Predicting Phase Transitions in Hypergraph q-Coloring with the Cavity Method

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CSPs, complexity and random instances

A constraint satisfaction problem (CSP) is a set of variables under a set of constraints.

Question :

Is there an assignement satisfying all the constraints ?

- Worst case instances require an extensive search to be answered: exponential run time and NP complexity.

Studying random CSPs consists in assigning a probability to each instance.

Question :

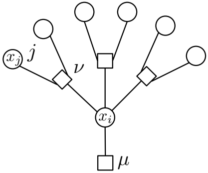
Are typical instances satisfiable ?

- Average-case properties can be studied with analytical and numerical methods.

Hypergraph q-coloring

Definition

- N Potts variables x_i on vertices : $x_i \in \{1, 2, ..q\}$
- M constraints Δ_μ on hyperedges :



$$\Delta_\mu(\{x_i\}_{i\in\partial\mu}) = \begin{cases} 0 \text{ if all variables } x_i \text{ are equal} \\ 1 \text{ otherwise} \end{cases}$$

Solutions:

- Coloring $\mathbf{x}$ is solution $=$ no hyperedge is monochromatic
- The set of solutions is $\mathcal{S} = \{\mathbf{x}/\Delta_\mu(\mathbf{x}_{\partial\mu}) = 1, \forall\mu\}$
- Define $Z = |\mathcal{S}| = \#$ proper colorings

Random instances of hypergraph q-coloring

Random ensembles

- K-**uniform** ℓ-**regular**: choose uniformly among hypergraphs with M hyperedges of size K and N nodes of degree ℓ
- K-**uniform Erdős-Réyni, average degree** $\ell = KM/N$: choose uniformly M hyperedges from all the $\binom{N}{K}$ possibility (i.e. $d \sim \text{Poisson}(\ell)$)

All results are derived

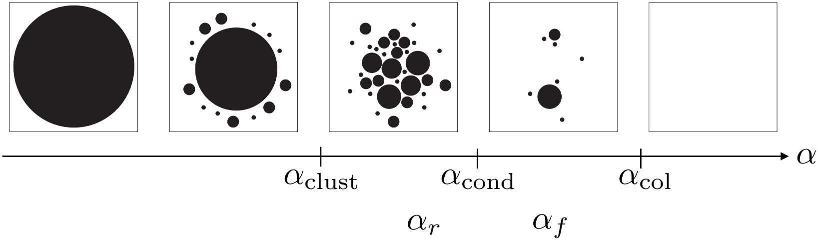
- in the **large size limit** $N \to \infty$, $M \to \infty$
- with **finite density of constraints** $\alpha = M/N = \ell/K$ (i.e. sparse hypergraphs)

Density α controls how difficult is the coloring problem

- Existence of a **sharp** satistifiability/colorability threshold $\alpha_{\rm col}$
- Existence of several structural changes of $\mathcal{S}$ for $\alpha < \alpha_{\rm col}$

Phase transitions in random CSPs with the cavity method

[Krzakala, Montanari,Ricci-Tersenghi, Semerjian,Zdeborova '07]



- Clusters: connected subset of solutions in configuration space.
- Frozen variables: take same value in all solutions of a cluster.

Previous works around q-coloring of hypergraphs

Physics conjectures have refined progressively the picture of the phase transitions

- Coloring on graph [Krzakala et al. '04, Zdeborová et al '07]
- Bicoloring of regular hypergraphs [Castellani et al '03, Dall'Asta et al '08, Braunstein et al '16]

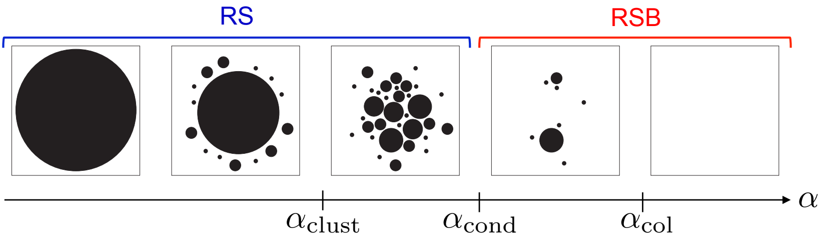
Rigorous works

- hypergraph bicoloring and NAE-SAT [Achlioptas et al '06 (algorithmic barriers and phase transitions,freezing), Coja-Oghlan et al 12' (condensation), Bapst et al '14 (existence and asymptotic location of condensation at positive temperature),Achlioptas et al 06' (leading order of colorability),Coja-Oghlan et al 12' (additional term to asymptotics of col),Ding et al '13 (location of colorability)]
- hypergraph q-coloring: [Krivelevich et al '97 (asymptotics of the chromatic number),Dyer et al '15 (generalisation to hypergraph of graph asymptotics of colorability(Achlioptas and Naor)), Ayre et al '15 (improving last results).]

Two different regimes for the cavity method

For the uniform probability distribution over valid colorings

- **Replica Symmetric regime:** correlations decay fast (shortest loops $O(\ln N)$)
- **Replica Symmetry Breaking:** long range dependencies



- **1RSB** computation assumes there is no clusters within clusters. Believed to be the relevant regime for a large class of CSPs (large enough q or large enough K).

RS formula and RSB complexity

RS equations / first moment: $Z_{RS} = q^N \left(1 - \frac{q}{q^K}\right)^M$

RSB equation: $Z_{1RSB} = \int \mathrm{d}s \, e^{N(s+\Sigma(s))} = e^{N(s^\star + \Sigma(s^\star))}$

- clusters of size e^{Ns} are $e^{\Sigma(s)}$ many
- Σ is called the **complexity**
- **typical solutions** = solutions in clusters of size $e^{Ns^\star}$
  (exponential approximation for $N \to \infty$)

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$$\Rightarrow \mathbb{E}\left[\ln Z/N\right] \leq \ln q + \alpha \ln\left(1 - 1/q^{K-1}\right)$$

$$\Rightarrow \quad \alpha_{\rm col} \leq \alpha_{\rm RS} = -\ln q / \ln\left(1 - 1/q^{K-1}\right) \text{ upper bound}$$

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Asymptotics of phase transitions thresholds - RS

Reasoning in terms of average connectivities $\ell = \alpha K$

$\ell_{\mathrm{RS}} = -K \ln q / \ln\left(1 - 1/q^{K-1}\right)$

Expanding at q fixed, $K \to \infty$

$$\ell_{\mathrm{RS}} = K q^{K-1} \ln q - \frac{K}{2} \ln q + O\left(\frac{K \ln q}{q^{K-1}}\right)$$

- Expansion in $\mathbf{1/q^{K-1}}$, small when either $K \to \infty$ or $q \to \infty$

NB: ER and regular expected to give same asymptotic behavior !

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$q{=}2 \quad \ell_{\mathrm{RS}} = K 2^{K-1} \ln 2 - \frac{K}{2} \ln 2 + O\left(\frac{K}{2^{K-1}}\right)$

$K{=}2 \quad \ell_{\mathrm{RS}} = 2q \ln q - \ln q + O\left(\frac{\ln q}{q}\right)$

- Expansion in $\mathbf{1/q^{K-1}}$, small when either $K \to \infty$ or $q \to \infty$

NB: ER and regular expected to give same asymptotic behavior !

Asymptotics - Condensation and Colorability

Leading orders $\ell_{\rm cond} \sim \ell_{\rm col} \sim \ell_{\rm RS}$ proven [Dyer et al & Ayre et al '15]

Expanding further at q fixed, $K \to \infty$, in $1/q^{K-1}$

- Condensation ($\Sigma(s*) = 0$)
  $$\ell_{\rm cond} = Kq^{K-1} \ln q - \frac{K}{2} \ln q - \frac{K}{2} \left(1 - \frac{1}{q}\right) 2 \ln 2 + \tilde{O}\left(\frac{1}{q^{K-1}}\right)$$
- Colorability ($\sup_s \Sigma(s) = 0$)
  $$\ell_{\rm col} = Kq^{K-1} \ln q - \frac{K}{2} \ln q - \frac{K}{2} \left(1 - \frac{1}{q}\right) + \tilde{O}\left(\frac{1}{q^{K-1}}\right)$$

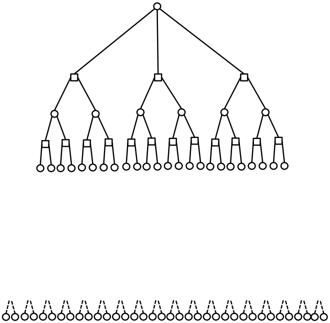
with $\tilde{O}\left(\frac{1}{q^{K-1}}\right) = O\left(\frac{poly(K, lnq)}{q^{K-1}}\right)$

Recovers

- known subleading orders at $q = 2$ or $K = 2$ [Zdeborová et al '07, Braunstein et al '16]
- rigorous results for hypergraphs [Dyer et al '15, Ayre et al '15]

Asymptotics - Rigidity (1/2)

Apparition of frozen variables in typical solutions, conjectured to be equivalent to a naive reconstruction problem.



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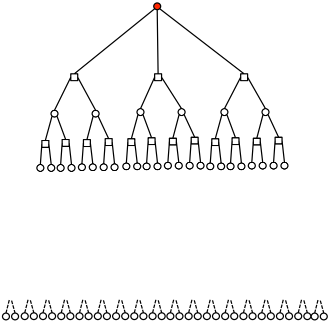

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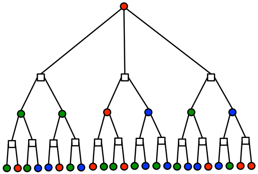

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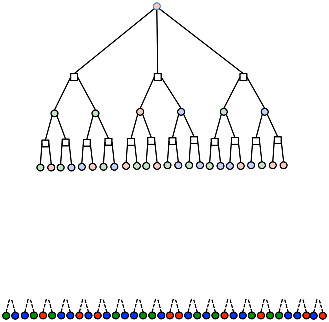


- Probability η_L for the root to be frozen in a color given the coloring at generation L

$$\eta_L = f(\eta_{L-1})$$

- Allow $L \to \infty$ for fixed point η
- Rigidity $\eta > 0$

Asymptotics - Rigidity (2/2)

Expanding at q fixed, $K \to \infty$, in $1/q^{K-1}$

$$\ell_r = q^{K-1}\left(\ln\left((K-1)(q-1)\right) + \ln\ln Kq + 1 + o(1)\right)$$

Recovers

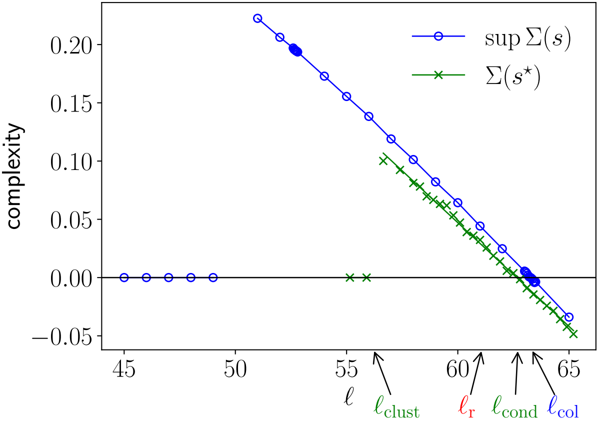
- Asymptotics for graph coloring and hypergraph bicoloring

$$\begin{aligned} q{=}2 &\quad \ell_r{=}2^{K-1}\Big(\ln K{+}\ln\ln K{+}1{+}O\Big(\tfrac{\ln\ln K}{\ln K}\Big)\Big) \\ K{=}2 &\quad \ell_r{=}q(\ln q{+}\ln\ln q{+}1{+}o(1)) \end{aligned}$$

- Gap in regimes $\ell_r \ll \ell_{\rm cond} \sim \ell_{\rm col} \sim \ell_{\rm RS} \sim q^{K-1} K \ln q$

Numerical evaluations for ER small q and K - Generic scenario

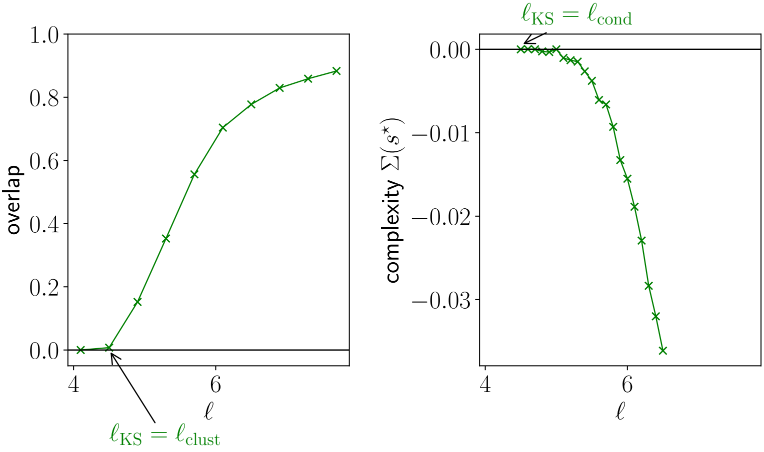
$q = 4$, $K = 3$, $\alpha = \ell / K$



Similar behavior for all cases $q \geq 3$ and $K \geq 3$, or $q \geq 2$ and $K \geq 5$

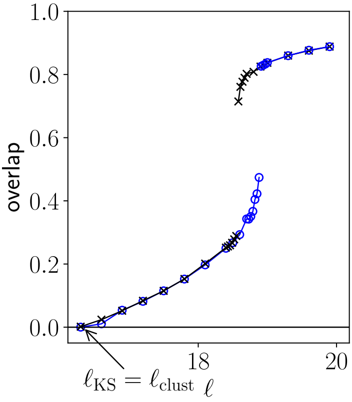
Numerical evaluations for ER - $q = 2$ $K = 3$

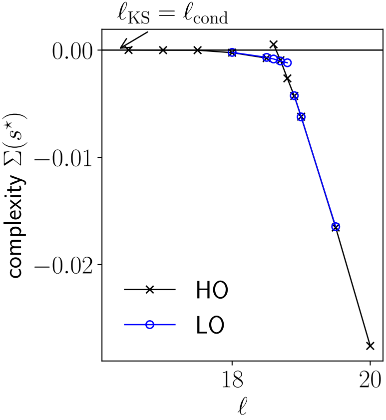
- 1RSB local instability makes $\ell_{\rm col}$ not tight, only upper bound
- Continuous transition $\ell_{\rm clust} = \ell_{\rm cond} = \ell_{\rm KS}$
  (Similar behavior for $q = 3$ and $K = 2$ [Zdeborová et al '07])



Numerical evaluations for ER - $q = 2$ $K = 4$

- 1RSB local instability makes $\ell_{\rm col}$ not tight, only upper bound
- Continuous transition $\ell_{\rm clust} = \ell_{\rm cond} = \ell_{\rm KS}$, but discontinuity in condensed phase
- Coexistence of two 1RSB solutions (planted models)





Conclusions

Main results

- Generalized asymptotics $O(\frac{poly(K,lnq)}{q^{K-1}})$
- Numerical evaluations

	$\ell_{\rm clust}$	$\ell_r(m=1)$	$\ell_{\rm cond}$	$\ell_{\rm col}$	$\ell_{\rm RS}$	$\ell_{\rm stab}^{\rm RS}$	$\ell_{\rm I}^{\rm SP}$	$\ell_{\rm II}^{\rm SP}$
$q=2$ $K=3$	4.50	(7.37)	4.50	(6.32)	7.23	**4.50**	**6.07**	6.16
$q=2$ $K=4$	16.33	(21.62)	16.33	(19.62)	20.76	**16.33**	**19.35**	17.84
$q=2$ $K=5$	47.4	(52.63)	51.5	52.32	53.70	56.25	59.42	43.9
$q=3$ $K=3$	25.06	(28.07)	26.2	26.92	27.98	32.00	33.62	23.9
$q=3$ $K=4$	97.7	105.88	114.3	115.04	116.44	225.33	225.51	90.7
$q=4$ $K=3$	56.20	61.09	62.7	63.3	64.44	112.50	112.78	52.7

- Two cases where 1RSB predictions of $\ell_{\rm col}$ is only an upper bound (not tight) because of local instability toward more steps of symmetry breaking

Publication preprint: arXiv:1707.01983 !

Thanks for your attention !