THE INTERPOLATION METHOD

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If \( p \) is a “combinatorially defined polynomial” and \( p(z_1, \ldots, z_n) \neq 0 \) for all \((z_1, \ldots, z_n)\) in some domain \( \Omega \subset \mathbb{C}^n \), then \( p(z_1, \ldots, z_n) \) can be efficiently approximated in a slightly smaller domain \( \Omega' \subset \Omega \).
Example: Permanent

The permanent of an \( n \times n \) matrix \( A = (a_{ij}) \),

\[
\text{per } A = \sum_{\sigma \in S_n} \prod_{i=1}^{n} a_{i\sigma(i)},
\]

where \( S_n \) is the symmetric group of the \( n! \) permutations of the set \( \{1, \ldots, n\} \).

Responsible for counting perfect matchings in graphs (combinatorics) and boson sampling (physics).
Theorem

Let $A = (a_{ij})$ be an $n \times n$ complex matrix such that

$$|1 - a_{ij}| \leq 0.5 \quad \text{for all} \quad i, j.$$

Then

$$\text{per} \ A \neq 0.$$
Let $A = (a_{ij})$ be an $n \times n$ complex matrix such that $|1 - a_{ij}| \leq 0.49$ for all $i, j$.

Then $\text{per } A$ can be approximated within relative error $0 < \epsilon < 1$ in $n^{O(\ln n - \ln \epsilon)}$ time. More precisely,

$$|\ln \text{per } A - p(A)| \leq \epsilon$$

for some polynomial $p = p_{n,\epsilon}$ of deg $p = O (\ln n - \ln \epsilon)$ in the matrix entries $a_{ij}$. 

**Lemma**

Let \( g : \mathbb{C} \rightarrow \mathbb{C} \) be a polynomial and let \( \beta > 1 \) be real such that

\[
g(z) \neq 0 \quad \text{for all} \quad |z| < \beta.
\]

Let \( f(z) = \ln g(z) \) for \( |z| \leq 1 \) and let

\[
T_m(z) = f(0) + \sum_{k=1}^{m} \frac{f^{(k)}(0)}{k!} z^k
\]

be the Taylor polynomial of degree \( m \) of \( f \) computed at \( z = 0 \).

Then

\[
|f(1) - T_m(1)| \leq \frac{\deg g}{(m + 1)\beta^m(\beta - 1)}
\]

The error goes down exponentially in \( m \).
If \( g(z) \) has no zeros in the circle of radius \( \beta > 1 \) then to approximate \( f(z) = \ln g(z) \) for \( |z| \leq 1 \) within an additive error \( \epsilon \) one can use the Taylor polynomial of \( f(z) \) at \( z = 0 \) of degree \( m = O_\beta (\ln \deg g - \ln \epsilon) \).
The main lemma

Proof.

We factor $g(z) = g(0) \prod_{i=1}^{n} \left(1 - \frac{z}{\alpha_i}\right)$ where $|\alpha_i| \geq \beta$ for $i = 1, \ldots, n$ and $n = \deg g$. Then

$$f(z) = f(0) + \sum_{i=1}^{n} \ln \left(1 - \frac{z}{\alpha_i}\right)$$

and

$$\ln \left(1 - \frac{z}{\alpha_i}\right) = -\sum_{k=1}^{m} \frac{z^k}{k\alpha_i^k} + \xi_{i,m}$$

where

$$|\xi_{i,m}| \leq \frac{1}{(m + 1)\beta^m(\beta - 1)}$$

for all $i, m$ provided $|z| \leq 1$. \hfill \square
How to compute $f^{(k)}(0)$?

We have

$$f(z) = \ln g(z) \quad \text{so that} \quad f'(z) = \frac{g'(z)}{g(z)} \quad \text{and} \quad g'(z) = f'(z)g(z).$$

Therefore,

$$g^{(k)}(0) = \sum_{j=1}^{k} \binom{k-1}{j-1} f^{(j)}(0) g^{(k-j)}(0).$$

Suffices to compute $g^{(k)}(0)$ for $k = 0, \ldots, m$ and solve a non-degenerate triangular system of linear equations for $f^{(j)}(0)$. 

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The interpolation method
How does it work for the permanent?

Define

$$g(z) = \text{per} \left( \mathbf{1}_{n \times n} + z (A - \mathbf{1}_{n \times n}) \right),$$

so that

$$g(0) = \text{per} (\mathbf{1}_{n \times n}) = n!, \quad g(1) = \text{per} A$$

and

$$g(z) \neq 0 \quad \text{provided} \quad |z| \leq \beta = \frac{0.50}{0.49}.$$

Let

$$f(z) = \ln g(z) \quad \text{for} \quad |z| \leq 1.$$ 

Then

$$g^{(k)}(0) = \frac{d^k}{dz^k} \sum_{\sigma \in S_n} \prod_{i=1}^{n} \left( 1 + z \left( a_{i\sigma(i)} - 1 \right) \right) \bigg|_{z=0}$$

is a polynomial of degree $k$ in the entries of $A$ and hence $f^{(k)}(0)$ is.
For example:

\[ f^{(1)}(0) = \frac{1}{n} \sum_{i,j=1}^{n} (a_{ij} - 1) \]

and

\[ f^{(2)}(0) = \frac{1}{n(n-1)} \sum_{(i_1,j_1), (i_2,j_2)} (a_{i_1j_1} - 1)(a_{i_2j_2} - 1) - \left( \frac{1}{n} \sum_{i,j} (a_{ij} - 1) \right)^2, \]

e tc.
A trickier case

Theorem

Let $A = (a_{ij})$ be an $n \times n$ complex matrix such that

$$\delta \leq \Re a_{ij} \leq 2 - \delta \quad \text{and} \quad |\Im a_{ij}| \leq \frac{1}{2} \delta^2 \quad \text{for all} \quad i,j$$

and some $0 < \delta \leq 1$. Then

$$\text{per } A \neq 0.$$
Corollary

Fix $0 < \delta \leq 1$. Let $A = (a_{ij})$ be an $n \times n$ real matrix such that

$$\delta \leq a_{ij} \leq 2 - \delta$$

for all $i, j$.

Then $\text{per } A$ can be approximated within relative error $0 < \epsilon < 1$ in $n^{O(\ln n - \ln \epsilon)}$ time.

More precisely,

$$|\ln \text{per } A - p(A)| \leq \epsilon$$

for some polynomial $p = p_{\delta,n,\epsilon}$ of degree $p = O_\delta (\ln n - \ln \epsilon)$. 
As before, define

\[ g(z) = \text{per} \left( \mathbf{1}_{n \times n} + z(A - \mathbf{1}_{n \times n}) \right), \]

so that \( g(0) = n! \) and \( g(1) = \text{per} \ A. \)

This time, \( g(z) \neq 0 \) in a neighborhood of \([0, 1] \subset \mathbb{C} \).

What do we do?
Construct a disc of radius $\beta > 1$ and a polynomial $\phi$ such that $\phi(0) = 0$, $\phi(1) = 1$ and $\phi$ maps the disc inside the region where $g(z) \neq 0$.

Apply the main Lemma to $g(\phi(z))$. 

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The interpolation method
Let $g(z)$ be a polynomial of degree $n$ such that $g^{(k)}(0)$ can be computed in $n^{O(k)}$ time. Suppose we can throw a sufficiently wide sleeve from 0 to 1 completely avoiding the complex zeros of $g$.

Then $g(1)$ can be approximated within relative error $0 < \epsilon < 1$ in $n^{O(\ln n - \ln \epsilon)}$ time.
Interpretation: we can efficiently interpolate from a high (in fact, infinitely high) temperature to a cooler temperature, provided there is no phase transition on the way.

Phase transition: complex zeros of the partition function approaching the positive real axis, [Yang and Lee, 1952].

In a sense, this is “simulated annealing” without simulation. We, however, are allowed to use complex temperatures.
More examples: matching polynomial

Let $G = (V, E)$ be a graph and let

$$p_G(z) = 1 + \sum_{k > 0} (\text{the number of matchings with } k \text{ edges}) z^k$$

be its matching polynomial.
The roots of $p_G(z)$ are negative real with the largest root

$$z_0 \leq -\frac{1}{4(\Delta - 1)},$$

where $\Delta > 1$ is the largest degree of a vertex of $G$ [Heilmann and Lieb, 1972].

In the shaded area, $\ln p_G(z)$ can be approximated within an additive error $\epsilon > 0$ in $n^{O(\ln n - \ln \epsilon)}$ time.
When $z$ approaches the positive real axis, the complexity improves and eventually matches that of [Bayati, Gamarnik, Katz, Nair and Tetali, 2007] for $z > 0$.

When the largest degree $\Delta$ of a vertex is fixed in advance, the algorithm can be run in polynomial time $(n/\epsilon)^{O(1)}$ [Patel and Regts, 2016].
Let $G = (V, E)$ be a graph. A set $S \subseteq V$ is independent if no two vertices of $S$ span an edge of $G$. 

![Graph with independent set highlighted]
More examples: independence polynomial

Let

\[ g(z) = 1 + \sum_{k>0} (\text{the number of independent sets with } k \text{ vertices}) z^k. \]

In general, we have a) as it is hard to count independent sets. But sometimes (claw-free graphs, [Chudnovsky and Seymour, 2007]), we have b) and then it is easy to count.
Let $\Delta$ be the largest degree of a vertex of $G = (V, E)$. There are no roots of the independence polynomial in the disc $|z| \leq \frac{(\Delta - 1)(\Delta - 1)}{\Delta^{\Delta}} \approx \frac{1}{e\Delta}$ for large $\Delta$ [Dobrushin 1996], [Scott and Sokal 2005].

There are no roots in the strip $0 \leq \Re z \leq (1 - \delta)\frac{(\Delta - 1)(\Delta - 1)}{(\Delta - 2)^{\Delta}} \approx \frac{e}{\Delta}, \quad |\Im z| \leq \epsilon$, [Peters and Regts, 2017].
Consequently, there is a quasi-polynomial (genuinely polynomial if $\Delta$ is fixed – [Patel and Regts, 2016]) approximation algorithm for the independence polynomial inside the Dobrushin - Scott - Sokal disc and up to the correlation decay – [Weitz 2006] – bound if $z$ is real.
For an \( n \times n \times n \) tensor \( A = (a_{ijk}) \),

\[
\text{PER } A = \sum_{\sigma_1,\sigma_2 \in S_n} \prod_{i=1}^{n} a_{i\sigma_1(i)\sigma_2(i)}
\]

(counts perfect matchings in 3-partite hypergraph).
Efficiently approximable for complex \( A \) provided

\[
|1 - a_{ijk}| \leq \frac{\sqrt{6}}{9} \approx 0.27 \quad \text{for all } i, j, k.
\]
Efficiently approximable for real \( A \) provided

\[
|1 - a_{ijk}| \leq \sqrt{2} - 1 \approx 0.41 \quad \text{for all } i, j, k.
\]
For a graph $G = (V, E)$ and a $k \times k$ symmetric matrix $A$, let

$$\text{hom}_G(A) = \sum_{\phi: V \rightarrow \{1, \ldots, k\}} \prod_{\{u, v\} \in E} a_{\phi(u)\phi(v)}.$$ 

Efficiently approximable for complex $A$ provided

$$|1 - a_{ij}| < \max_{0 \leq \theta < 2\pi/3\Delta} \sin \frac{\theta}{2} \cos \frac{\theta\Delta}{2}$$

for all $i, j$ (for $\Delta = 3$, we get 0.18).

Efficiently approximable for real $A$ provided

$$|1 - a_{ij}| < \tan \frac{\pi}{9} \approx 0.36$$

for all $i, j$ if $\Delta = 3$

and

$$|1 - a_{ij}| < \tan \frac{\pi}{4(\Delta - 1)}$$

for all $i, j$ if $\Delta \geq 4$. 

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