Order Detection under Pairwise Measurements

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Joint work with
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Yihong Wu (Yale)

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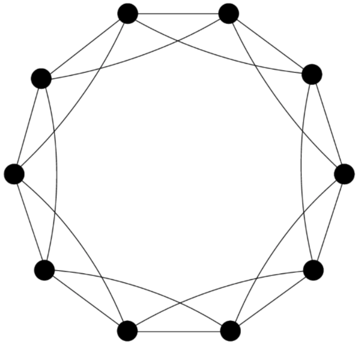
Order detection in small-world networks [Cai-Liang-Rakhlin '16]



4-circulant graph

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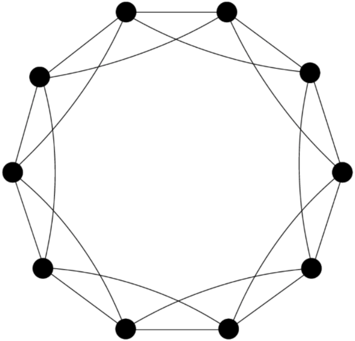

4-circulant graph

- Edge becomes non-edge with probability $1 - p$
- Non-edge becomes edge with probability q

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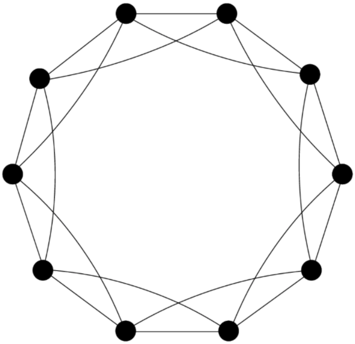

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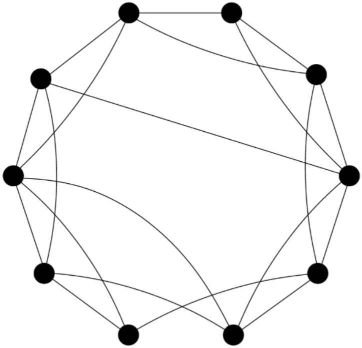

small-world graph

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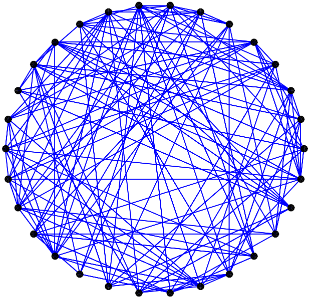
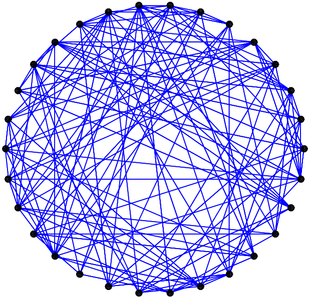
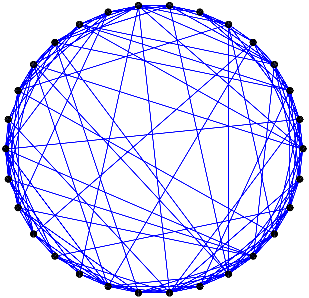
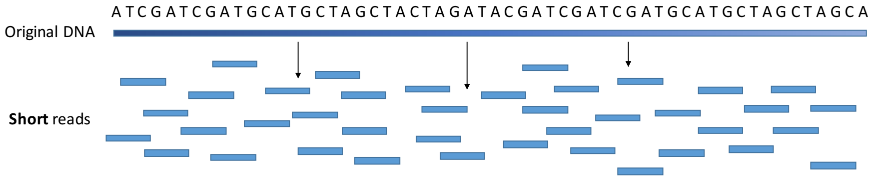






Goal: recover the underlying vertex ordering from observed graph

Ordering DNA scaffolds with Chicago reads



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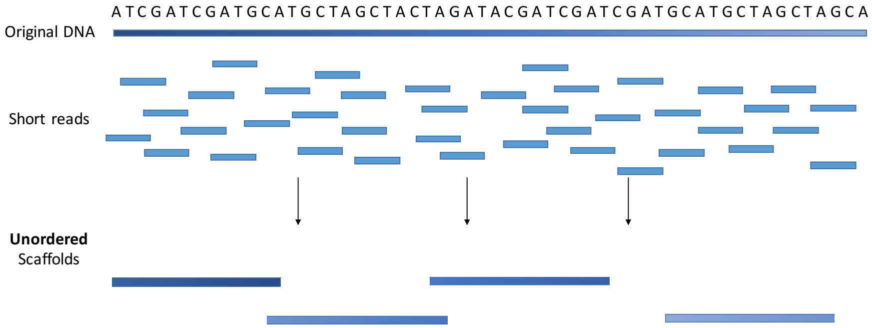

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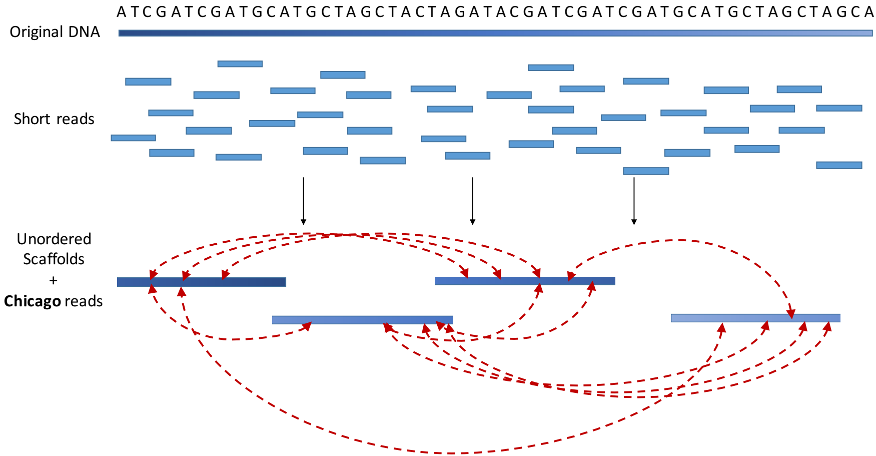

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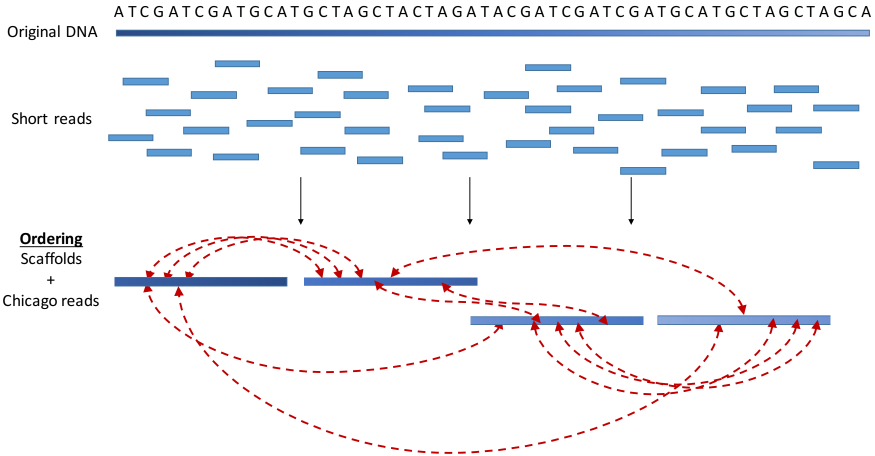

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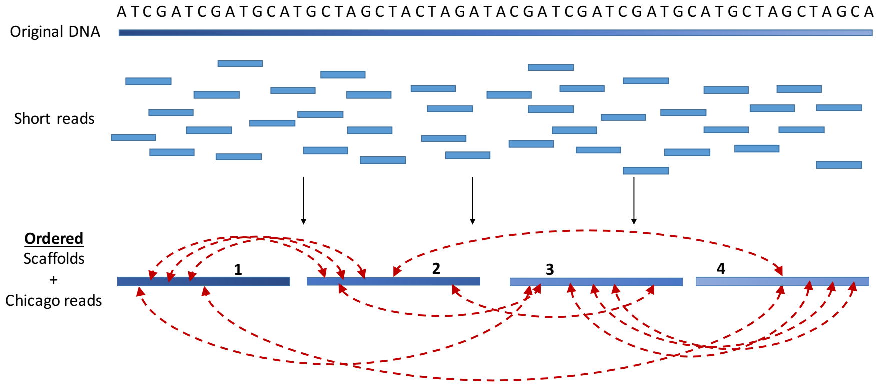

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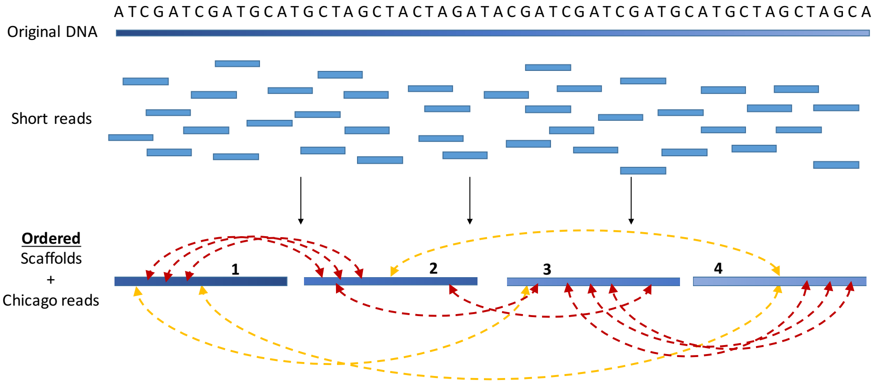

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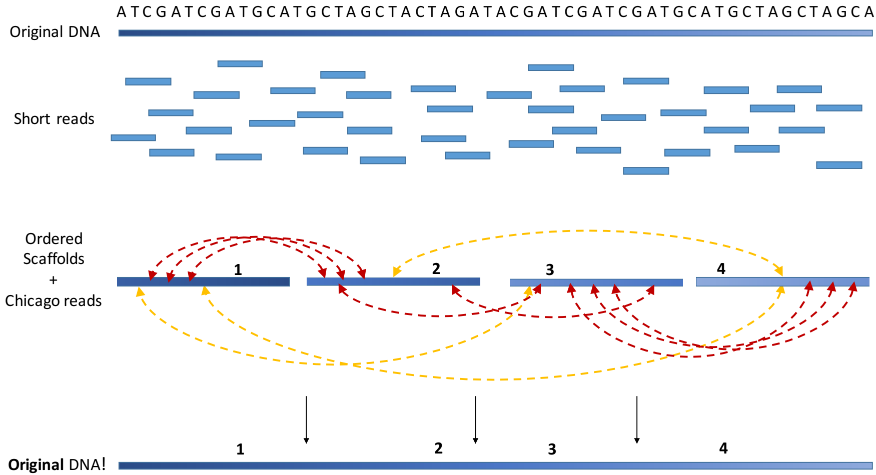

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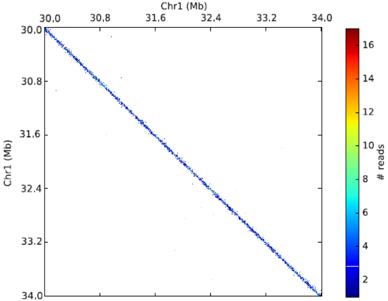

Figure S3 from [Putnam et al. 16]
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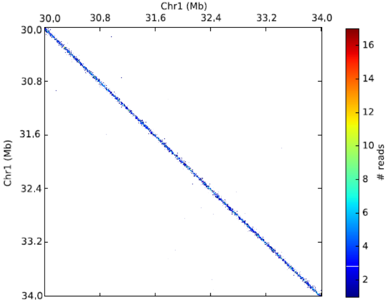

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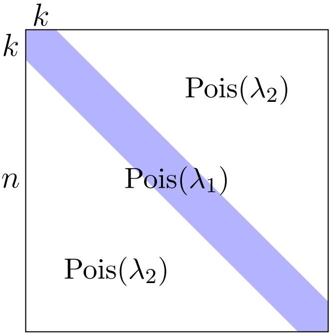

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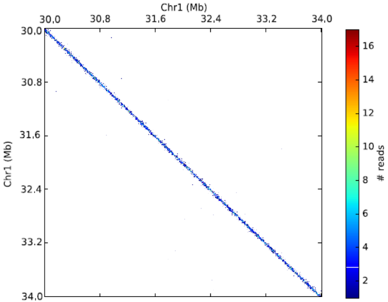

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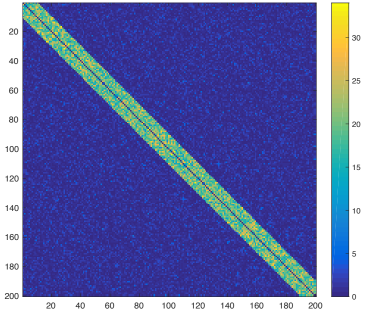
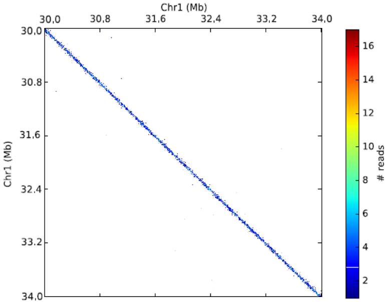


$n = 200,\ k = 10,\ \lambda_1 = 20,\ \lambda_2 = 1$

Ordering DNA scaffolds with Chicago reads



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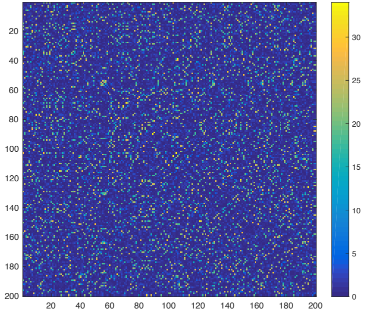
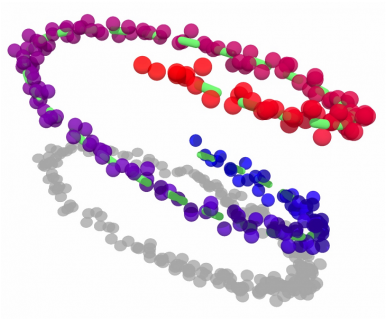


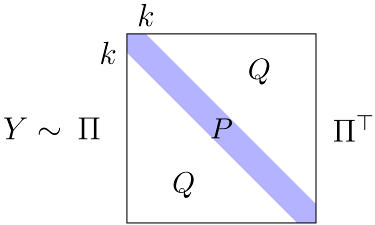
Goal: recover hidden permutation

Data seriation (stringing) [Kendall 71']

- Given a similarity matrix Y for n objects
- Ordering the n objects so that similar objects are near each other

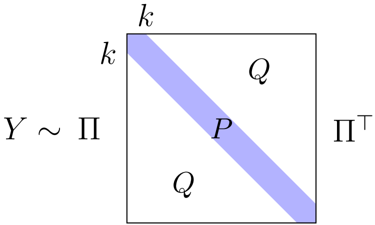


A Planted Ordering Model



- Π is the permutation matrix corresponding to ordering π

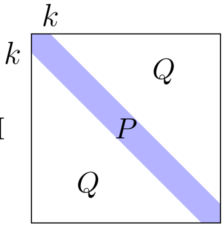
A Planted Ordering Model



- Π is the permutation matrix corresponding to ordering π
- $Y_{ii} = 0$ and for $i \neq j$:

$$Y_{ij} \sim \begin{cases} P & \text{if } |\pi(i) - \pi(j)| \leq k \\ Q & \text{otherwise} \end{cases}$$

A Planted Ordering Model

$Y \sim \Pi$  $\Pi^\top$

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- Goal: Learn π from observation of Y

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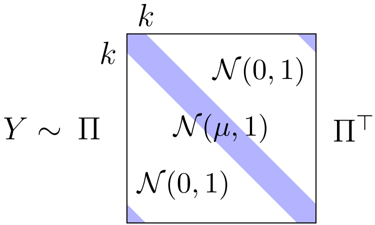

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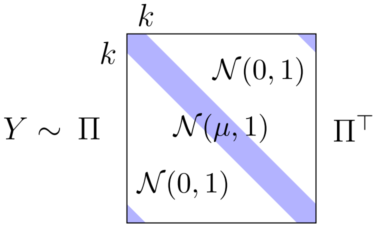

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- Goal: Learn π from observation of Y
- When $k = 1$, reduces to hidden Hamiltonian cycle model [Broder-Frieze-Shamir 06]

Statistical tasks

- Exact recovery:

$$\mathbb{P}\left\{\hat{\pi} = \pi\right\} \xrightarrow{n\to\infty} 1$$

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- Detection:

$$\mathcal{H}_0 : \mu = 0 \quad \text{v.s.} \quad \mathcal{H}_1 : \mu > 0$$

Type-I + Type-II error probabilities $\to 0$

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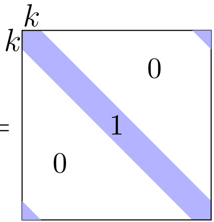
Main Questions

- When is recovery or detection informationally possible?
- Is IT-limit achievable in polynomial-time?

Outline of the remainder

1. Exact recovery
2. Detection
3. Weak recovery
4. Summary and concluding remarks

Exact recovery: maximum likelihood estimation

$A =$ 

$$\begin{aligned} \max \quad & \langle Y, \Pi A \Pi^\top \rangle \\ \text{s.t.} \quad & \Pi \in S_n \end{aligned}$$

- S_n: set of $n \times n$ permutation matrices
- When $k = 1$, maximum weighted Hamiltonian cycle problem

Exact recovery: necessary condition

Theorem (Necessary condition)

Exact recovery *is information-theoretically impossible if*

$$\mu^2 < 2\log n$$

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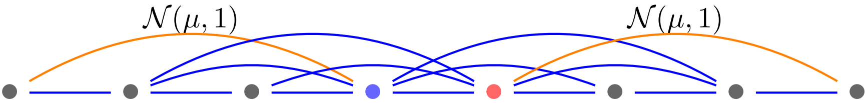
Independent of bandwidth k



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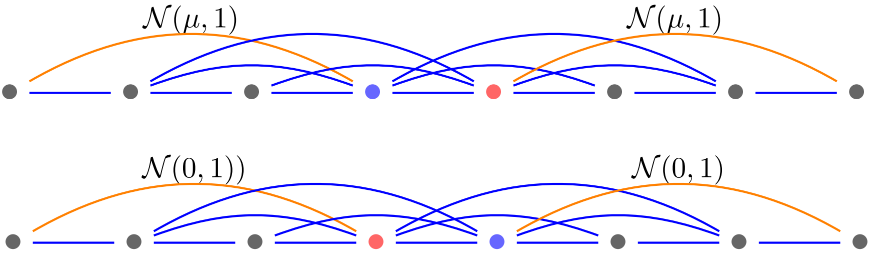

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Remarks

- MLE fails on the event

$$\mathcal{F} \triangleq \cup_{j>i}\{Y_{i-1,j} + Y_{i,j+1} > Y_{i-1,i} + Y_{j,j+1}\}$$

- $|\{i : Y_{i,i+1} \approx \mu/2\}| \approx ne^{-\mu^2/8}$
- $\mathbb{P}\{Y_{i-1,j} + Y_{i,j+1} > \mu\} \approx e^{-\mu^2/4}$

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- $\mathbb{P}\{Y_{i-1,j} + Y_{i,j+1} > \mu\} \approx e^{-\mu^2/4}$
- The necessary condition is tight

Exact recovery: naïve thresholding $k = 1$

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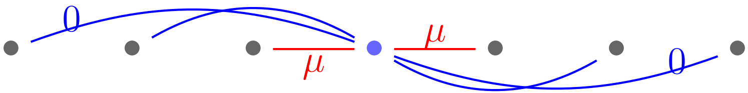

Theorem (naïve thresholding $k = 1$)

When $k = 1$, the naïve thresholding achieves **exact recovery** *if*

$$\mu^2 > 8 \log n$$

Exact recovery: naïve thresholding for general k

- A naïve thresholding algorithm for general k:
  For every vertex, keep the $2k$ edges with the largest weights

Theorem (naïve thresholding for general k)

When $k = 1$, the naïve thresholding exactly recovers $2k$-NN graph if

$$\mu^2 > 8 \log n + 4 \log k$$

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Remarks

When $k = 1$, a factor of 2 gap to the IT limit $\mu^2 = 4 \log n$

Exact recovery: greedy merging $k = 1$

Greedy merging [Motahari-Bresler-Tse '13]

1. Initialize the set of edges to be empty
2. Among all vertices with degree less than 2, connect two vertices i, j with largest Y_{ij}
3. Repeat Step 2

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Remarks

i and j will not be connected if

$$Y_{ij} < \min\{Y_{i-1,i}, Y_{i,i+1}\} \quad \text{or} \quad Y_{ij} < \min\{Y_{j-1,j}, Y_{j,j+1}\}$$

Exact recovery: greedy merging for genearl k

Greedy merging for general k

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The greedy merging exactly recovers the $2k$-NN graph if

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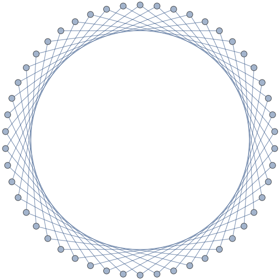
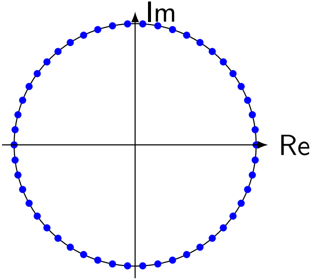
From $2k$-NN graph to vertex ordering



$2k$-NN graph



Eigenvector v_2 of circulant graph

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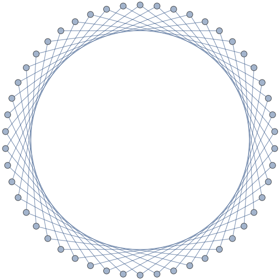

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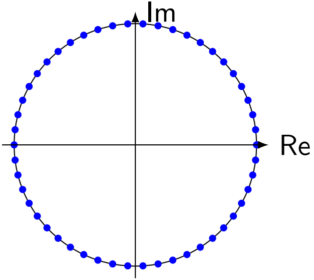


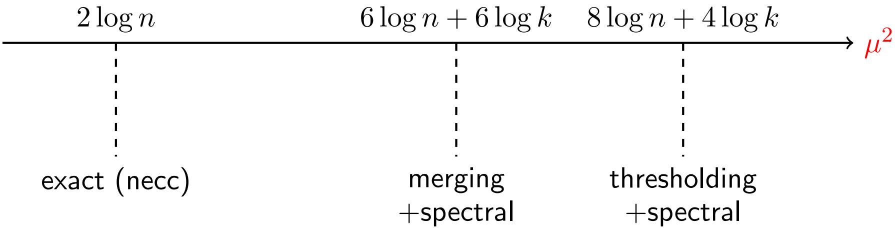
Eigenvector v_2 of circulant graph

$$v_2 = (\omega^{\pi(1)}, \ldots, \omega^{\pi(n)}),$$

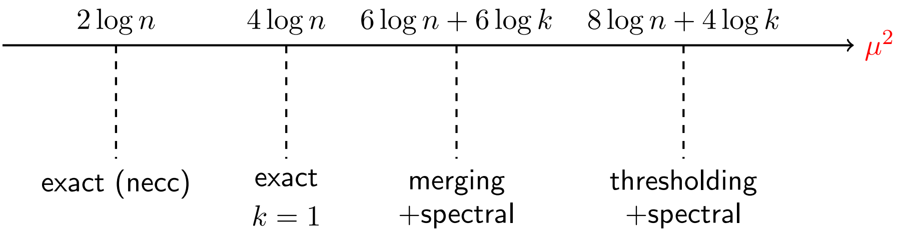
where $\omega = \exp\left(\frac{2\pi i}{n}\right)$ is the n^{th} root of unity

Exact recovery of vertex ordering for genearl k

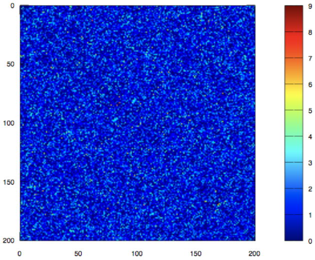
1. Estimate $2k$-NN graph A
2. Let v_2 denote the (complex) eigenvector of A corresponding to the 2nd largest eigenvalue
3. Sort the phase of v_2 and output the ordering


$2 \log n$
$6 \log n + 6 \log k$
$8 \log n + 4 \log k$
μ^2
exact (necc)
merging
+spectral
thresholding
+spectral

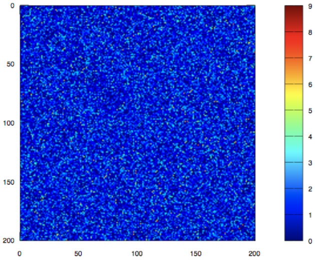
Summary for exact recovery



Detection threshold

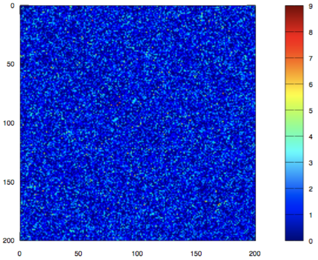


H_0

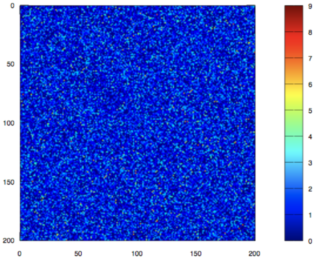


H_1

Detection threshold



H_0



H_1

Theorem

Detection *is possible if and only if*

$$k^2\mu^2 \to \infty$$

Proof of detection threshold

- Upper bound: sum statistic $\sum_{i<j} Y_{ij}$
- Lower bound: bounded second moment

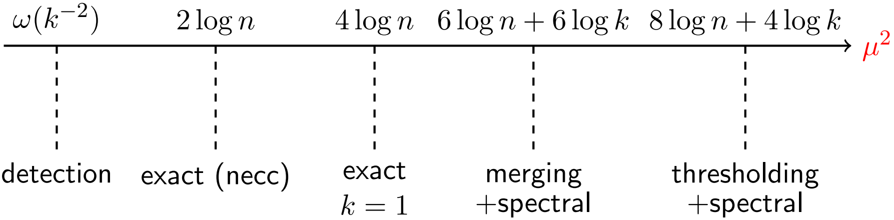
$$\mathbb{E}_{Y\sim\mathbb{Q}}\left[\left(\frac{\mathbb{P}(Y)}{\mathbb{Q}(Y)}\right)^2\right] = \mathbb{E}_{\pi,\pi'} \exp\left(\mu^2 \omega(\pi,\pi')\right),$$

where

$$\omega(\pi,\pi') = \sum_{i<j} \mathbf{1}_{\{|\pi(i)-\pi(j)|\le k,\ |\hat{\pi}(i)-\hat{\pi}(j)|\le k\}}$$

- Heuristically, $\omega(\pi,\pi') \sim \mathrm{Pois}(2k^2)$
- Hence, if $k^2\mu^2 = O(1)$, then the second moment is bounded

Summary for exact recovery and detection



Weak recovery

Weak recovery:

$$\underbrace{\frac{1}{nk}\sum_{i<j}\mathbf{1}_{\{|\pi(i)-\pi(j)|\le k,\ |\hat{\pi}(i)-\hat{\pi}(j)|\le k\}}}_{\text{overlap}} \to 1$$

Weak recovery for $k = 1$

Theorem

When $k = 1$, weak recovery is information-theoretically possible if and only if

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Remarks

- Upper bound: analysis of MLE
- Lower bound: rate distortion argument

Proof of lower bound for weak recovery

-

$$\begin{aligned} I(Y;\pi) &\ge I(\hat{\pi};\pi) \\ &\ge \min_{\mathbb{E}[\omega(\tilde{\pi},\pi)]=(1+o(1))n} I(\tilde{\pi};\pi) \\ &\approx H(\pi) \approx n \log n \end{aligned}$$

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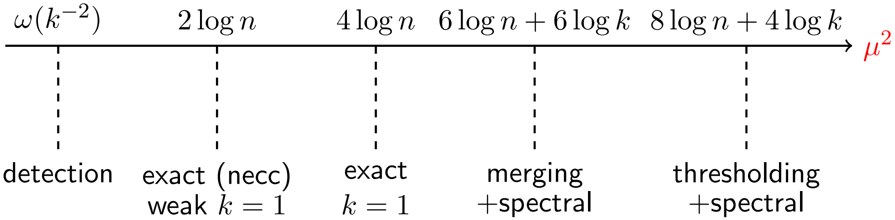
\end{aligned}
$$

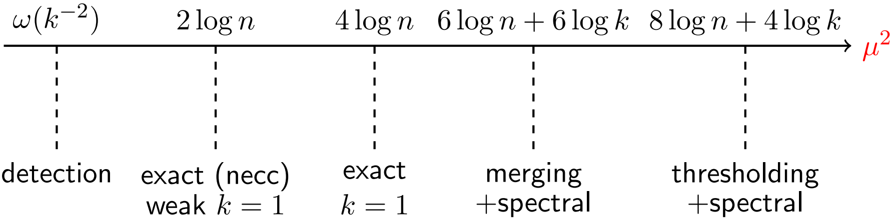
-

$$
\begin{aligned}
I(Y;\pi) &= \min_{\mathbb{Q}} D(\mathbb{P}_{Y|\pi} \| \mathbb{Q} \mid \mathbb{P}_{\pi}) \\
&\leq D(P_{Y|\pi^*} \| \mathcal{N}(0,1)^{\otimes \binom{n}{2}} | \mathbb{P}_{\pi}) \\
&= n\mu^2/2
\end{aligned}
$$

Conclusion and remarks



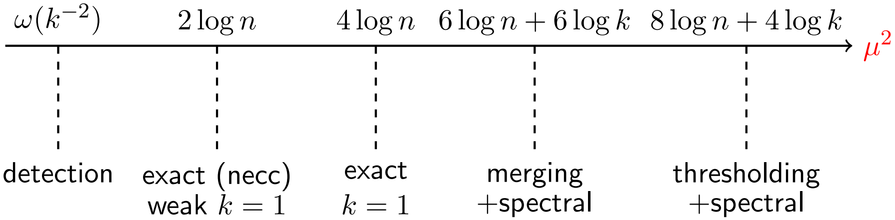
Conclusion and remarks



Future work

- Recovery threshold for general k
- SDP relaxation of MLE

Conclusion and remarks



Future work

- Recovery threshold for general k
- SDP relaxation of MLE
- Real data experiment