Dichotomy for Real Holant^c Problems

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Outline

- Definitions and results
- A journey over seven years
- Tractable families
- Hardness proof

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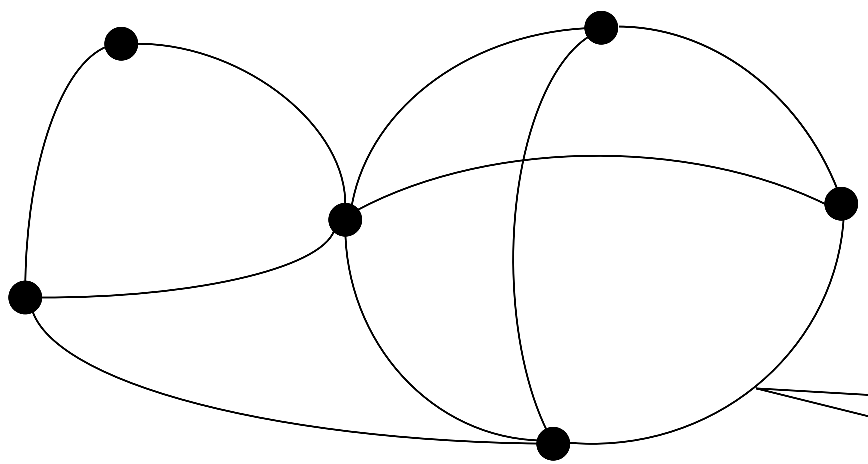
Holant

$$\sum_{x_1,x_2,\ldots,x_m \in [q]} \prod_{v \in V} F_v(x\,|_{\mathrm{v}})$$

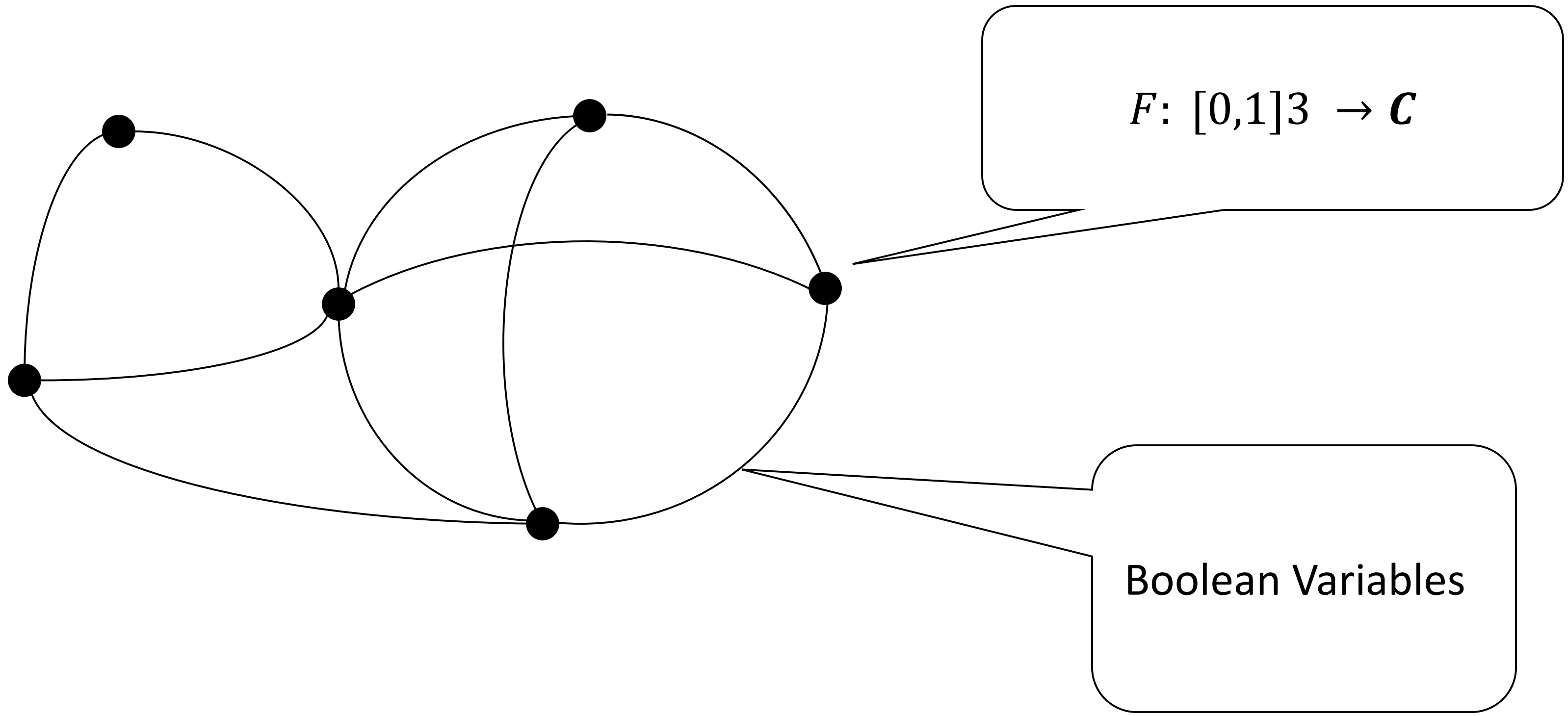


$F\colon\ [0,1,\ldots q-1]3\ \rightarrow \boldsymbol{C}$

Variables of $[0,1,\ldots q-1]$

Holant

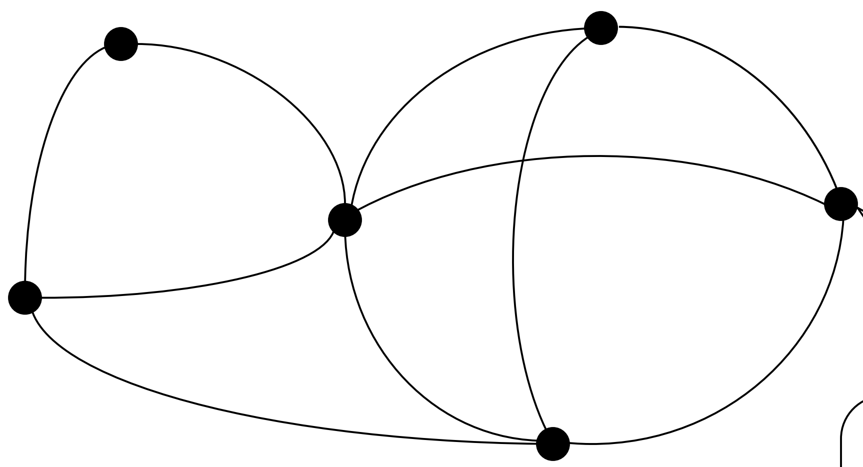
$$\sum_{x_1,x_2,\ldots,x_m\in\{0,1\}} \prod_{v\in V} F_v(x\,|_{\mathrm{v}})$$



Examples

$$\sum_{x_1,x_2,\ldots,x_m\in\{0,1\}}\prod_{v\in V}F_v(x\,|_{\mathrm{v}})$$

Perfect Matching



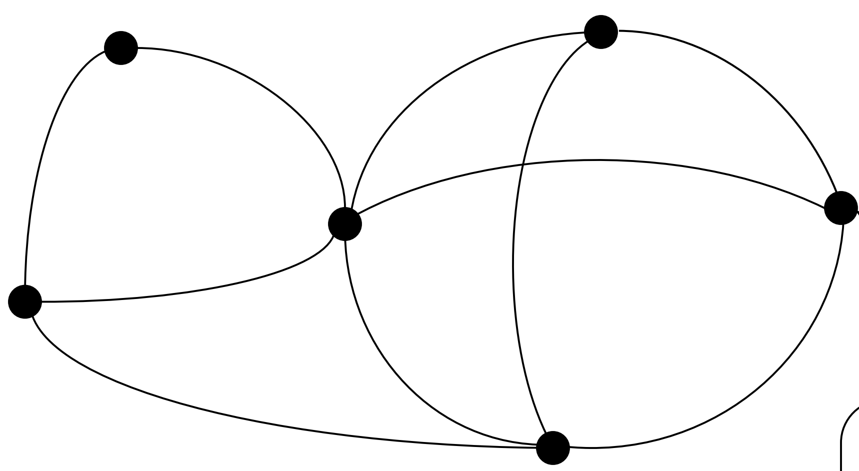
$$wt(\sigma) = number\ of\ \ 1s\ in\ \sigma$$

$$F_v(\sigma) = \begin{cases} 1 & wt(\sigma) = 1, \\ 0 & otherwise \end{cases}$$

Examples

$$\sum_{x_1,x_2,\ldots,x_m\in\{0,1\}} \prod_{v\in V} F_v(x|_{\mathrm{v}})$$

Matching



$$F_v(\sigma) = \begin{cases} 1 & wt(\sigma) \le 1, \\ 0 & otherwise \end{cases}$$

$wt(\sigma) = number\ of\ \ 1s\ in\ \sigma$

Decision Holant(F)

A partial classification [Feder 2001]

- Tractable: Horn, dual-Horn, 2SAT, affine, 0-valid, and 1-valid
- Delta-Matroid Parity problems   (Perfect Matching problem is a special case)
- NP-Complete

- Dichotomy for Boolean CSP was known. [Schaefer 78]
- Even Delta-Matroids is in P. [Kazda, Kolmogorov, Rolínek 2017]

New interesting tractable problems: an example

- NTW_3 is the Not-Two function of arity 3:

$$F_v(\sigma) = \begin{cases} 0 & wt(\sigma) = 2, \\ 1 & otherwise \end{cases}$$

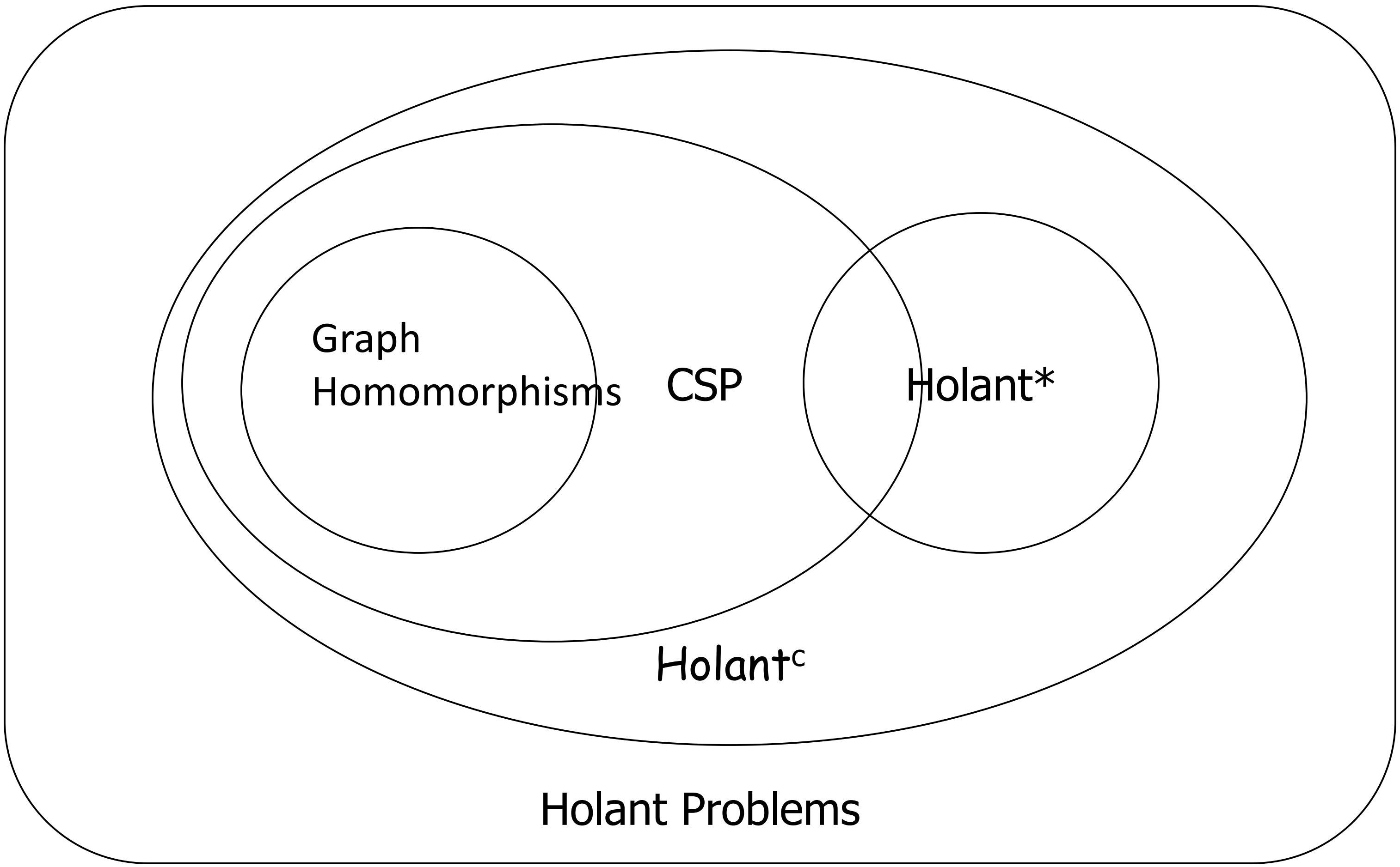
- #CSP(NTW_3) is #P-complete

- Counting Holant(NTW_3) is in P. (why? An exercise.)

Other Sub-Frameworks

- CSP($\mathcal{F}$)=Holant($\mathcal{F}$ ∪ Equalities)
- Holant*($\mathcal{F}$)= Holant($\mathcal{F}$ ∪ U)
- Holant^c($\mathcal{F}$) = Holant($\mathcal{F}$ ∪ $\{\Delta_0, \Delta_1\}$)
- CSP_2^c($\mathcal{F}$)=Holant($\mathcal{F}$ ∪ $\{\Delta_0, \Delta_1, =_2, =_4, =_6, \ldots\}$)
- …

Relations



Counting Dichotomies

- Symmetric Complex Holant* [CLX 09]
- Symmetric Real Holant^c [CLX 09]
- Complex CSP [CLX09]
- Symmetric Complex Holant^c [CHL 10]
- Complex Holant* [CLX 11]
- Symmetric Real Holant [HL 12]
- Symmetric Complex Holant [CGW 13]

Our results

- A dichotomy for real Holant^c
- A dichotomy for complex CSP_2^c

Two independent works

- A dichotomy for non-negative Holant [Lin, Wang 17]
- A dichotomy for complex Holant^+. [Backens 17]
- $\text{Holant}^+(\boldsymbol{\mathcal{F}}) = \text{Holant}(\boldsymbol{\mathcal{F}} \cup \{\Delta_0, \Delta_1, (1,1), (1,-1)\})$

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A journey over seven years

- We initialed this project seven years ago when Jin-Yi visited Mingji and myself in Beijing in 2010.
- We made some progress and got stuck.
- During the later six years, we came back to the problem from time to time, but no big progress made.
- We continued the problem during the Simons semester here, and discovered a new tractable family.
- A complete dichotomy was proved.

Some functions

- f_3 is the parity function of arity 3: $f_3(\mathrm{x}_1\mathrm{x}_2\mathrm{x}_3) = 1$ if $x_3 = x_1 + x_2$; otherwise, the function value is 0.
- $f_7(\mathrm{x}_1\mathrm{x}_2\mathrm{x}_3\mathrm{x}_4\mathrm{x}_5\mathrm{x}_6\mathrm{x}_7) = 1$ if $x_4 = x_1 + x_2, x_5 = x_1 + x_3, x_6 = x_2 + x_3, \mathrm{x}_7 = \mathrm{x}_1 + \mathrm{x}_2 + \mathrm{x}_3$; otherwise, the function value is 0.
- f_{31} is function of arity 31. It is the 0-1 indicator function of a (particular kind of) 5-dimensional linear subspace of Z_2^{31}.
- These are pure affine functions and known to be tractable alone.

Some more functions

- $f_7^{\alpha}(\mathrm{x}_1\mathrm{x}_2\mathrm{x}_3\mathrm{x}_4\mathrm{x}_5\mathrm{x}_6\mathrm{x}_7) = (-1)^{\mathrm{x}_1\mathrm{x}_2\mathrm{x}_3}$ if $x_4 = x_1 + x_2, x_5 = x_1 + x_3, x_6 = x_2 + x_3, \mathrm{x}_7 = \mathrm{x}_1 + \mathrm{x}_2 + \mathrm{x}_3$; otherwise, the function value is 0.
- $f_7^{\alpha}(\pm)$ is a function of arity 14. On the input $x_{1\sim 14}$, the function value is $f_7^{\alpha}(\mathrm{x}_1\mathrm{x}_2\mathrm{x}_3\mathrm{x}_4\mathrm{x}_5\mathrm{x}_6\mathrm{x}_7)$ if $x_{i+7} = \overline{x_i}$ for $i = 1, \dots, 7$; otherwise, the function value is 0.

Seven years ago

- We observed that a dichotomy for CSP_2^c is the main missing component for Holant^c
- We found a P-time algorithm for $CSP_2^c\left(f_7^\alpha(\pm)\right)$
- $CSP_2^c\left(f_7^\alpha(\pm), f_{2^k-1}\right)$ is #P-hard for k=1,2,3,4
- But we got stuck on $CSP_2^c(f_7^\alpha(\pm), f_{31})$
- We conjectured that it is also #P-hard, but failed to prove it in six years.
- At Simons, we figured out that it is tractable, and $CSP_2^c\left(f_7^\alpha(\pm), f_{2^k-1}\right)$ is tractable for all $k \geq 5$.

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Tractable family P

- Product type P denotes the class of functions which can be expressed as a product of unary functions, binary equality functions and binary disequality functions.

Tractable family A

- Affine type A denotes all functions f : $\{x_1, x_2, \dots, x_n\} \rightarrow C$ satisfying the following conditions:
    - $supp(f)$ is affine $\chi(Ax = b)$.
    - Assume $x_1, x_2, \dots, x_r$ are free variables.

$\mathrm{f(x)} = \lambda\chi(Ax = b)\cdot i^{L(x_1,\dots,x_r)+2Q(x_1,\dots,x_r)}$, where L is an integer coefficient linear polynomial and Q is an integer coefficient multilinear polynomial where each monomials has degree 2.

Dichotomy for CSP

- {Equality, Δ_0, Δ_1} $\subset (A \cap P)$

- $\#CSP(F)$ is #P-hard unless $F \subset P$ or $F \subset A$, in which case the problem is in P. [Cai, L. , Xia 2009]

Tractable family A^α

- $\alpha = \sqrt{i}, M_s = \begin{pmatrix} 1 & 0 \\ 0 & s \end{pmatrix}, M_\alpha = \begin{pmatrix} 1 & 0 \\ 0 & \alpha \end{pmatrix}$
- Dual Affine type A^α denotes all functions f satisfying that $M_\alpha f \in A$.
- $\text{Holant}(A^\alpha)$ is in P.
- $\{\Delta_0, \Delta_1, =_2, =_4, =_6, \dots\} \subset A^\alpha$, so $\text{CSP}_2^c(A^\alpha)$ is in P.
- Comparing to CSP, this is the only new trackable family for CSP_2^c of symmetry functions.
- $f_7^\alpha \in A^\alpha$, but $f_7^\alpha(\pm) \notin P \cup A \cup A^\alpha$.

The new tractable family

- (Local affine L). A function f is in L, if and only if for each $\sigma = s_1 s_2 \cdots s_n \in \{0,1\}^n$ in the support of f, $(M_{\alpha^{s_1}} \otimes M_{\alpha^{s_2}} \otimes \cdots \otimes M_{\alpha^{s_n}}) f$ is in A .

- $\{\Delta_0, \Delta_1, =_2, =_4, =_6, \ldots\} \subset L$

- $f_7^{\alpha}(\pm)$, $f_{31} \in L$. (why?)

The algorithm

- Solve the linear system to get an assignment which is on the support for all functions.

- Apply M_α transformation on edges with value 1.

- By definition of L, all the functions are in A after the transformation.

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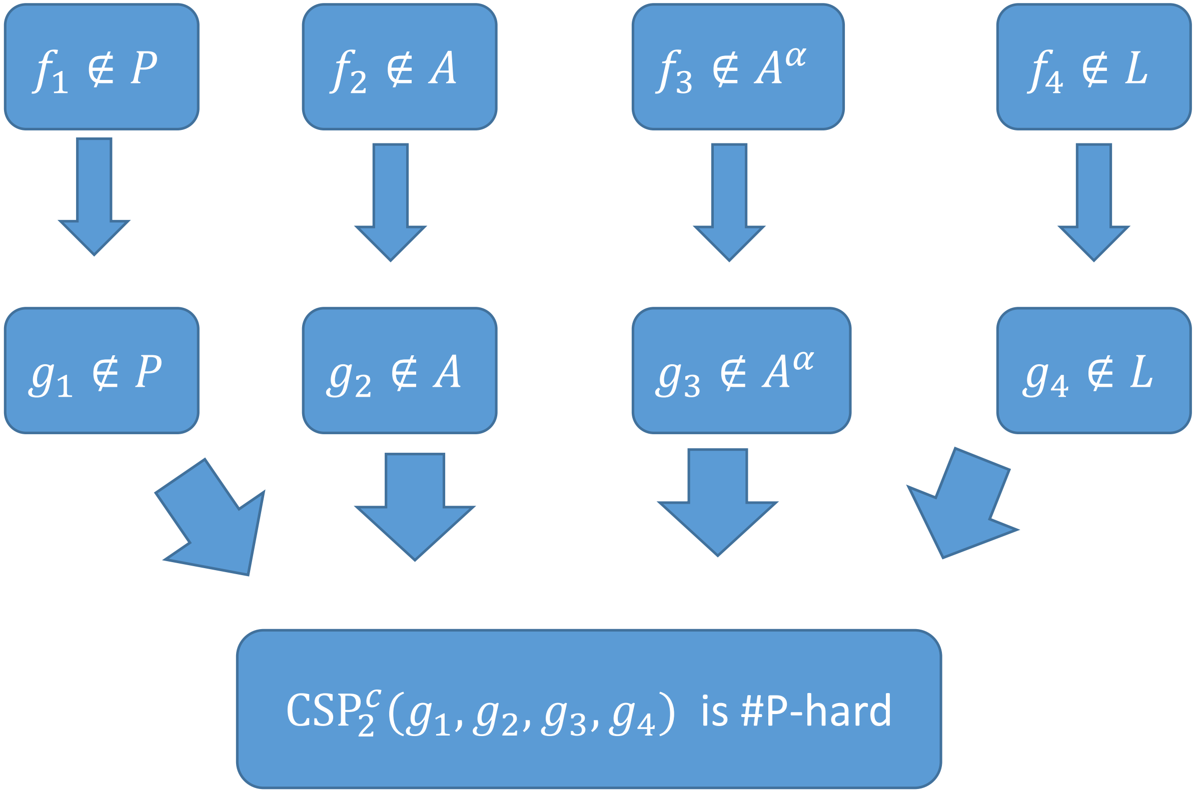
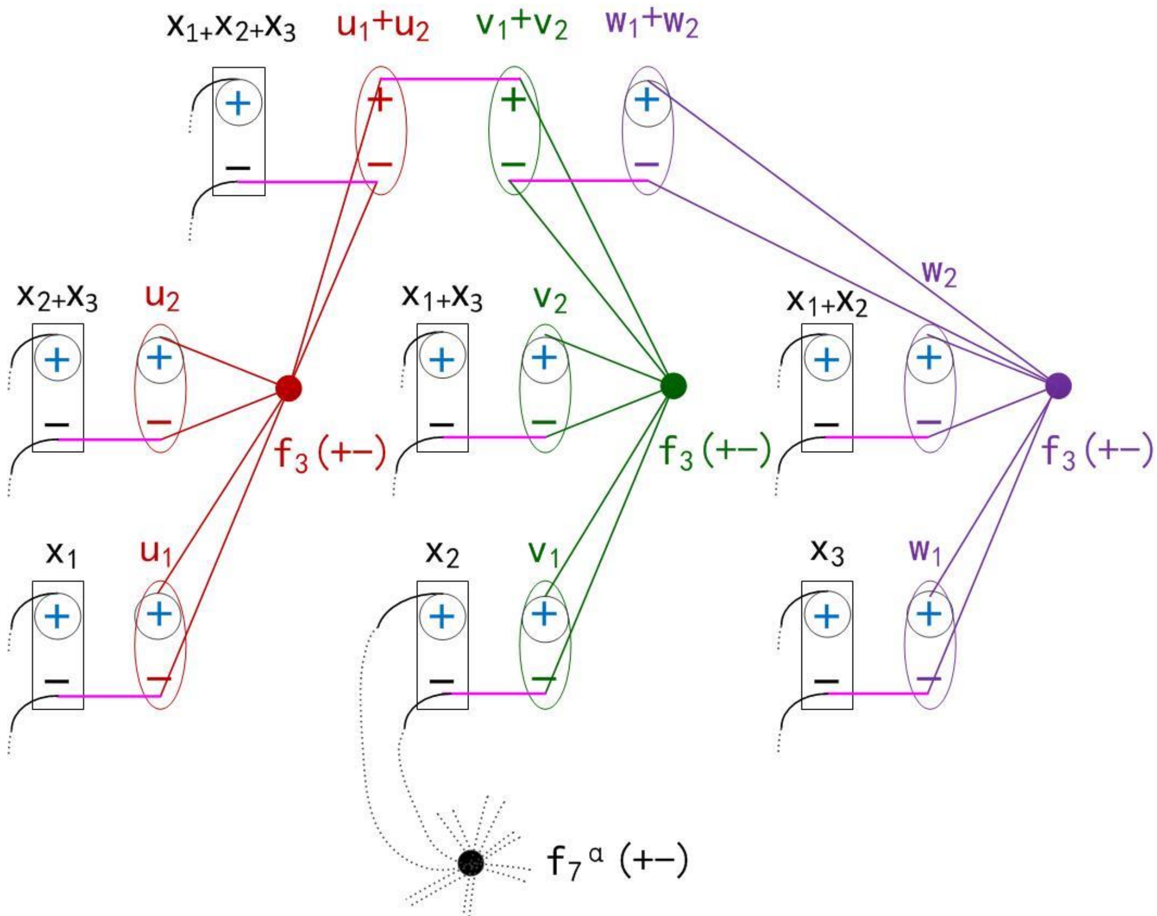
Hardness proof for $\mathbf{CSP}_2^c$



Reduction by an example



Hardness proof for Real Holant^c

- If we can interpolate all the unary functions, we can call the dichotomy for Holant*.
- If we can realize (or interpolate) $=_3$, we can realize all the equality and call the dichotomy for CSP.
- If we can realize (or interpolate) $=_4$, we can realize all the equality of even arity and call the dichotomy for CSP_2^c.
- Also need holographic reduction in the proof.
- A dichotomy for complex Holant^c. [Backens 17]

Thank You!