A constant-factor approximation algorithm for the asymmetric travelling salesman problem

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Travelling salesman problem

Given $n$ cities and their pairwise distances, find a shortest tour visiting all $n$ cities.

- One of the best known NP-hard optimization problems
- Studied since the 19$^{th}$ century
- Symmetric TSP: $d(i, j) = d(j, i) \ \forall i, j$
- Asymmetric TSP: $d(i, j) \neq d(j, i)$ is possible

Triangle inequality:

$d(i, j) \leq d(i, k) + d(k, j) \ \forall i, j, k$

UK pub tour
[Cook et al., 2015]
Symmetric vs Asymmetric TSP

Symmetric TSP

• 1.5-approximation algorithm [Christofides ’76]

• Graphic TSP: unweighted graph shortest path metric
  • Current best 1.4 [Sebő & Vygen ’14], following
    [Oveis Gharan, Saberi & Singh ’11]
    [Mömke & Svensson ’11]
    [Mucha ’12]
Symmetric vs Asymmetric TSP

Asymmetric TSP

- $\log_2 n$-approximation algorithm [Frieze, Galbiati & Maffioli ’82]
- $0.99 \log_2 n$ [Bläser ’03]
- $0.84 \log_2 n$ [Kaplan, Lewenstein, Shafrir & Sviridenko ’03]
- $0.67 \log_2 n$ [Feige & Singh ’07]

- $O \left( \frac{\log n}{\log \log \log n} \right)$ [Asadpour, Goemans, Mądry, Oveis Gharan & Saberi ’10]
  
  via thin trees.
Asymmetric TSP – recent developments

• $O(poly \log \log n)$ bound on integrality gap of LP
  [Anari & Oveis Gharan ’15]

Constant-factor approximations:
• Bounded genus graphs [Oveis Gharan & Saberi ’11]
• Node-weighted graphs [Svensson ’15]
• Graphs with 2 edge weights [Svensson, Tarnawski & V. ’16]

Our result: constant-factor approximation for general ATSP with respect to the Held-Karp relaxation.
ATSP – Graphic formulation

Input: directed graph $G = (V, E)$, edge weights $w: E \rightarrow \mathbb{R}_+$
Find a minimum weight tour $F$.

• Tour = closed walk visiting every vertex at least once = Eulerian and connected edge multiset
• Eulerian: $\delta_F^{in}(v) = \delta_F^{out}(v) \ \forall v \in V$
• Subtour = closed walk (not necessarily connected)

In-degree & out-degree in F
ATSP – Graphic formulation

Input: directed graph $G = (V, E)$, edge weights $w: E \to \mathbb{R}_+$
Find a minimum weight tour $F$.

- **Tour** = closed walk visiting every vertex at least once =
  = Eulerian and connected edge multiset
- **Eulerian**: $\delta_F^{in}(v) = \delta_F^{out}(v) \forall v \in V$
- **Subtour** = closed walk (not necessarily connected)

In-degree & out-degree in $F$
Held-Karp relaxation

- **Input:** $G = (V, E)$, edge weights $w: E \to \mathbb{R}_+$.
- **Variables** $x_e : E \to \mathbb{R}_+$: multiplicity of selecting edge $e$.

\[
\begin{align*}
\text{minimize} \quad & w^T x \\
\text{subject to} \quad & x(\delta^{in}(v)) = x(\delta^{out}(v)) \quad \forall v \in V \\
& x(\delta(S)) \geq 2 \quad \forall S \subseteq V, S \neq \emptyset \\
& x \geq 0
\end{align*}
\]

- **Eulerian degree constraints**
- **Subtour elimination constraints**

Undirected degree:
\[
\delta(S) = \delta^{in}(S) + \delta^{out}(S)
\]
Held-Karp relaxation

• Input: $G = (V, E)$, edge weights $w: E \to \mathbb{R}_+$. 
• Variables $x_e: E \to \mathbb{R}_+$: multiplicity of selecting edge $e$.

minimize $w^\top x$

subject to $x(\delta^{\text{in}}(v)) = x(\delta^{\text{out}}(v)) \quad \forall v \in V$

$x(\delta(S)) \geq 2 \quad \forall S \subseteq V, S \neq \emptyset$

$x \geq 0$

• Can be solved in polynomial time
• Integrality gap $\geq 2$ [Charikar, Goemans & Karloff '06]
Pick any two...
Pick any two...

- integral
- connecting Eulerian cycle
- spanning tree
- cycle cover
- ATSP
- Held-Karp
- connected
Repeated cycle cover algorithm
[Frieze, Galbiati & Maffioli ’82]

Relaxing connectivity:
1. Find minimum weight cycle cover
2. Contract and repeat

• Each cycle cover has cost $\leq OPT$
• Overall $\log_2 n$ rounds
• $\log_2 n$ approximation
Node-weighted case [Svensson’15]

Directed graph $G = (V, E)$, node weights $h: V \rightarrow \mathbb{R}_+$

$w(u, v) = h(u) + h(v) \quad \forall u, v \in E$

Local-Connectivity ATSP: relaxing connectivity constraints to “local”

\[ \alpha \]-light algorithm for Local-Connectivity ATSP

\[ (9 + \varepsilon)\alpha \]-approximation for ATSP

Theorem [Svensson’15]
There exists a polytime $(27 + \varepsilon)$-approximation for node-weighted ATSP.
Roadmap

General ATSP

Laminarily weighted ATSP

LP duality + uncrossing

Irreducible instances

Graph theory: contractions

Node weighted algorithm + contractions

Vertebrate pairs

O(1)-light lcATSP algorithm in vertebrate pairs

Local-connectivity ATSP

[SVENSSON ’15]
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[Svensson ’15]
Dual of the Held-Karp relaxation

minimize \( w^T x \)
subject to
\[
x(\delta^{in}(v)) = x(\delta^{out}(v)) \quad \forall v \in V
\]
\[
x(\delta(S)) \geq 2 \quad \forall \emptyset \neq S \subseteq V
\]
\[
x \geq 0
\]

maximize \( 2 \sum_{\emptyset \neq S \subseteq V} y_S \)
subject to
\[
\sum_{S: (u,v) \in \delta(S)} y_S + \alpha_u - \alpha_v \leq w(u,v) \quad \forall (u,v) \in E
\]
\[
y \geq 0
\]

• Dual can be solved in polynomial time.
• One can efficiently find an optimal \((\alpha, y)\) such that the support of \(y\) is a laminar family of sets.
  Efficient uncrossing [Karzanov’96]
Laminarly weighted ATSP: \( J = (G, \mathcal{L}, x, y) \)

\[
\begin{align*}
\text{minimize} & \quad w^T x \\
\text{subject to} & \quad x(\delta^{\text{in}}(v)) = x(\delta^{\text{out}}(v)) \quad \forall v \in V \\
& \quad x(\delta(S)) \geq 2 \quad \forall \emptyset \neq S \subsetneq V \\
& \quad x \geq 0
\end{align*}
\]

\[
\begin{align*}
\text{maximize} & \quad 2 \sum_{\emptyset \neq S \subsetneq V} y_S \\
\text{subject to} & \quad \sum_{S: (u,v) \in \delta(S)} y_S + \alpha_u - \alpha_v \leq w(u,v) \quad \forall (u, v) \in E \\
& \quad y \geq 0
\end{align*}
\]

- \( G \): directed graph
- \( \mathcal{L} \): laminar family of sets
- \( x \): feasible Held-Karp solution
  - tight on every set in \( \mathcal{L} \):
    \( x(\delta(S)) = 2 \ \forall S \in \mathcal{L} \)
- \( y \): \( \mathcal{L} \rightarrow \mathbb{R}_+ \)
Laminarly weighted ATSP: $I = (G, \mathcal{L}, x, y)$

minimize $w^T x$
subject to
\[
x(\delta^{in}(v)) = x(\delta^{out}(v)) \quad \forall v \in V
\]
\[
x(\delta(S)) \geq 2 \quad \forall \emptyset \neq S \subsetneq V
\]
\[
x \geq 0
\]

maximize $2 \sum_{\emptyset \neq S \subsetneq V} y_S$
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\]
\[
y \geq 0
\]

- $G$: directed graph
- $\mathcal{L}$: laminar family of sets
- $x$: feasible Held-Karp solution
  
  tight on every set in $\mathcal{L}$: $x(\delta(S)) = 2 \ \forall S \in \mathcal{L}$
- $y: \mathcal{L} \to \mathbb{R}_+$

Induced weight function: $w(u,v) = \sum_{S: (u,v) \in \delta(S)} y_S$
Laminarly weighted ATSP: \( J = (G, \mathcal{L}, x, y) \)

- \( G \): directed graph
- \( \mathcal{L} \): laminar family of sets
- \( x \): feasible Held-Karp solution tight on every set in \( \mathcal{L} \): \( x(\delta(S)) = 2 \ \forall S \in \mathcal{L} \)
- \( y \): \( \mathcal{L} \to \mathbb{R}_+ \)

**minimize** \( w^T x \)

**subject to**

\[
x(\delta^{in}(v)) = x(\delta^{out}(v)) \quad \forall v \in V \\
x(\delta(S)) \geq 2 \quad \forall \emptyset \neq S \subsetneq V \\
x \geq 0
\]

**maximize** \( 2 \sum_{\emptyset \neq S \subsetneq V} y_S \)

**subject to**

\[
\sum_{S: (u,v) \in \delta(S)} y_S + \alpha_u - \alpha_v \leq w(u, v) \quad \forall (u, v) \in E \\
y \geq 0
\]

Induced weight function: \( w(u, v) = \sum_{S: (u,v) \in \delta(S)} y_S \)
Reduction to laminarily weighted ATSP

- Start with any $G$ and $w$.
- Compute Held-Karp optimal solution $x$ and dual $y$ supported on laminar family $\mathcal{L}$
- Delete all edges with $x_e = 0$.

\[
\begin{align*}
\text{maximize} & \quad 2 \sum_{\emptyset \neq S \subseteq V} y_S \\
\text{subject to} & \quad \sum_{S:(u,v) \in \delta(S)} y_S + \alpha_u - \alpha_v \leq w(u,v) \quad \forall (u,v) \in E \\
& \quad y \geq 0
\end{align*}
\]

**Observations:**
- Optimal solutions and optimum value are the same for $w$ and for $w'(u,v) = w(u,v) + \alpha_v - \alpha_u$
- All remaining edges have $w'(u,v) = \sum_{S:(u,v) \in \delta(S)} y_S$
Roadmap

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Graph theory: contractions

Node weighted algorithm + contractions

Vertebrate pairs

O(1)-light lcATSP algorithm in vertebrate pairs

Local-connectivity ATSP

[ Svensson ’15 ]
Vertebrate pairs

Vertebrate pair \((J, B)\)

- \(J = (G, \mathcal{L}, x, y)\) instance
- \(B\): backbone = subtour that crosses every nonsingleton set in \(\mathcal{L}\)
Vertebrate pairs

• We will reduce general ATSP to solving ATSP for a vertebrate pair \((J, B)\) with \(w(B) = \Theta(OPT)\) (more or less...)

• Solve Local-Connectivity ATSP on such instances, and apply [Svensson’15]
Local-Connectivity ATSP [Svensson’15]

Instance $I = (G, \mathcal{L}, x, y)$ with induced weights $w: E \to \mathbb{R}_+$

Lower bound function $lb: V \to \mathbb{R}_+$ with $\sum_{v \in V} lb(v) = OPT$

Input: partition of the vertex set $V = V_1 \cup V_2 \cup \cdots \cup V_k$
Local-Connectivity ATSP [Svensson’15]

Instance $\mathcal{I} = (G, \mathcal{L}, x, y)$ with induced weights $w: E \rightarrow \mathbb{R}_+$

Lower bound function $\text{lb}: V \rightarrow \mathbb{R}_+$ with $\sum_{v \in V} \text{lb}(v) = OPT$

Input: partition of the vertex set $V = V_1 \cup V_2 \cup \cdots \cup V_k$

Output: Eulerian edge set $F$ with $|\delta(V_i) \cap F| > 0$ for each $V_i$
Local-Connectivity ATSP [Svensson’15]

Instance $I = (G, \mathcal{L}, x, y)$ with induced weights $w: E \to \mathbb{R}_+$

Lower bound function $lb: V \to \mathbb{R}_+$ with $\sum_{v \in V} lb(v) = OPT$

Input: partition of the vertex set $V = V_1 \cup V_2 \cup \ldots \cup V_k$

Output: Eulerian $F$ with $|\delta(V_i) \cap F| > 0$ for each $V_i$

$\alpha$-light algorithm: for every component $C$ of $F$,

$$\frac{w(E(C))}{lb(V(C))} \leq \alpha$$

“Every component pays for itself locally”
Local-Connectivity ATSP [Svensson’15]

Theorem [Svensson’15]
There exists a polytime \((27 + \varepsilon)\)-approximation for node-weighted ATSP.

\(\alpha\)-light algorithm for Local-Connectivity ATSP

\((9 + \varepsilon)\alpha\)-approximation for ATSP
Local-Connectivity ATSP: node-weighted case

• Instance $\mathcal{I} = (G, \mathcal{L}, x, y)$, with $\mathcal{L}$ containing only singletons (ignore $B$)
  \[ w(u, v) = y_u + y_v \]

• Define $lb(u) = 2y_u \quad \forall u \in V$

• Partition $V = V_1 \cup V_2 \cup \cdots \cup V_k$ all strongly connected

• Modify $G$ and $x$, and solve an integer circulation problem
Local-Connectivity ATSP: node-weighted case

• Instance $I = (G, \mathcal{L}, x, y)$, with $\mathcal{L}$ containing only singletons (ignore $B$)
  
  \[ w(u, v) = y_u + y_v \]

• Define $lb(u) = 2y_u$ $\forall u \in V$

• Partition $V = V_1 \cup V_2 \cup \cdots \cup V_k$ all strongly connected

• Modify $G$ and $x$, and solve an integer circulation problem

  • For each $V_i$, create auxiliary vertex $a_i$
  
  • Reroute 1 fractional unit of incoming and outgoing flow $x$ to $a_i$
  
  • Solve integer circulation problem routing =1 unit through each $a_i$
  
  • Map back to original $G$
Local-Connectivity ATSP: node-weighted case

• The rerouted $x$ is feasible to the circulation problem of weight $OPT$
• Flow integrality: there exists integer solution of weight $\leq OPT$
• After mapping back, every vertex with $y_v > 0$ has in-degree $\leq 2$
• For a component $C$, $w(E(C)) = \sum_{(u,v)\in E(C)} y_u + y_v \leq 4 \sum_{v\in C} y_v$
• $lb(V(C)) = 2 \sum_{v\in C} y_v \implies 2$-light algorithm
Local-Connectivity ATSP: one nonsingular set in $\mathcal{L}$

- Vertebrate pair $(J, B)$. Assume $\mathcal{L}$ has a single non-singleton component $S$. Thus,

$$w(u, v) = \begin{cases} y_u + y_v + y_s & \text{if } (u, v) \in \delta(S) \\ y_u + y_v & \text{if } (u, v) \notin \delta(S) \end{cases}$$

- Define

$$lb(u) = \begin{cases} 2y_u & \text{if } u \in V \setminus V(B) \\ \frac{w(B)}{|V(B)|} & \text{if } u \in V(B) \end{cases}$$

- $\sum_{v \in V} lb(v) = O(OPT)$, since $w(B) = \Theta(OPT)$
Local-Connectivity ATSP: one nonsingular set in $\mathcal{L}$

- By assumption, $x(\delta^{in}(S)) = x(\delta^{out}(S)) = 1$
- Backbone property: there is a node $s \in V(B) \cap S$
- Simple flow argument: we can route the incoming 1 unit of flow to $S$ to $s$
Local-Connectivity ATSP: one nonsingular set in $\mathcal{L}$

- Partition $V = V_1 \cup V_2 \cup \cdots \cup V_k$
- Add backbone $B$ into Eulerian set $F$.
- Via flow splitting, "force" all edges entering $S$ to proceed to $s \in V(B)$
- Create auxiliary vertices $a_i$ as before
- Solve integral circulation problem, and add solution to $F$. 
Local-Connectivity ATSP: one nonsingular set in $\mathcal{L}$

Analysis

- For all components $C$ not crossing $S$, $w(E(C))/\text{lb}(V(C)) \leq 2$ exactly as in the node-weighted case
- Giant component $C_0$ containing $B$.
  - Contains all edges crossing $S$
  - Has lower bound $\text{lb}(V(C_0)) \geq \text{lb}(V(B)) = \Theta(\text{OPT})$
  - $w(E(C_0)) \leq w(F) \leq O(\text{OPT})$
- Therefore solution is $O(1)$-light.
- Same approach extends to arbitrary $\mathcal{L}$: enforce that every subtour crossing a set in $\mathcal{L}$ must intersect the backbone.
Roadmap

- General ATSP
- Laminarly weighted ATSP
- Irreducible instances
  - LP duality + uncrossing
  - Graph theory: contractions
- Node weighted algorithm + contractions
- Vertebrate pairs
- \( O(1) \)-light lcATSP algorithm in vertebrate pairs
- Local-connectivity ATSP
- [Svensson’15]
Motivation: reducing by contraction

- All sets in the family $\mathcal{L}$ are singletons: node-weighted ATSP
- Would like to reduce the problem by contracting nonsingleton sets in $\mathcal{L}$
Motivation: reducing by contraction

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Motivation: reducing by contraction

- All sets in the family $\mathcal{L}$ are singletons: node-weighted ATSP
- Would like to reduce the problem by contracting nonsingleton sets in $\mathcal{L}$
Irreducible instances

- **Irreducible set** $S \in \mathcal{L}$: There exists $u, v \in S$ such that the shortest path between $u$ and $v$ inside $S$ visits "almost all" sets $X \subseteq S, X \in \mathcal{L}$

- **Irreducible instance** $I = (G, \mathcal{L}, x, y)$: all sets in $\mathcal{L}$ are irreducible
Irreducible instances

- Irreducible set $S \in \mathcal{L}$: There exists $u, v \in S$ such that the shortest path between $u$ and $v$ inside $S$ visits “almost all” sets $X \subseteq S, X \in \mathcal{L}$

- Irreducible instance $I = (G, \mathcal{L}, x, y)$: all sets in $\mathcal{L}$ are irreducible
Irreducible instances

• Reducible set $S \in \mathcal{L}$: For every pair $u, v \in S$, there is a “cheap” path connecting them (if they are connected).
• Reducible sets can be contracted.

**Theorem:**
polytime $\rho$-approximation for irreducible instances

$\Rightarrow$

polytime $8\rho$-approximation for arbitrary instances
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[Svensson ’15]
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Vertebrate pair \((J, B)\)

- \(J = (G, \mathcal{L}, x, y)\) instance
- \(B\): backbone = subtour that crosses every nonsingleton set in \(\mathcal{L}\)
Finding a vertebrate pair in an irreducible instance \( J = (G, \mathcal{L}, x, y) \)

1. Obtain a node-weighted instance by contracting all maximal sets in \( \mathcal{L} \)
2. Use [Svensson ‘15] to find a tour here, and blow it back to a subtour \( B \) in the original instance \( J \) in a pessimistic way:
   inside each maximal \( S \in \mathcal{L} \), \( B \) crosses \( \geq 0.75 \text{value}(S) \)
3. If it crosses every set in \( \mathcal{L} \), then \( (J, B) \) is a vertebrate pair
4. Otherwise, recurse by contracting all maximal sets in \( \mathcal{L} \) not crossed by \( B \).
   This works because their total weight is \( \leq 0.25 \text{value}(J) \)
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[Svensson ’15]
Summary

• Via all these reductions, we obtain an \( 5500 \)-approximation algorithm for ATSP.
• Squeezing the arguments a bit more and opening up black boxes, can be probably decreased to a few hundreds.
• Still very far from lower bound 2 on the integrality gap of Held-Karp

Open questions

• Improve to a constant \(< 100 \)
• Thin tree conjecture is still open.
• Bottleneck ATSP.
• Better than 3/2 approximation for symmetric TSP.
SCALEOPT
Scaling Methods for Discrete and Continuous Optimization

- ERC Starting Grant 2018-22
- Openings for post docs and PhD students

http://personal.lse.ac.uk/veghl/scaleopt.html

Thank you!
Simplifying assumption *for the talk*

**Assumption:** all sets in the family $\mathcal{L}$ are strongly connected in $G$.

Not true in general, but the connected components have a nice path structure:
Paths traversing a set

• How much is the weight of connecting an incoming and an outgoing edge in a set $S \in \mathcal{L}$?

$$D_S(u, v) =$$
Paths traversing a set

- How much is the weight of connecting an incoming and an outgoing edge in a set $S \in \mathcal{L}$?

$$D_S(u, v) = \sum_{R: u \in R, R \subseteq S} y_R$$
Paths traversing a set

- How much is the weight of connecting an incoming and an outgoing edge in a set $S \in \mathcal{L}$?

$$D_S(u, v) = \sum_{R: u \in R, R \subset S} y_R + d_S(u, v)$$

Min weight path inside $S$. 
Paths traversing a set

• How much is the weight of connecting an incoming and an outgoing edge in a set $S \in \mathcal{L}$?

$$D_S(u, v) = \sum_{R:u \in R, R \subsetneq S} y_R + d_S(u, v) + \sum_{R:v \in R, R \subsetneq S} y_R = 18$$
Paths traversing a set

- How much is the weight of connecting an incoming and an outgoing edge in a set $S \in \mathcal{L}$?

$$D_S(u, v) = \sum_{R: u \in R, R \not\subseteq S} y_R + d_S(u, v) + \sum_{R: v \in R, R \not\subseteq S} y_R = 18 \leq 38$$

Lemma:

$$D_S(u, v) \leq 2 \sum_{R \not\subseteq S} y_R = \text{value}(S)$$
Irreducible instances

• Reducible set $S \in \mathcal{L}$:
  $$\text{Max}_{u,v \in S} \ D_S(u, v) \leq \frac{3}{4} \text{value}(S)$$

• Irreducible instance $I = (G, \mathcal{L}, x, y)$:
  no set $S \in \mathcal{L}$ is reducible

Lemma:
$$D_S(u, v) \leq 2 \sum_{R \subseteq S} y_R = \text{value}(S)$$

Theorem:
polytime $\rho$-approximation for irreducible instances $\Rightarrow$
polytime $8\rho$-approximation for arbitrary instances
Recursive algorithm via contractions

- Instance $I = (G, \mathcal{L}, x, y)$
- $\text{value}(I) = 2 \sum_{R \subseteq V} y_R = \text{Held-Karp optimum}$
- $S$: minimal reducible set in $\mathcal{L}$.

$8\rho$-approximation for $I = 8\rho$-approximation on instance by contracting $S$
$\rho$-approximation of irreducible instance “inside” $S$
Recursive algorithm via contractions

- Instance $I = (G, L, x, y)$
- $value(I) = 2 \sum_{R \subseteq V} y_R$
  $= $ Held-Karp optimum
- $S$: minimal reducible set in $L$.
- $I' = I/S$: contract $S$ in $I$.
- $S \rightarrow s$
- $y_s = y_S + \frac{3}{8} value(S)$
- $value(I') = value(I) - \frac{1}{4} value(S)$

$\text{value}(I) = 64$

$\text{value}(I') = 58$

$11 = 2 + \frac{3}{8} \cdot 24$
Recursive algorithm via contractions

**Inductive assumption:** We have a polytime $8\rho$-approximation for smaller instances

- Apply recursively on $J'$ to obtain tour $T'$
  
  \[ w(T') \leq 8\rho \cdot \text{value}(J') \]
  \[ = 8\rho \left( \text{value}(J) - \frac{1}{4} \text{value}(S) \right) \]

\[ \text{value}(J) = 64 \]

\[ \text{value}(J') = 58 \]

\[ 11 = 2 + \frac{3}{8} \cdot 24 \]

\[ \leq 464\rho \]
Contracting $S$

**Inductive assumption:** We have a polytime $8\rho$-approximation for smaller instances.

- Apply recursively on $I'$ to obtain tour $T'$
  
  
  
  $w(T') \leq 8value(I')$

  $= 8\rho(value(I) - \frac{1}{4}value(S))$

- Map back to subtour $T$ in $I$ with $w(T) \leq w(T')$

\[
value(I) = 64 \\
value(I') = 58 \\
11 = 2 + \frac{3}{8} \cdot 24 \\
\leq 464\rho
\]
Inducing on $S$

- We add a tour $F_S$ inside $S$, using the $\rho$-approximation on irreducible instances.
- $I''$: remove $S$, and contract $V \setminus S$ to $\bar{s}$, with $y_{\{\bar{s}\}} = \text{value}(S)/2$
- $I''$ is irreducible.
Inducing on $S$

- We add a tour $F_S$ inside $S$, using the $\rho$-approximation on irreducible instances.
- $J''$: remove $S$, and contract $V \setminus S$ to $\bar{S}$, with
  \[
y_{\{\bar{S}\}} = \frac{\text{value}(S)}{2} \]
- $J''$ is irreducible.
- Find tour $F''$ in $J''$ with weight
  \[
  w(F'') \leq \rho \text{value}(J'') = 2\rho \text{value}(S) \]

\[\text{value}(J) = 32\]
Inducing on $S$

- Find tour $F''$ in $J''$ with weight $w(F'') \leq \rho \text{value}(J'') = 2\rho \text{value}(S)$
- Map back $F''$ to $F_S$ in $J$ with $w(F_S) \leq w(F) \leq 2\rho \text{value}(S)$
- $T \cup F_S$ is a tour in $J$

$$w(T \cup F_S) \leq 8\rho \left( \text{value}(J) - \frac{1}{4} \text{value}(S) \right) + 2\rho \text{value}(S) = 8\rho \text{value}(J)$$

$\text{value}(J) = 32$