The interpolation method

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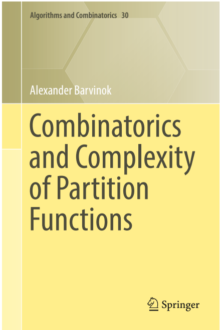
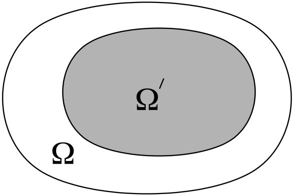


The general idea

If p is a "combinatorially defined polynomial" and $p(z_1, \ldots, z_n) \neq 0$ for all $(z_1, \ldots, z_n)$ in some domain $\Omega \subset \mathbb{C}^n$, then $p(z_1, \ldots, z_n)$ can be efficiently approximated in a slightly smaller domain $\Omega' \subset \Omega$.



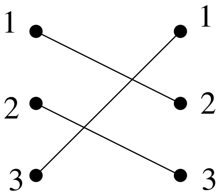
Example: Permanent

The permanent of an $n \times n$ matrix $A = (a_{ij})$,

$$\operatorname{per} A = \sum_{\sigma \in S_n} \prod_{i=1}^{n} a_{i\sigma(i)},$$

where S_n is the symmetric group of the $n!$ permutations of the set $\{1, \ldots, n\}$.

Responsible for counting perfect matchings in graphs (combinatorics) and boson sampling (physics).



$a_{12} a_{23} a_{31}$

Zero-free region for the permanent

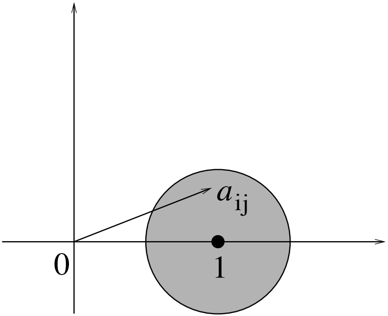
Theorem

Let $A = (a_{ij})$ be an $n \times n$ complex matrix such that

$$|1 - a_{ij}| \leq 0.5 \quad \text{for all} \quad i, j.$$

Then

$$\operatorname{per} A \neq 0.$$



Approximating the permanent in a slightly smaller region

Corollary

Let $A = (a_{ij})$ *be an* $n \times n$ *complex matrix such that*

$$|1 - a_{ij}| \ \leq \ 0.49 \quad \textit{for all} \quad i, j.$$

Then $\mathrm{per}\ A$ *can be approximated within relative error* $0 < \epsilon < 1$ *in* $n^{O(\ln n - \ln \epsilon)}$ *time.*

More precisely,

$$|\ln \mathrm{per}\ A - p(A)| \ \leq \ \epsilon$$

for some polynomial $p = p_{n,\epsilon}$ *of* $\deg p = O\left(\ln n - \ln \epsilon\right)$ *in the matrix entries* a_{ij}*.*

The main lemma

Lemma

Let $g : \mathbb{C} \longrightarrow \mathbb{C}$ *be a polynomial and let* $\beta > 1$ *be real such that*

$$g(z) \neq 0 \quad \text{for all} \quad |z| < \beta.$$

Let $f(z) = \ln g(z)$ *for* $|z| \leq 1$ *and let*

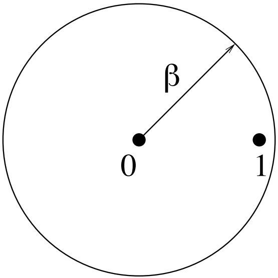
$$T_m(z) = f(0) + \sum_{k=1}^{m} \frac{f^{(k)}(0)}{k!} z^k$$

be the Taylor polynomial of degree m *of* f *computed at* $z = 0$. *Then*

$$|f(1) - T_m(1)| \ \leq \ \frac{\deg g}{(m+1)\beta^m(\beta - 1)}$$

The error goes down exponentially in m.

The main lemma



If $g(z)$ has no zeros in the circle of radius $\beta > 1$ then to approximate $f(z) = \ln g(z)$ for $|z| \leq 1$ within an additive error ϵ one can use the Taylor polynomial of $f(z)$ at $z = 0$ of degree $m = O_\beta(\ln \deg g - \ln \epsilon)$.

The main lemma

Proof.

We factor $g(z) = g(0) \prod_{i=1}^{n} \left(1 - \frac{z}{\alpha_i}\right)$ where $|\alpha_i| \geq \beta$ for $i = 1, \ldots, n$ and $n = \deg g$. Then

$$f(z) = f(0) + \sum_{i=1}^{n} \ln\left(1 - \frac{z}{\alpha_i}\right)$$

and

$$\ln\left(1 - \frac{z}{\alpha_i}\right) = -\sum_{k=1}^{m} \frac{z^k}{k\alpha_i^k} + \xi_{i,m}$$

where

$$|\xi_{i,m}| \leq \frac{1}{(m+1)\beta^m(\beta - 1)} \quad \text{for all} \quad i, m$$

provided $|z| \leq 1$. □

How to compute $f^{(k)}(0)$?

We have

$$f(z) = \ln g(z) \quad \text{so that} \quad f'(z) = \frac{g'(z)}{g(z)} \quad \text{and} \quad g'(z) = f'(z)g(z).$$

Therefore,

$$g^{(k)}(0) = \sum_{j=1}^{k} \binom{k-1}{j-1} f^{(j)}(0) g^{(k-j)}(0).$$

Suffices to compute $g^{(k)}(0)$ for $k = 0, \ldots, m$ and solve a non-degenerate triangular system of linear equations for $f^{(j)}(0)$.

How does it work for the permanent?

Define

$$g(z) = \operatorname{per}\left(\mathbf{1}_{n\times n} + z\left(A - \mathbf{1}_{n\times n}\right)\right),$$

so that

$$g(0) = \operatorname{per}\left(\mathbf{1}_{n\times n}\right) = n!, \quad g(1) = \operatorname{per}\ A$$

and

$$g(z) \neq 0 \quad \text{provided} \quad |z| \ \leq\ \beta = \frac{0.50}{0.49}.$$

Let

$$f(z) = \ln g(z) \quad \text{for} \quad |z| \leq 1.$$

Then

$$g^{(k)}(0) = \frac{d^k}{dz^k} \sum_{\sigma \in S_n} \prod_{i=1}^{n} \left(1 + z\left(a_{i\sigma(i)} - 1\right)\right) \Big|_{z=0}$$

is a polynomial of degree k in the entries of A and hence $f^{(k)}(0)$ is.

For example:

$$f^{(1)}(0) = \frac{1}{n} \sum_{i,j=1}^{n} (a_{ij} - 1)$$

and

$$f^{(2)}(0) = \frac{1}{n(n-1)} \sum_{\substack{(i_1,i_2),\\(j_1,j_2)}} (a_{i_1 j_1} - 1)(a_{i_2 j_2} - 1) - \left(\frac{1}{n} \sum_{i,j} (a_{ij} - 1) \right)^2,$$

etc.

A trickier case

Theorem

Let $A = (a_{ij})$ *be an* $n \times n$ *complex matrix such that*

$$\delta \ \leq \ \Re\, a_{ij} \ \leq \ 2 - \delta \quad \text{and} \quad |\Im\, a_{ij}| \ \leq \ \frac{1}{2}\delta^2 \quad \text{for all} \quad i, j$$

and some $0 < \delta \leq 1$. *Then*

$$\operatorname{per} A \neq 0.$$

A trickier case

Corollary

Fix $0 < \delta \leq 1$. *Let* $A = (a_{ij})$ *be an* $n \times n$ *real matrix such that*

$$\delta \ \leq \ a_{ij} \ \leq \ 2 - \delta \quad \textit{for all} \quad i, j.$$

Then $\mathrm{per}\ A$ *can be approximated within relative error* $0 < \epsilon < 1$ *in* $n^{O(\ln n - \ln \epsilon)}$ *time.*
More precisely,

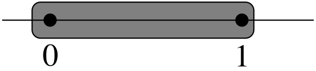
$$\left| \ln \mathrm{per}\ \mathrm{A} - p(A) \right| \ \leq \ \epsilon$$

for some polynomial $p = p_{\delta, n, \epsilon}$ *of* $\deg p = O_\delta\left(\ln n - \ln \epsilon\right)$.

As before, define

$$g(z) = \mathrm{per}\ \left(\mathbf{1}_{n\times n} + z\left(A - \mathbf{1}_{n\times n}\right)\right),$$

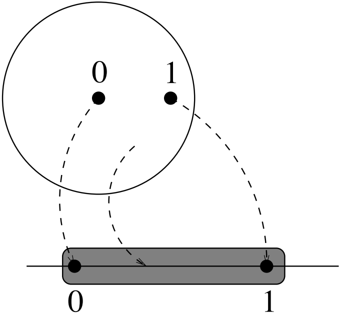
so that $g(0) = n!$ and $g(1) = \mathrm{per}\ A$.
This time, $g(z) \neq 0$ in a neighborhood of $[0, 1] \subset \mathbb{C}$.



What do we do?

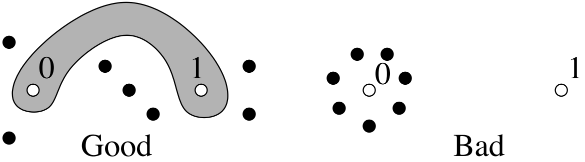
from Theorem to Corollary

Construct a disc of radius $\beta > 1$ and a polynomial ϕ such that $\phi(0) = 0$, $\phi(1) = 1$ and ϕ maps the disc inside the region where $g(z) \neq 0$.



Apply the main Lemma to $g(\phi(z))$.

Back to the general principle



Let $g(z)$ be a polynomial of degree n such that $g^{(k)}(0)$ can be computed in $n^{O(k)}$ time. Suppose we can throw a sufficiently wide sleeve from 0 to 1 completely avoiding the complex zeros of g.

Then $g(1)$ can be approximated within relative error $0 < \epsilon < 1$ in $n^{O(\ln n - \ln \epsilon)}$ time.

A bit about physics

Interpretation: we can efficiently interpolate from a high (in fact, infinitely high) temperature to a cooler temperature, provided there is no phase transition on the way.

Phase transition: complex zeros of the partition function approaching the positive real axis, [Yang and Lee, 1952].

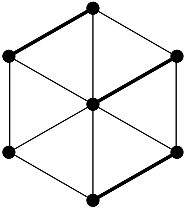
In a sense, this is "simulated annealing" without simulation. We, however, are allowed to use complex temperatures.

More examples: matching polynomial

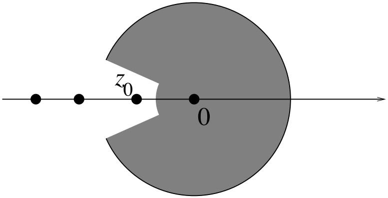
Let $G = (V, E)$ be a graph and let

$$p_G(z) = 1 + \sum_{k>0} (\text{the number of matchings with } k \text{ edges})\, z^k$$

be its *matching polynomial*.



More examples: matching polynomial



The roots of $p_G(z)$ are negative real with the largest root

$$z_0 \leq -\frac{1}{4(\Delta - 1)},$$

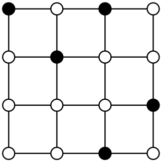
where $\Delta > 1$ is the largest degree of a vertex of G [Heilmann and Lieb, 1972].
In the shaded area, $\ln p_G(z)$ can be approximated within an additive error $\epsilon > 0$ in $n^{O(\ln n - \ln \epsilon)}$ time.

More examples: matching polynomial

When z approaches the positive real axis, the complexity improves and eventually matches that of [Bayati, Gamarnik, Katz, Nair and Tetali, 2007] for $z > 0$.

When the largest degree Δ of a vertex is fixed in advance, the algorithm can be run in polynomial time $(n/\epsilon)^{O(1)}$ [Patel and Regts, 2016].

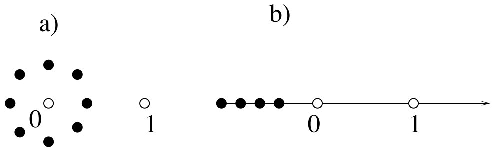
Let $G=(V,E)$ be a graph. A set $S \subset V$ is *independent* if no two vertices of S span an edge of G.



More examples: independence polynomial

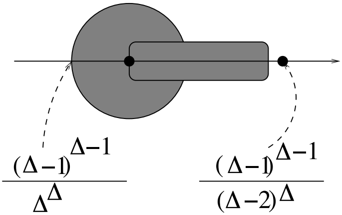
Let

$$g(z) = 1 + \sum_{k>0} (\text{the number of independent sets with } k \text{ vertices})\, z^k.$$



In general, we have a) as it is hard to count independent sets. But sometimes (claw-free graphs, [Chudnovsky and Seymour, 2007]), we have b) and then it is easy to count.

Independence polynomial: a closer look



Let Δ be the largest degree of a vertex of $G=(V,E)$.
There are no roots of the independence polynomial in the disc

$$|z| \leq \frac{(\Delta-1)^{(\Delta-1)}}{\Delta^\Delta} \approx \frac{1}{e\Delta} \quad \text{for large} \quad \Delta$$

[Dobrushin 1996], [Scott and Sokal 2005].
There are no roots in the strip

$$0 \leq \Re z \leq (1-\delta)\frac{(\Delta-1)^{(\Delta-1)}}{(\Delta-2)^\Delta} \approx \frac{e}{\Delta}, \quad |\Im z| \leq \epsilon,$$

[Peters and Regts, 2017].

Independence polynomial: a closer look

Consequently, there is a quasi-polynomial (genuinely polynomial if Δ is fixed – [Patel and Regts, 2016]) approximation algorithm for the independence polynomial inside the Dobrushin - Scott - Sokal disc and up to the correlation decay – [Weitz 2006] – bound if z is real.

More examples: 3-dimensional permanent

For an $n \times n \times n$ tensor $A = (a_{ijk})$,

$$\mathrm{PER}\ A = \sum_{\sigma_1, \sigma_2 \in S_n} \prod_{i=1}^{n} a_{i\sigma_1(i)\sigma_2(i)}$$

(counts perfect matchings in 3-partite hypergraph).
Efficiently approximable for complex A provided

$$|1 - a_{ijk}| \ \leq \ \frac{\sqrt{6}}{9} \approx 0.27 \quad \text{for all} \quad i, j, k.$$

Efficiently approximable for real A provided

$$|1 - a_{ijk}| \ \leq \ \sqrt{2} - 1 \approx 0.41 \quad \text{for all} \quad i, j, k.$$

More examples: graph homomorphisms (Potts model)

For a graph $G=(V,E)$ and a $k \times k$ symmetric matrix A, let

$$\hom_G(A) = \sum_{\phi: V \longrightarrow \{1,\ldots,k\}} \prod_{\{u,v\}\in E} a_{\phi(u)\phi(v)}.$$

Efficiently approximable for complex A provided

$$|1-a_{ij}| \ < \ \max_{0 \leq \theta < 2\pi/3\Delta} \sin\frac{\theta}{2}\cos\frac{\theta\Delta}{2} \quad \text{for all} \quad i,j$$

(for $\Delta = 3$, we get 0.18).
Efficiently approximable for real A provided

$$|1-a_{ij}| \ < \ \tan\frac{\pi}{9} \approx 0.36 \quad \text{for all} \quad i,j \quad \text{if} \quad \Delta = 3$$

and

$$|1-a_{ij}| \ < \ \tan\frac{\pi}{4(\Delta-1)} \quad \text{for all} \quad i,j \quad \text{if} \quad \Delta \geq 4.$$