The Non-Uniform k-Center Problem

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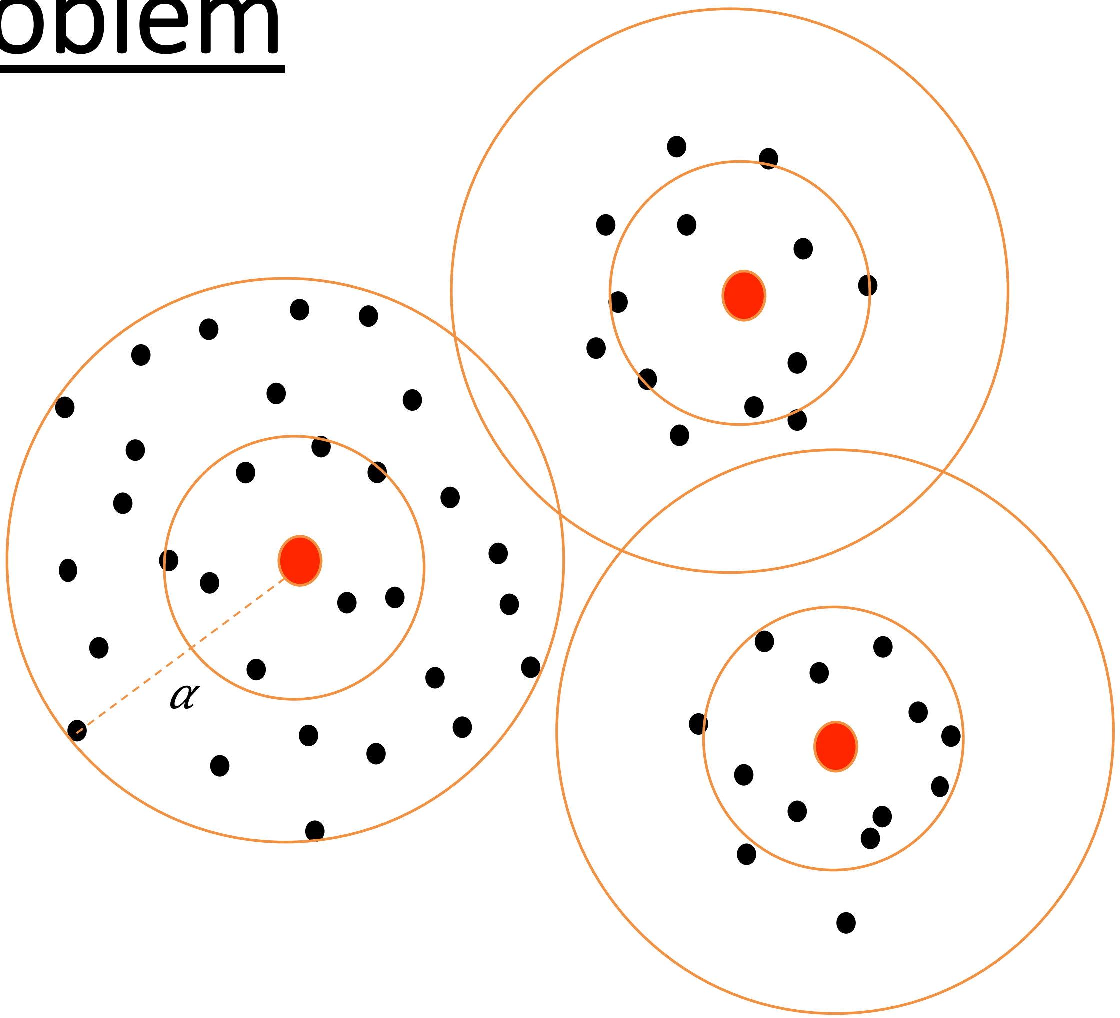
k-Center Problem

Input: n point in a metric space (X,d)

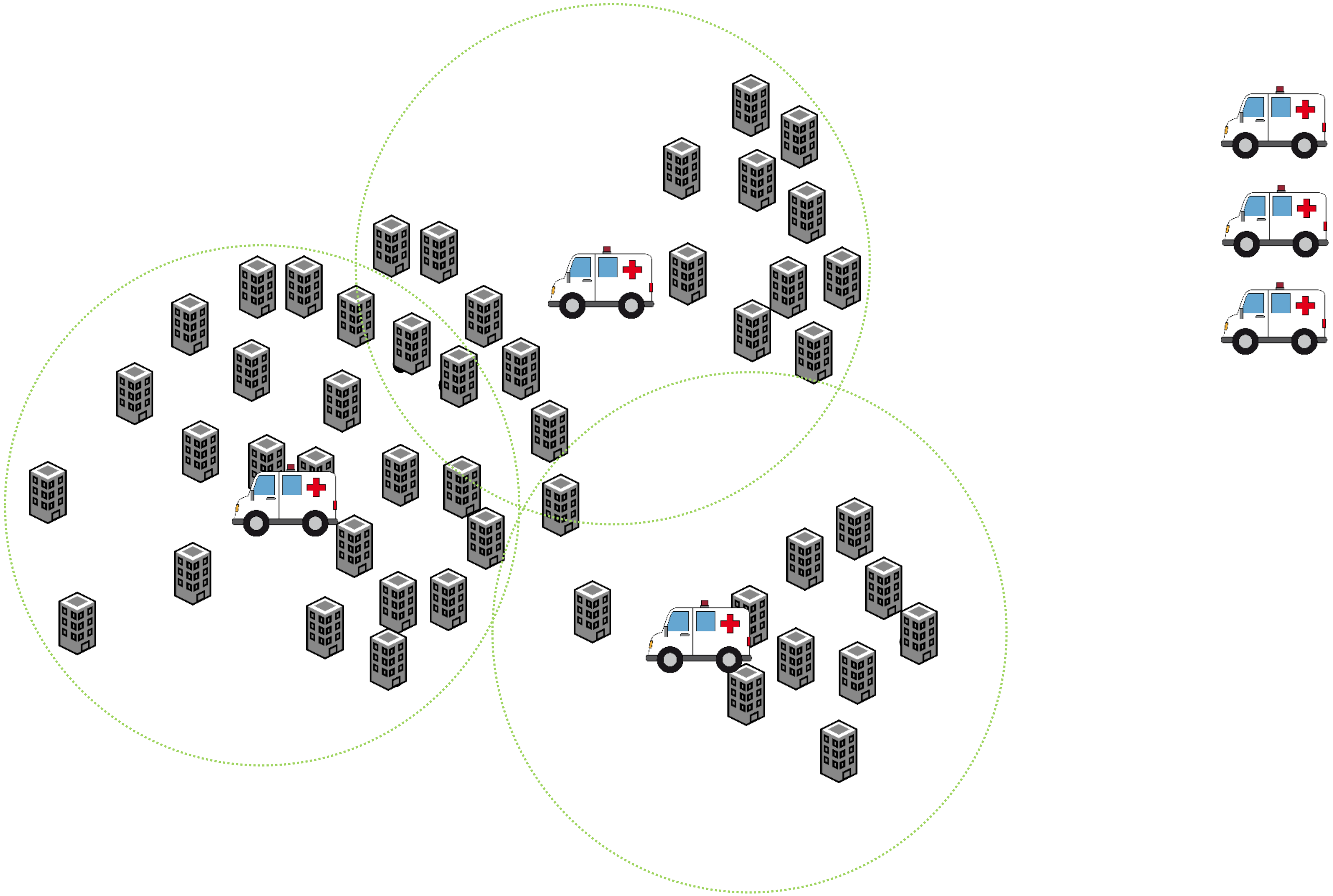
Integer k.

Output: Subset C, |C|=k to minimize $\max_{x\in X} \underbrace{\min_{c\in C} d(x,c)}_{d(x,C)}$



Find k centers of unit radius balls such that with **minimum dilation** α they cover the whole metric space.

Depot Location



The Optimum Location of Multi-centres on a Graph

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The location of a number of service centres on a network (graph) in such a way so that every node (demand point) lies within a critical distance of at least one of the centres appears often in problems involving emergency services. When the number p of centres is fixed and what is required is their location so that this critical distance is the smallest possible, the resulting location is called "the absolute p-centre of the graph". This paper presents an iterative algorithm for finding absolute p-centres in general (weighted or unweighted) graphs. The algorithm is shown to be computationally efficient for quite large graphs. Results (computing times and numbers of iterations) are given for 15 test graphs varying in size from 10 to 50 nodes and from 20 to 120 links.

INTRODUCTION

PROBLEMS involving the location of emergency-service centres on a graph (or network) are usually constrained by some "critical" time (or distance). The service centres must be located in such a way so that every demand point lies within this critical distance from at least one of the centres. Typical problems of this type arise in the location of ambulance, hospital, police or fire stations, the location of policemen on point duty and the location of mobile repair service units. In a rather different context the location of switching centres in a telephone, electricity or computer network also involve this basic problem.

The problem can generally be stated as follows:

(1) To find the optimal location of a given number (say p) of centres so that the time (distance) required to reach the most remote demand point from any one centre is a minimum.

Operational Research Quarterly (1970-1977)
Vol. 22, No. 2 (Jun., 1971), pp. 145-154

k-Center Problem

- Relation to Dominating Set Problem

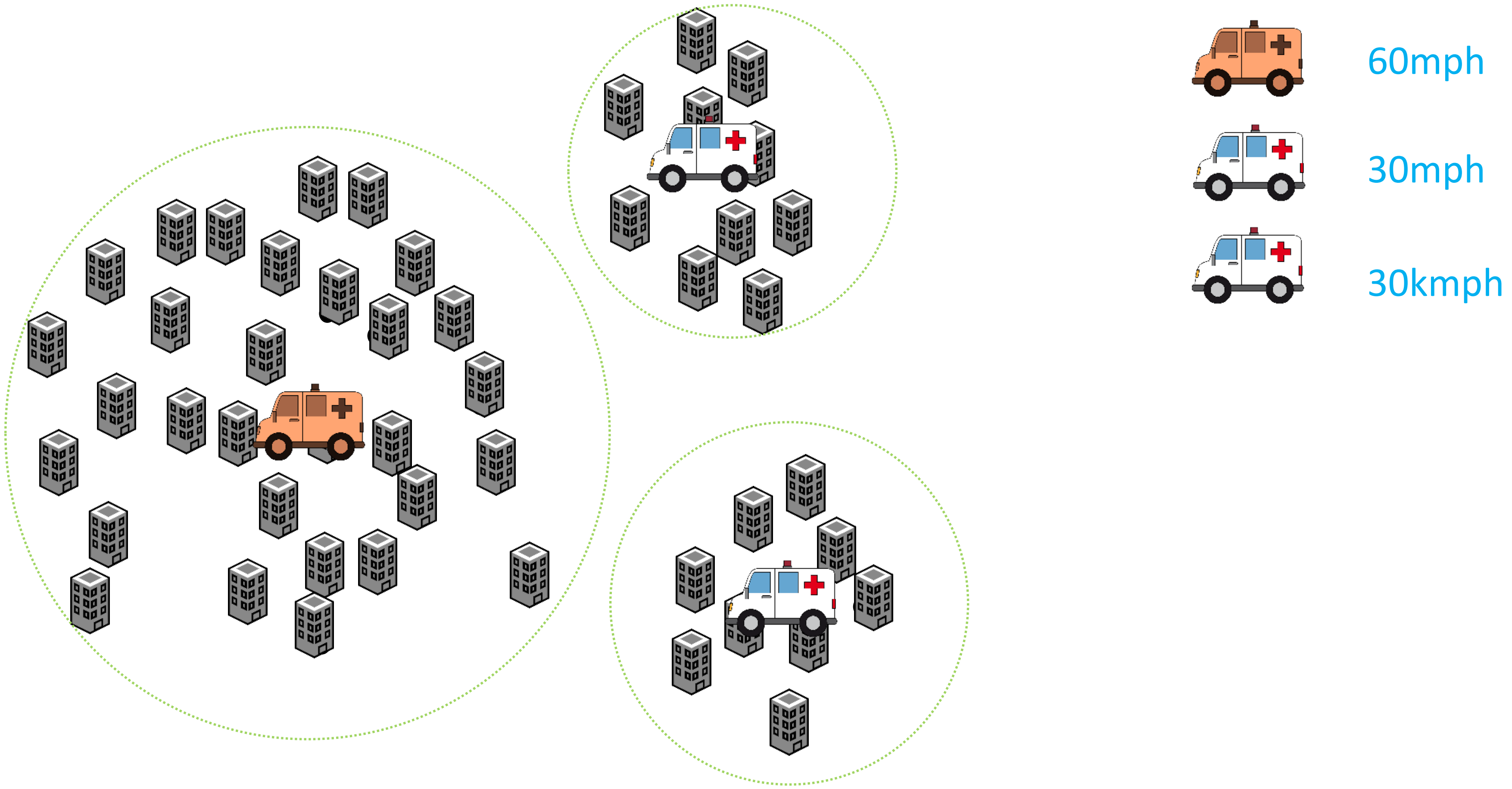
- NP-hard to get approximation factor < 2

Hsu, Nemhauser 1979

- Optimal 2-approximation algorithm

Hochbaum, Shmoys 1985

Depot Location with Non-Uniform Speeds



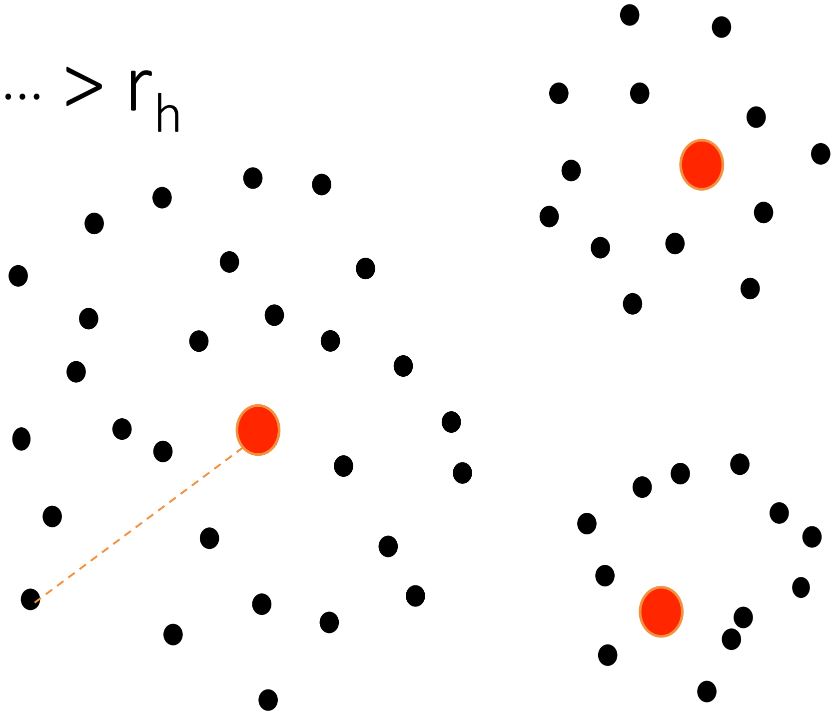
The Non-Uniform k-Center (NUkC) Problem

Input: n point in a metric space (X,d),

Tuples: $(k_1,r_1), \ldots , (k_h,r_h)$; $r_1 > r_2 > \ldots > r_h$

Output: Find locations of centers of these balls to minimize

$$\max_{x \in X} \ \min_{c \in C} d(x,c)/r_c$$



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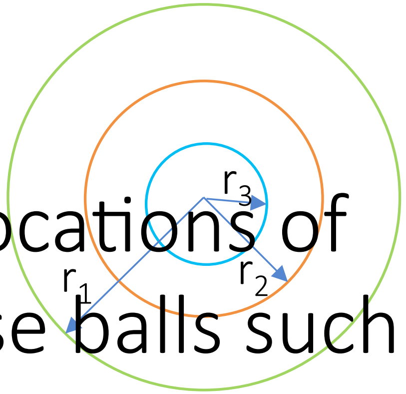
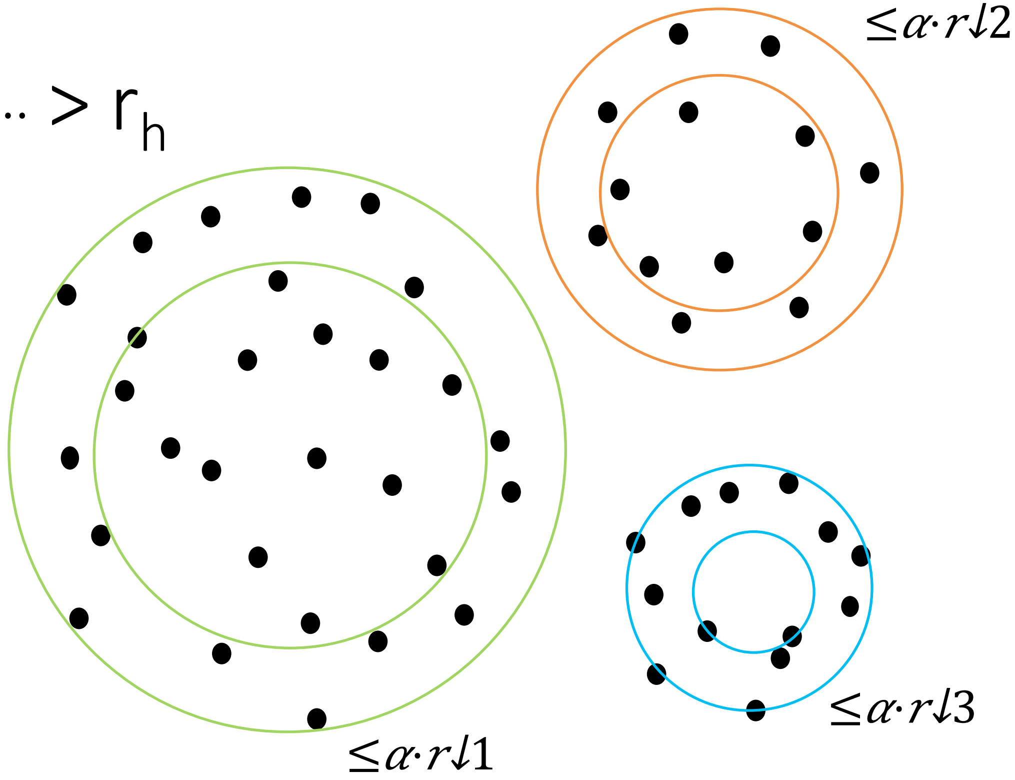




Output: Find locations of centers of these balls such that with **minimum dilation** α they cover the whole metric space.

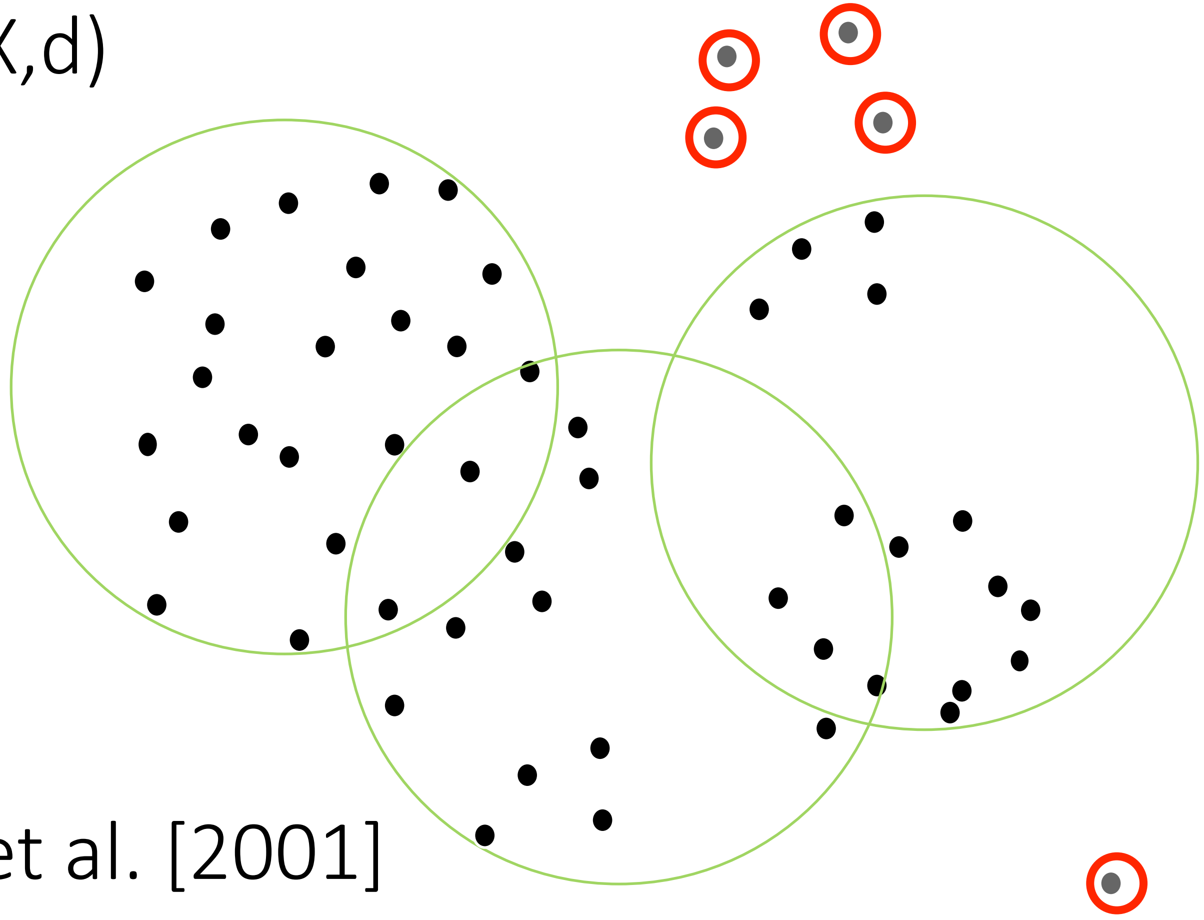
NUkC generalizes k-Center with Outliers

Input: n point in a metric space (X,d)
Integers k,z

Output: z **outliers**, and k-center solution on remaining points.

NUkC with ((k,1),(z,0))

3-approximation due to Charikar et al. [2001]



Results regarding NUkC

- O(1)-approximation with only 2-types of radii (h=2).
- 2-approximation for k-center with outliers.

- A “brittle” impossibility of uni-criteria algorithms

- (O(1),O(1))-bicriteria approximation.
  - cover space with $O(k_i)$ balls of radii $O(r_i)$.

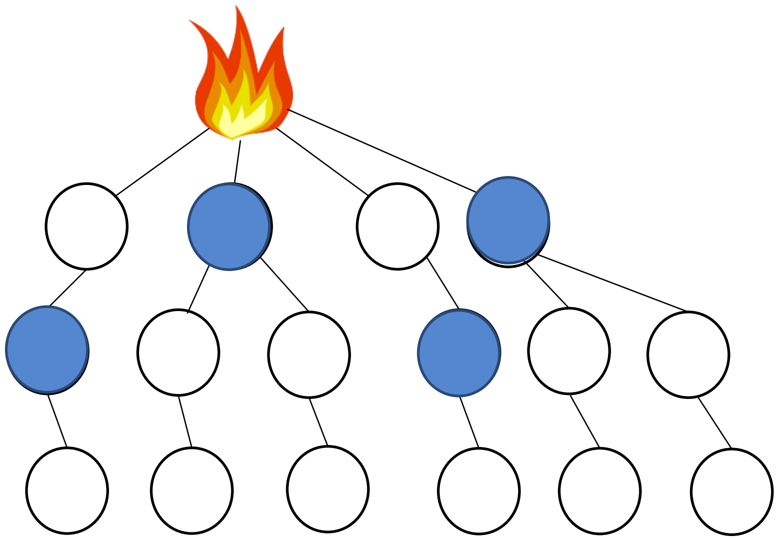
Connection to the Firefighter Problem

The Firefighter Problem

Input: Rooted (layered) Tree.

Feasible Solution: $N \subseteq V(T)$

- N hits all root leaf paths
- N contains $\leq$ B nodes from each layer

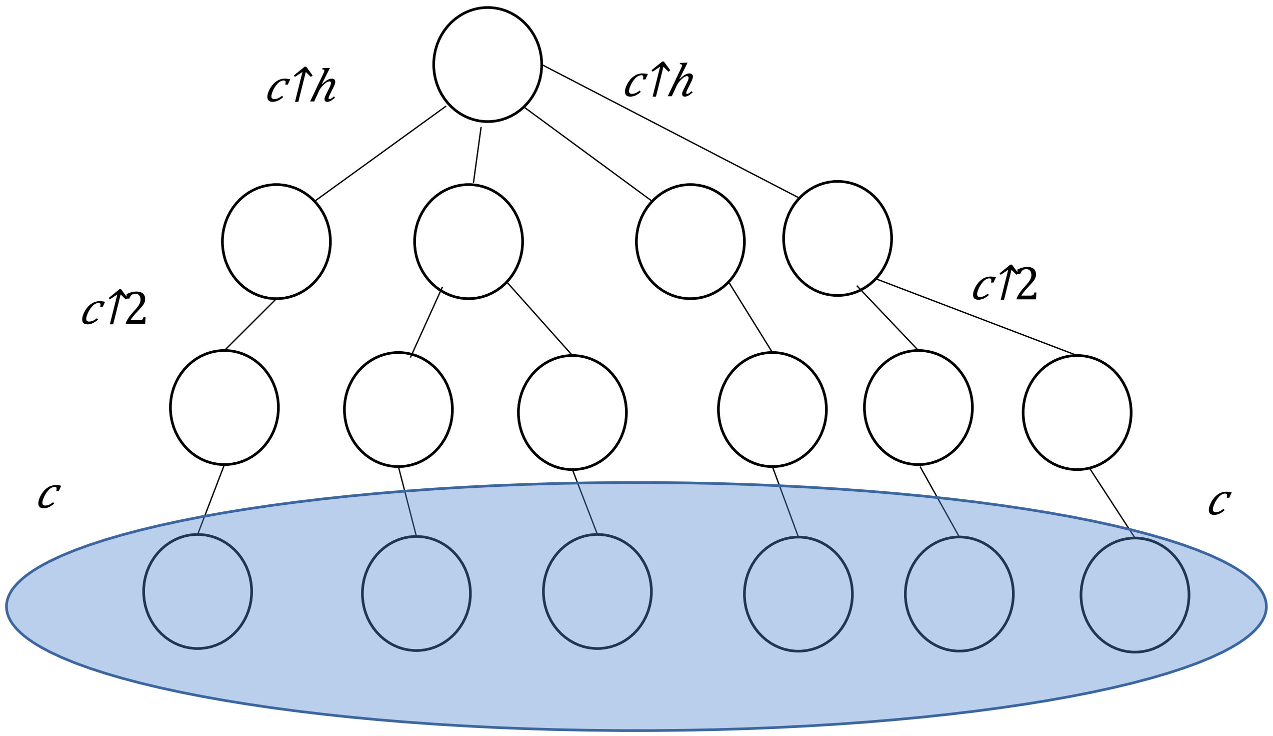


Goal: Feasible solution with minimum B.

NP-hard to decide if $B^* = 1$ or 2.
Finbow et al, 2007

Polytime solvable on trees of constant height via DP

No Unicriteria Approximation for NUkC



Hard Instance for Firefighter.

NUkC Instance:

X = Leaves of the Tree

d = HST style Metric

$(1,c^h), (1,c^{h-1}),\dots,(1,c)$

If FF-OPT = 1, NUkC-OPT < 2

If FF-OPT > 1, NUkC-OPT $\geq$ c

Firefighter to NUkC via LPs

“Natural” NUkC LP;
Feasible solution ***x***

→

Rooted Tree **T**;
Feasible solution ***y*** for
“natural” firefighter LP

LP-relaxation for NUkC

- $\forall q \in X,\ t \in [h]$
  - variable $x_{q,t} \geq 0$
  - extent to which we place a ball of radius r_t centered at q

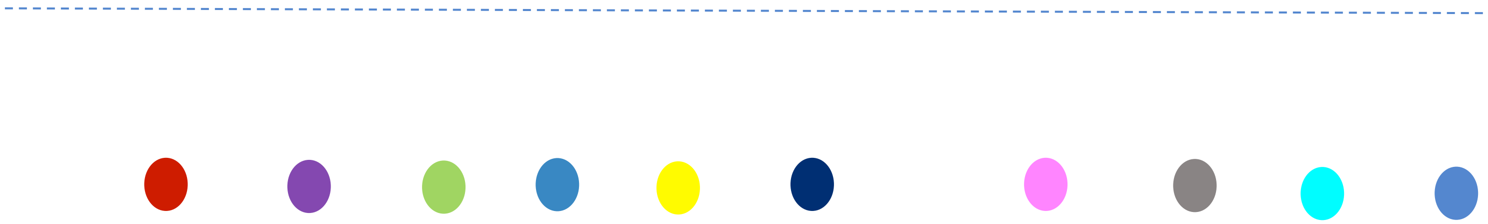
Assume optimal dilation to be 1

$$\forall p \in X, \sum_{t=1}^{h} \sum_{q \in B(p, r_t)} x_{q,t} \geq 1$$

$$\forall t \in [h],\ \sum_{q \in X} x_{q,t} \leq k_t$$

$r_1 \geq \cdots \geq r_h$

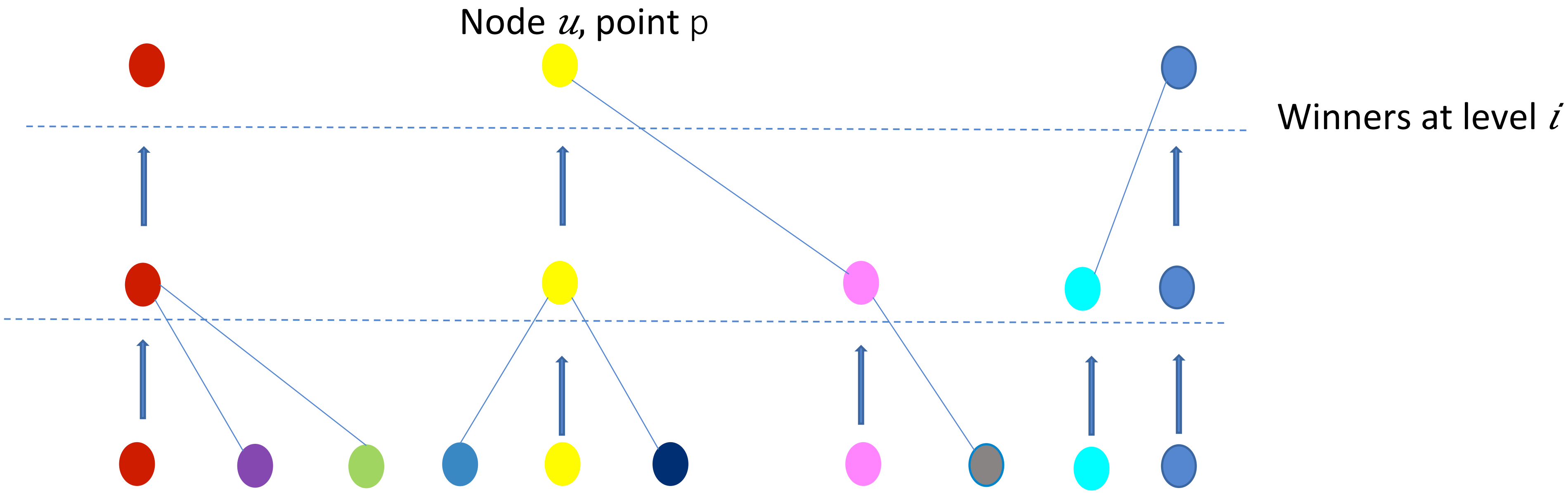
$y_w = 0$



Points in the metric

$$y_u = \sum_{q \in B(p, r_i)} x_{q,i}$$

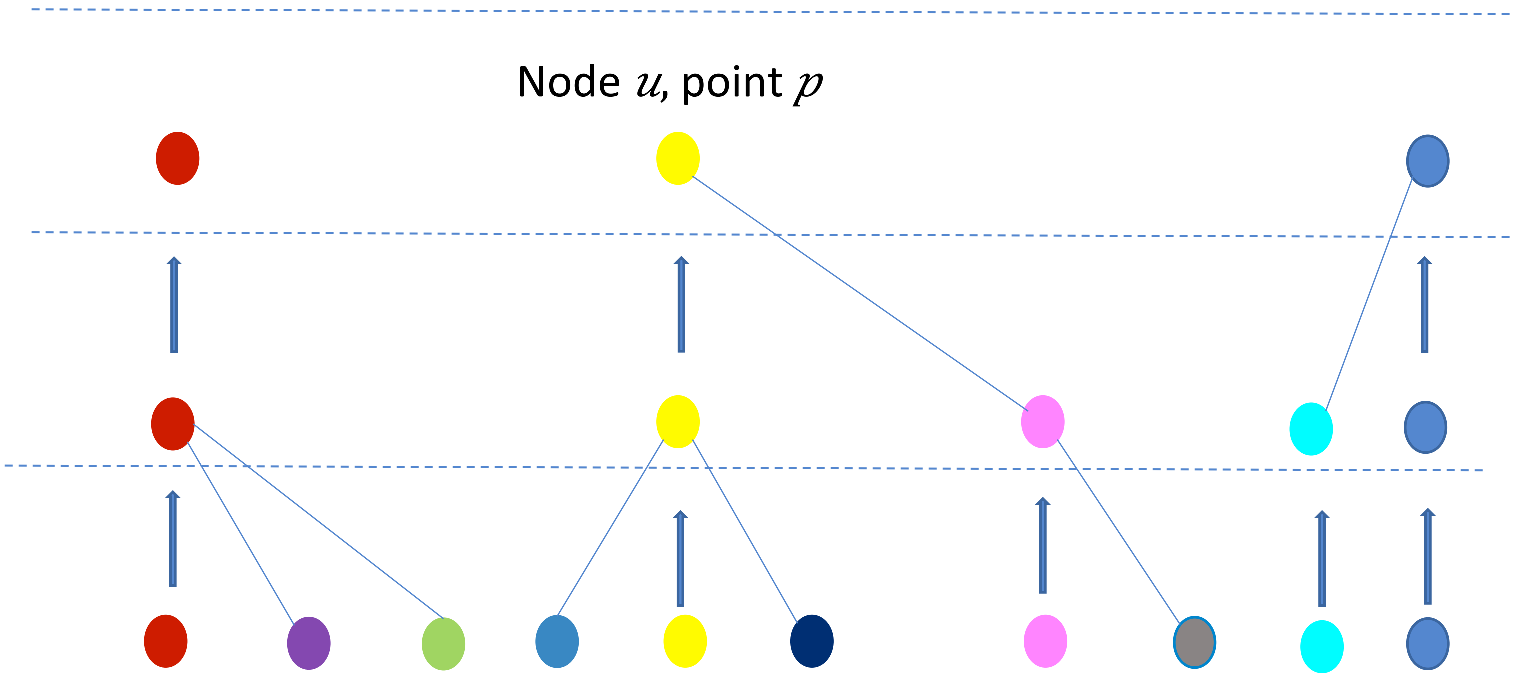
Coverage that p gets
at the current level



$$COV_{\geq i}(p)=\sum_{l=i}^{h}\sum_{q \in B(p,r_l)} x_{q,l}$$

Total coverage that p gets up to current level

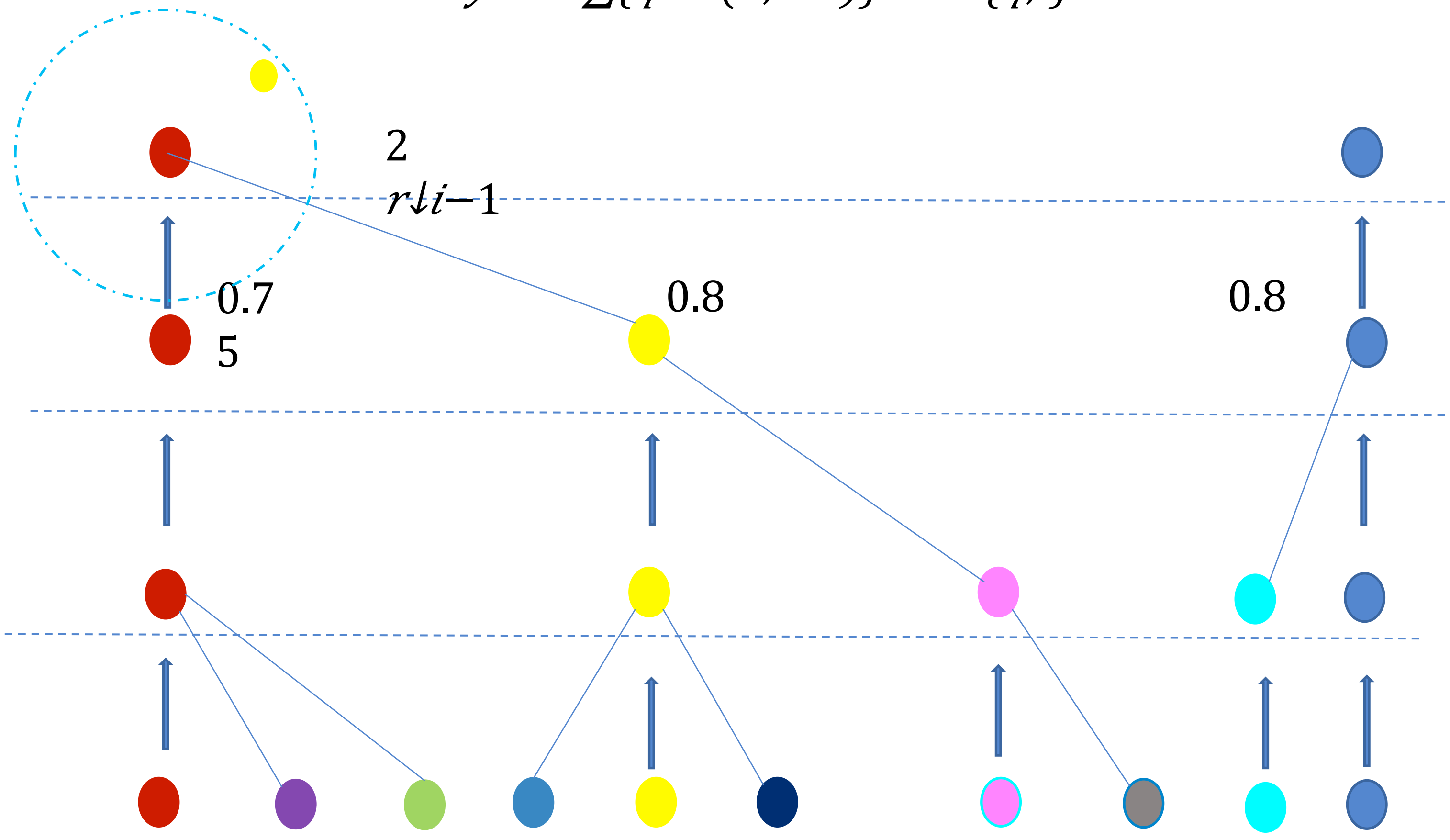
Propagate points with small total coverage from levels below



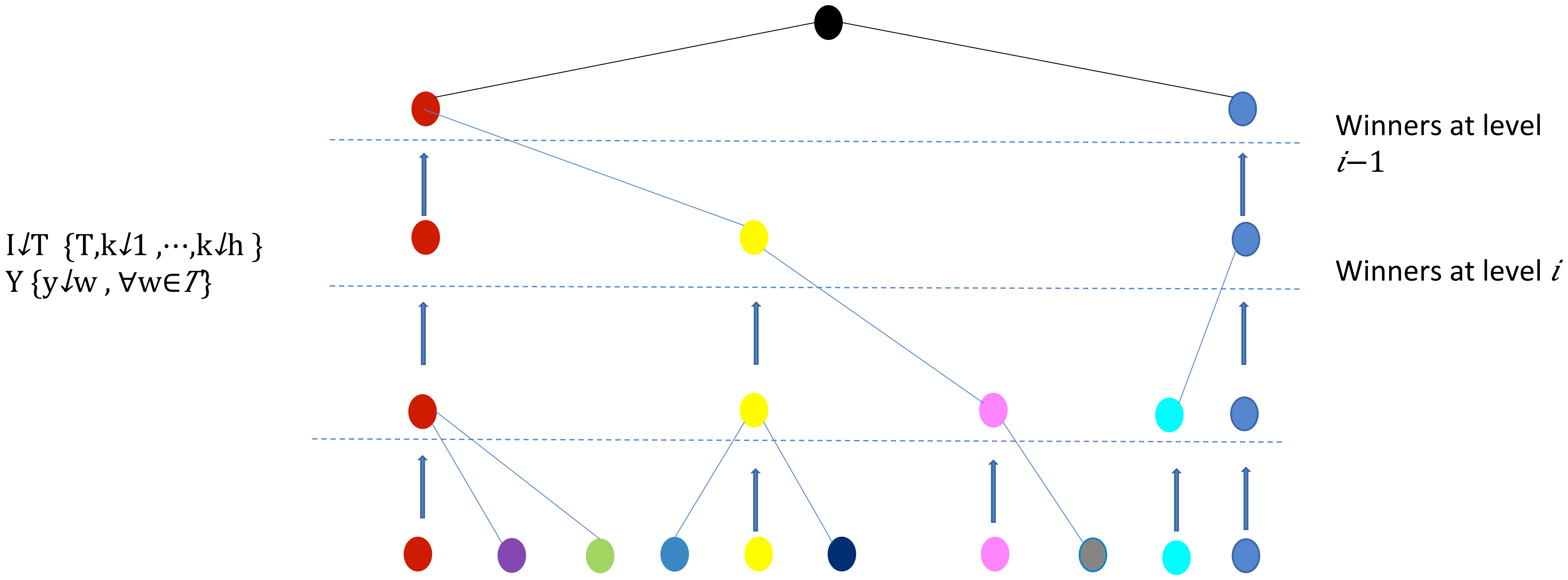
Winners at level i

$$y_u = \sum_{q \in B(u, r_i)} x_{q,i}$$

Propagate points with small total coverage from levels below



Winners at level i


$I_T\ \{T,k_1,\cdots,k_h\}$
$Y\ \{y_w, \forall w \in T\}$
Winners at level $i-1$
Winners at level i

Firefighter to NUkC

Lemma: If Firefighter-LP for height h trees has integrality gap ρ_h, then cover X by opening $\leq \rho_h k_t$ balls of radius $\leq 2\sum_{s \geq t} r_s$

Since $\rho_2 = 1$, this implies the results for the two types of radii.

Breaks down (!) even for three types of radii.

Immidiately implies $(O(\log^* n), 8)$-bicriteria approximation

Chalermsook, Chuzhoy [2010]

Lots more work on the lines of Adjiashvili, Baggio, Zenklusen'16 to get $(O(1), O(1))$-approximation.

Last Slide

- Non-homogeneous version of k-center
  - Non-trivial generalization (even to the outlier problem)
  - OPEN: Algorithm for constantly many types
  - Connection to Firefighter Problem

- Heterogeneous Resource Allocation
  - Heterogenous Capacitated k-center [C, Krishnaswamy, Kumar IPCO17]
  - OPEN: Algorithm for constantly many types
  - Connection to cardinality constrained scheduling