Small (Explicit) Extended Formulation for Knapsack Cover Inequalities from Monotone Circuits

Abbas Bazzi **Samuel Fiorini** Sangxia Huang Ola Svensson

&

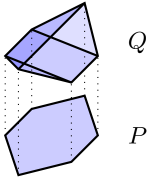
Samuel Fiorini Tony Huynh Stefan Weltge

Extended formulations vs circuit complexity

[Hrubeš '12, GJW '16]

∃ **connection** between circuit complexity and extended formulations

What is the **smallest LP** for solving problem A?



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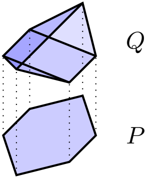

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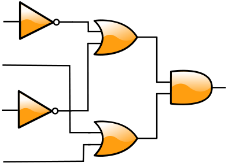

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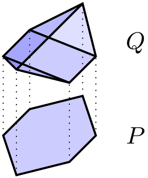

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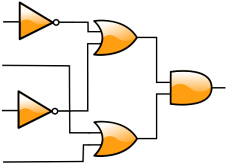

$2^{\Omega(n)}$ size LP lower bound for matchings $\implies$ matching requires monotone circuits of $\Omega(n)$ depth

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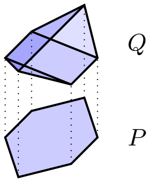

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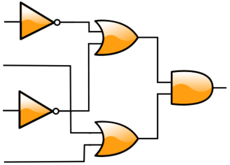

We use that certain functions have small depth monotone circuits to give small **explicit** LPs for covering problems

Our problem

1 "Simplest" binary integer program with single covering constraint:

$$100x_1 + 50x_2 + 50x_3 \geqslant 101$$
$$x_1, x_2, x_3 \in \{0, 1\}$$

2 Write down an LP relaxation:

$$100x_1 + 50x_2 + 50x_3 \geqslant 101$$
$$0 \leqslant x_1, x_2, x_3 \leqslant 1$$

inequalities = 7

3 Adversary finds worst objective function (= evaluates integrality gap)

If $\min x_2 + x_3$, then
IP optimum = 1
LP optimum = 1/50

$\Longrightarrow$ Integrality gap = 50

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$$x_1 = 1$$
$$x_2 + x_3 \geqslant 1$$
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inequalities = 5

3 Adversary finds worst objective function (= evaluates integrality gap)

For any objective fn
IP optimum = LP optimum

$\Longrightarrow$ **Integrality gap** = 1

Our problem — formally

1. "Simplest" $0/1$-set defined by single covering constraint

$$\sum_{i=1}^{n} a_i x_i \geqslant \beta \quad \text{where } a \in \mathbb{Z}_+^n \text{ and } \beta \in \mathbb{Z}_+$$
$$x \in \{0,1\}^n$$

convex hull of integer solutions = **min-knapsack polytope**

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$$Ax + By = c \quad \text{equality constraints}$$
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3. Evaluate the relaxation vs all possible objective functions in terms of
   - **size** = # inequalities
   - **integrality gap** = $\sup \frac{\text{IP optimum}}{\text{LP optimum}}$

State of the Art

Knapsack-cover ineqs [Balas '75, Hammer et al. '75, Wolsey '75, Carr et al. '06]

There are exponentially many (but approximately separable) inequalities that bring the integrality gap down to 2

Many applications:

- Network design
- Facility location
- Scheduling

Question

Is there a **poly-size** relaxation with any **constant** integrality gap?

Previously:

- [Bienstock and McClosky **'12**] can be done when the objective is sorted

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. . . our first feeling: maybe **there is no such relaxation**?

Our results (1/2)

Theorem (Existential – Bazzi, F, Huang, Svensson '17)

The min-knapsack polytope can be $(2+\varepsilon)$-approximated by an LP of size $(n/\varepsilon)^{O(1)} \cdot 2^{O(d)}$ where d ***is the minimum depth of a monotone circuit*** *that computes (truncactions of) the corresponding weighted threshold function*

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- [Beimel & Weinreb '**08**] Weighted threshold functions

$$f(x_1, \ldots, x_n) = \begin{cases} 1 & \text{if } \sum_{i=1}^{n} a_i x_i \geqslant \beta \\ 0 & \text{otherwise} \end{cases}$$

have monotone circuits of depth $O(\log^2 n)$

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Corollary (Existential – Bazzi, F, Huang, Svensson '17)

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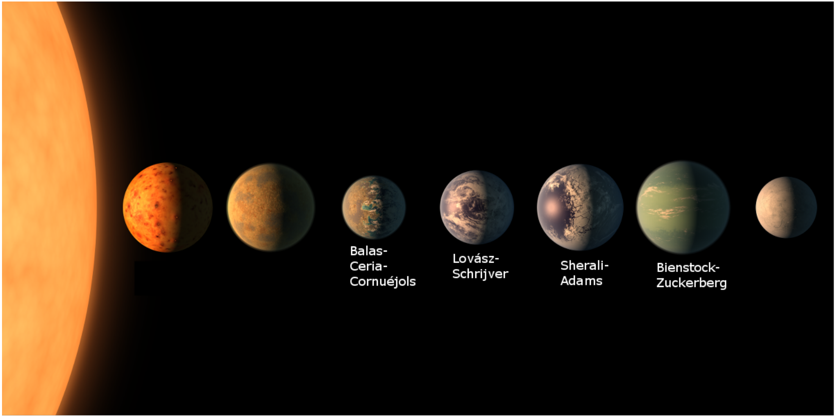
A galaxy of hierarchies



Strengthening relaxations using formulas

$S \subseteq \{0,1\}^n$ $0/1$-set

ϕ Boolean formula **defining** S

$Q \subseteq [0,1]^n$ convex relaxation of S

$\longrightarrow$

New relaxation $\phi(Q)$

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Starting from ϕ, recursively apply **rules**:

- Replace x_i by $Q \cap \{x : x_i = 1\}$
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- Replace $\phi_1 \wedge \phi_2$ by $\phi_1(Q) \cap \phi_2(Q)$
- Replace $\phi_1 \vee \phi_2$ by $\mathrm{conv}(\phi_1(Q) \cup \phi_2(Q))$

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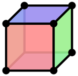
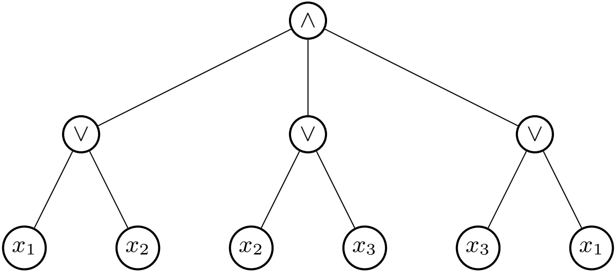
Relaxation: $Q := [0,1]^3$ 



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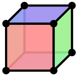

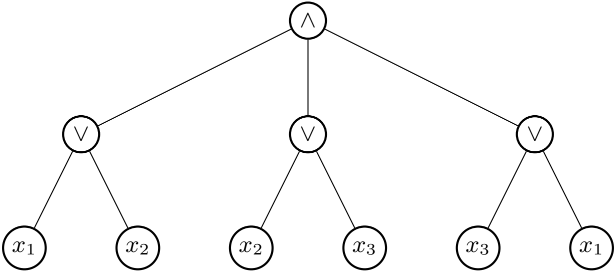

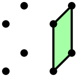

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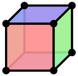
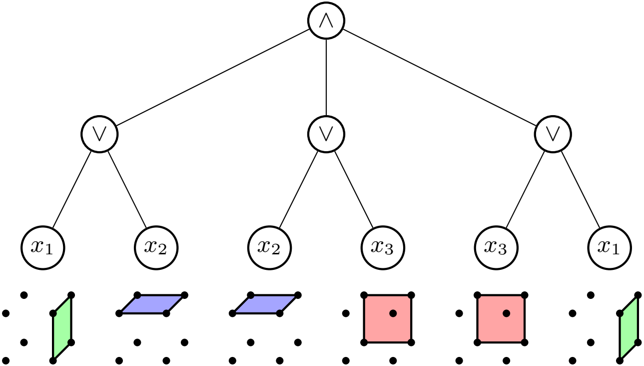
Relaxation: $Q := [0,1]^3$ 



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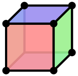

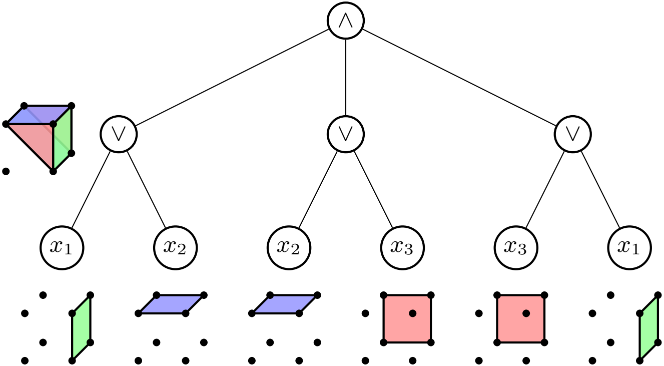

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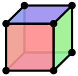
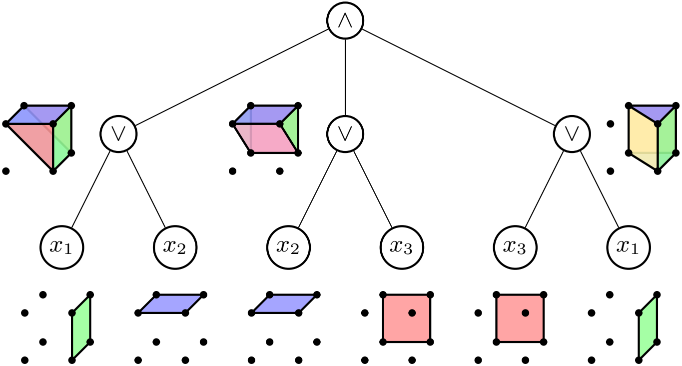
Relaxation: $Q := [0,1]^3$ 



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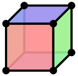
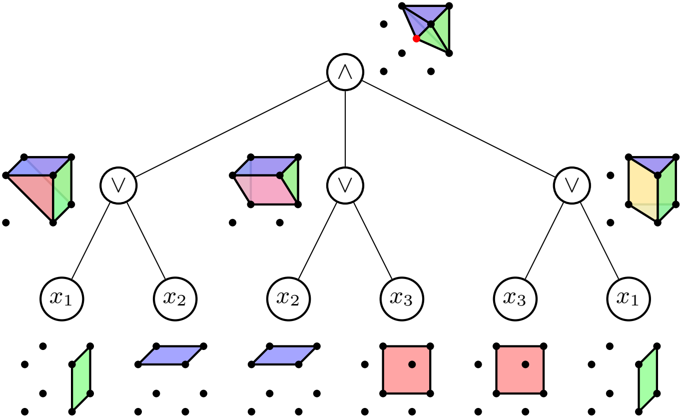
Relaxation: $Q := [0,1]^3$ 



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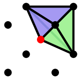

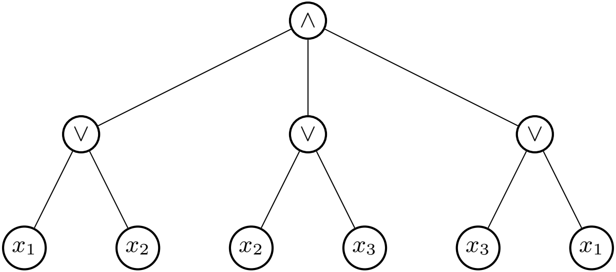

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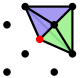

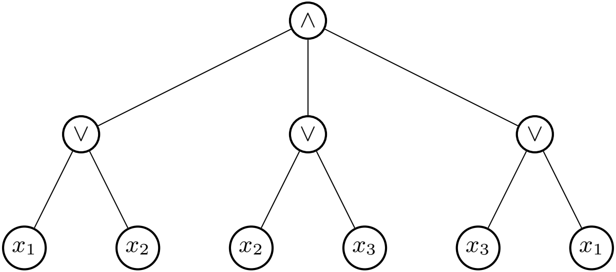

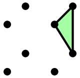

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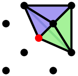

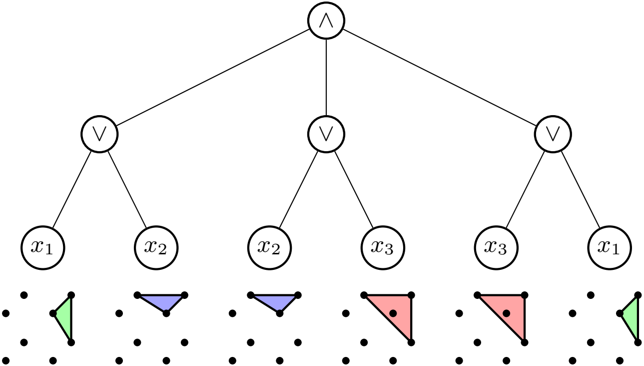

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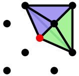
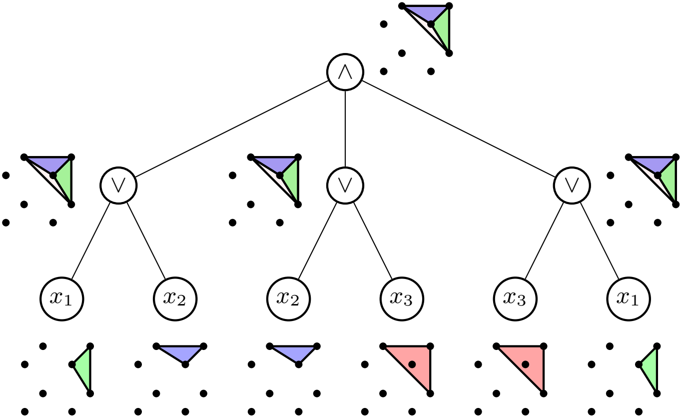
Relaxation: $Q' := \phi([0,1]^3)$ 



Our results (2/2)

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- $2x_3 + x_4 + 2x_7 \geqslant 3$ has pitch **2**

BZ's approximation of the CG-closures

Theorem (Bienstock & Zuckerberg '06)

If $Q = \{x \in [0,1]^n : Ax \geqslant b\}$ *for* A, b ***nonnegative***, *then*

$$Q \cap \{x \mid x \text{ satisfies all valid pitch} \leqslant k \text{ ineqs}\}$$

is $(1+\varepsilon)$*-approx of the* ℓ*-th CG-closure of* Q *whenever* $k = \Omega(\ell/\varepsilon)$

Comparison to BZ'04

Main theorem from Bienstock & Zuckerberg '04, where $g(k) = \Omega(k^2)$:

THEOREM 1.2. *Let $k \geq 1$ be a fixed integer. Consider a set-covering problem*

$$\min\{c^T x : Ax \geq e,\ x \in \{0,1\}^n\},$$

where A is an $m \times n$, 0-1 matrix and e is the vector of m 1s. Let P_k denote the set of all valid inequalities for $\{x \in \{0,1\}^n : Ax \geq e\}$ of pitch $\leq k$. Then there exists a positive integer $g(k)$, a polytope $\mathcal{Q}_k \subseteq R^n$, and a polytope $\bar{\mathcal{Q}}_k \subseteq R^{(m+n)^{g(k)}}$ satisfying the following:

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  - $\leqslant \mathrm{xc}(Q) + 2n \cdot |\phi|^k$ where ϕ is **any** formula deciding S

Theorem (F, Huynh & Weltge '17)

*Assuming ϕ **monotone**,*

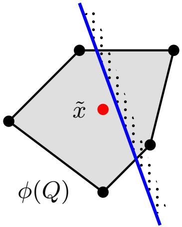
Q satisfies all valid pitch $\leqslant k$ inequalities	$\Longrightarrow$	*$\phi(Q)$ satisfies all valid pitch $\leqslant k+1$ inequalities*

Proof (inspired by Karchmer & Widgerson **'90**)

Assume $\sum_{i \in I^+} c_i x_i \geqslant \delta$ pitch-$(k+1)$ ineq **not valid** for $\phi(Q)$

Letting $a \in \{0,1\}^n$ with $a_i = 0 \iff i \in I^+$, have:

- $\phi(a) = 0$
- $\exists$ violator $\tilde{x} \in \phi(Q)$



If $\phi = \phi_1 \wedge \phi_2$ then

- $\phi_1(a) = 0$ **or** $\phi_2(a) = 0$
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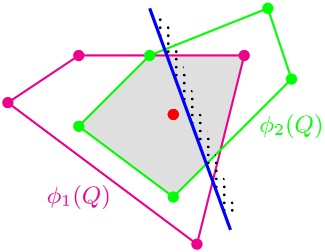

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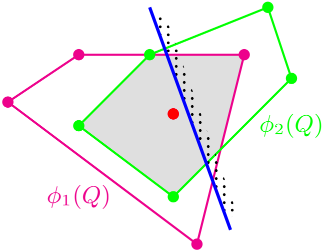

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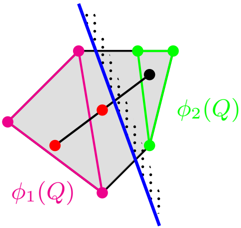

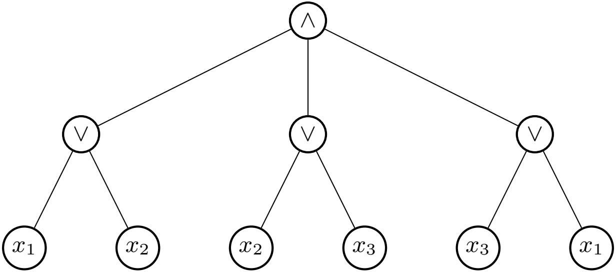
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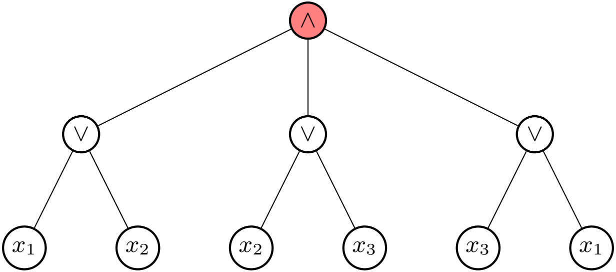
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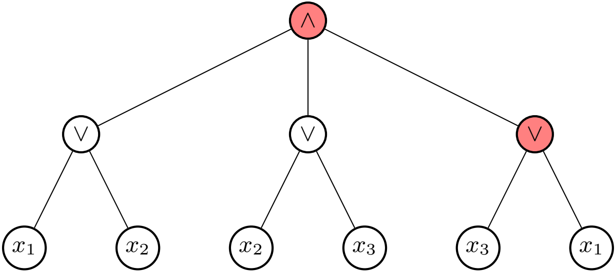

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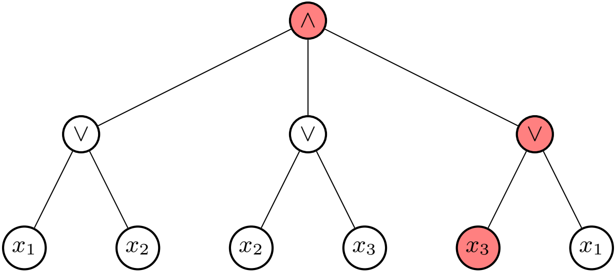
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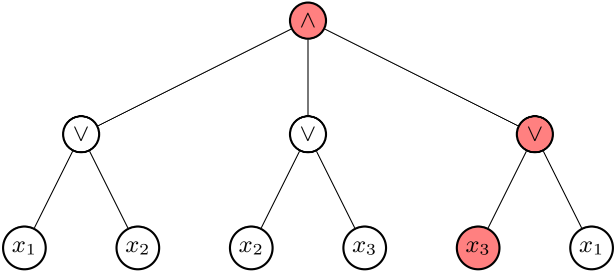

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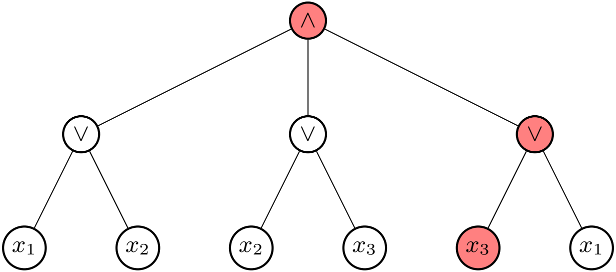

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contradicts hypothesis that Q satisfies pitch $\leqslant k$ ineq

$$\sum_{i \neq j} c_i x_i \geqslant \delta - c_j$$

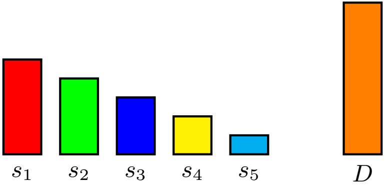
□


THE SIMPSONS™

Knapsack-cover inequalities

Given *sizes* $s_1, \dots, s_n \in \mathbb{Z}_+$ and *demand* $D \in \mathbb{Z}_+$:

$$f(x) = 1 \iff \sum_{i=1}^{n} s_i x_i \geqslant D$$



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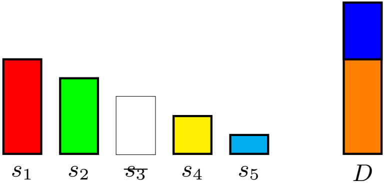

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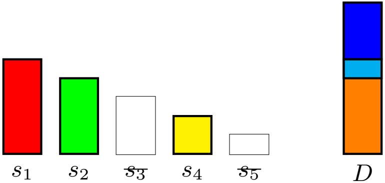

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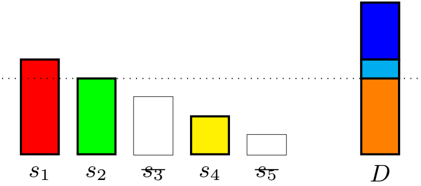

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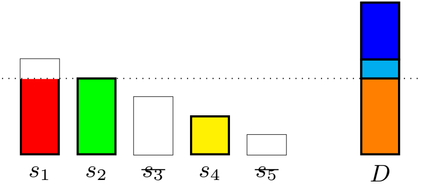

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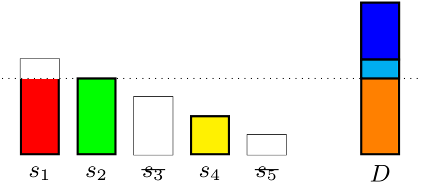

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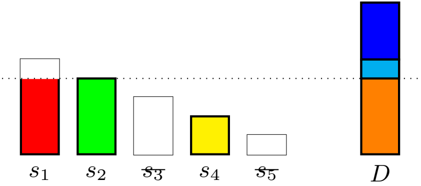

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Intuition: KC ineq is **pitch-**1 w.r.t. *large* items $\longleftarrow$ items i such that $s_i \geqslant D(a)$

The relaxation

1. **Sort** item sizes: $s_1 \geqslant s_2 \geqslant \cdots \geqslant s_n$

2. **Parametrize** the KC inequalities by:
   - $\alpha :=$ index of last **large** item
   - $\beta := \sum_{i \leqslant \alpha} s_i a_i$

3. **Construct** monotone formula $\phi_{\alpha,\beta}$ for threshold function

$$f_{\alpha,\beta}(x) = 1 \iff \sum_{i \leqslant \alpha} s_i x_i \geqslant \beta + 1$$

4. **Define** relaxation by the following formula:

$$\bigwedge_{\alpha,\beta} \left(\phi_{\alpha,\beta}(x) \quad \vee \quad \left(\sum_{i > \alpha} s_i x_i \geqslant D - \beta \right) \right)$$

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