} s_i a_i \)

3. **Construct** monotone formula \( \phi_{\alpha, \beta} \) for threshold function
   \[
   f_{\alpha, \beta}(x) = 1 \iff \sum_{i \leq \alpha} s_i x_i \geq \beta + 1
   \]

4. **Define** relaxation by the following formula:
   \[
   \bigwedge_{\alpha, \beta} \left( \phi_{\alpha, \beta}(x) \lor \left( \sum_{i > \alpha} s_i x_i \geq D - \beta \right) \right)
   \]
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— THANK YOU! —