Privacy and Geometry

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**A Statistical Database and Linear Queries**

- **Database**: $D \in U^n$.
  - collection of $n$ rows, one per individual
  - each row gives the type of the individual
  - universe $U$: all possible types
- E.g. $U = \{0, 1\}^d$: each individual described by $d$ binary attributes
A Statistical Database and Linear Queries

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- **Linear query**: for $q : \mathcal{U} \rightarrow [0, 1]$ and $D = \{r_1, \ldots, r_n\}$:

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- E.g. *counting queries* when $q : U \rightarrow \{0, 1\}$.
- **Workload**: a collection of linear queries $Q = \{q_1, \ldots, q_m\}$:
  
  $$Q(D) = \langle q_1(D), \ldots, q_m(D) \rangle.$$
Differential Privacy

Neighboring databases: $D$ and $D'$ that differ in at most one row.

Definition ([DMNS06])

An algorithm $A$ is $(\varepsilon, \delta)$-differentially private if for every two neighboring databases $D$, $D'$, and every measurable subset $S$ of the range of $A$, $A$ satisfies

$$\mathbb{P}[A(D) \in S] \leq e^{\varepsilon} \mathbb{P}[A(D') \in S] + \delta.$$
Measure of Error

Accuracy of algorithm $\mathcal{A}$ – *average (squared) error*:

$$\text{Err}(\mathcal{A}, \mathcal{Q}, n) = \max_{D \in \mathcal{U}^n} \left( \mathbb{E} \frac{1}{m} \sum_{i=1}^{m} (\mathcal{A}(\mathcal{Q}, D)_i - q_i(D))^2 \right)^{1/2};$$

$$= \max_{D \in \mathcal{U}^n} \left( \mathbb{E} \frac{1}{m} \|\mathcal{A}(\mathcal{Q}, D) - \mathcal{Q}(D)\|_2^2 \right)^{1/2};$$
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- Can define $\text{Err}$ with worst case error, i.e. $\| \cdot \|_\infty$. But average error is natural for geometric techniques.
Sample Complexity

What is the smallest database size for which we can guarantee error at most $\alpha$?

$$\text{sc}(A, Q, \alpha) = \min\{n : \text{Err}(A, Q, n) \leq \alpha\}$$

$$\text{sc}_{\varepsilon, \delta}(Q, \alpha) = \min\{\text{sc}(A, Q, \alpha) : A \text{ is } (\varepsilon, \delta) - \text{DP}\}$$
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Goal:
Characterize the sample complexity in terms of natural properties of $\mathcal{Q}$.

- Understand the “hardness” of $\mathcal{Q}$.
- Understand “optimal” algorithms.
The Sensitivity Polytope

**Sensitivity Polytope** $K_Q$ [HT10]:
- convex hull of $\pm Q(D)$ for all databases $D$ of size $n = 1$
- $D$ and $D'$ neighboring $\iff |D| \cdot Q(D) - |D'| \cdot Q(D') \in K_Q$.
- Describes how answers change between neighboring databases.
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- Describes how answers change between neighboring databases.

- Identify geometric measures of the “size” of $K_Q$ that characterize sample complexity/error.
Mean Point Problem

Can think of $D = \{r_1, \ldots, r_n\} \subseteq U$ as $\{x_1, \ldots, x_n\} \subset K_Q$:

$$x_i = Q(\{r_i\}) \quad Q(D) = \frac{1}{n} \sum_{i=1}^{n} x_i \in K_Q.$$ 

From now on we treat the Mean Point Problem (MPP):

- **Input:** $\{x_1, \ldots, x_n\} \subset K$
- **Approximate:** $\frac{1}{n} \sum_{i=1}^{n} x_i$

We will assume $K \subseteq [0, 1]^m$. Sample complexity is $\text{sc}_{\varepsilon, \delta}(K, \alpha)$. 

Algorithms for the MPP imply algorithms for query release. Sample complexity lower bounds for MPP imply lower bounds for query release, up to losing a $\text{poly}(1/\alpha)$ factor.
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Packing

- **α-Packing**: $y_1, \ldots, y_N \in K$ s.t. $i \neq j \implies \|y_i - y_j\|_2 \geq 2\alpha$.
- $N(K, \alpha) =$ size of the largest $\alpha$-packing.
**Packing**

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- \( N(K, \alpha) = \text{size of the largest } \alpha\text{-packing} \).
- **Claim**: \( sc_{\varepsilon,0}(K, \alpha) \geq \frac{1}{\varepsilon} \log\left( \frac{N(K, 2\alpha\sqrt{m})}{2} \right) \)
Packing Lower Bound

- $D_i = \{y_i, y_i \ldots, y_i\}$, where $\{y_1, \ldots, y_N\}$ is a $2\alpha\sqrt{m}$-packing.
Packing Lower Bound

\[ K \geq \frac{1}{2} e^{-\varepsilon n} \]

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Lower Bounds

**Packing Lower Bound**

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- Assume $\mathbb{P}(\frac{\|A(D_i) - y_i\|_2}{\sqrt{m}} \leq 2\alpha) \geq \frac{1}{2}$ for all $i$. By group privacy,

$$\mathbb{P}\left(\frac{\|A(D_1) - y_i\|_2}{\sqrt{m}} \leq 2\alpha\right) \geq e^{-\varepsilon n}\mathbb{P}\left(\frac{\|A(D_i) - y_i\|_2}{\sqrt{m}} \leq 2\alpha\right) \geq \frac{1}{2}e^{-\varepsilon n}.$$
### Packing Lower Bound

- **Diagrams and Formulas**
  - $D_i = \{y_i, y_i \ldots, y_i\}$, where $\{y_1, \ldots, y_N\}$ is a $2\alpha\sqrt{m}$-packing.
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    \]
  - Events are disjoint: $\frac{1}{2} Ne^{-\varepsilon n} \leq 1$. 

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Tightness of the Lower Bound

Using the exponential mechanism:

\[
\frac{1}{\varepsilon} \log N(K, 2\alpha \sqrt{m}) \lesssim sc_{\varepsilon,0}(K, \alpha) \lesssim \frac{1}{\varepsilon} \log N(K, \alpha \sqrt{m}/4)
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Lower Bounds

**Tightness of the Lower Bound**

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- The lower bounds hold for MPP, not query release. We lose a \( \frac{1}{\alpha^2} \) factor for query release.
- The upper bound is certified by an exponential time algorithm. A polynomial time algorithm [HT10, BDKT12, NTZ13] achieves the same result up to factors \( \text{poly}(\log m, \log |U|) \).
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- Recent: Analogous characterization using packing numbers for Concentrated Differential Privacy. (with Blasiok, Bun, Kattis, Steinke)
Kolmogorov width

\[ d_k(K) = \inf \{ \text{radius}(P_W K) : \text{co} - \dim(W) < k \} , \]

where \( P_W \) is the orthogonal projection onto \( W \).
Kolmogorov width and Approximate DP

\[ \max_{k \leq \varepsilon n} \frac{\sqrt{kd_k(K)}}{\varepsilon \sqrt{mn}} \lesssim \inf \{ \text{Err}(A, K, n) : A \text{ is } (\varepsilon, \delta) \text{ - DP} \} \]

\[ \lesssim (\log n)(\log 1/\delta)^{1/4}(\log |U|)^{1/4} \cdot \max_{k \leq \varepsilon n} \frac{\sqrt{kd_k(K)}}{\varepsilon \sqrt{mn}} \]
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[NTZ13, Nik15]:

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- The lower bound is a reconstruction attack.
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Kolmogorov width and Approximate DP

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\lesssim (\log n)(\log 1/\delta)^{1/4}(\log |U|)^{1/4} \cdot \max_{k \leq \varepsilon n} \frac{\sqrt{kd_k(K)}}{\varepsilon \sqrt{mn}}
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- The lower bound is a reconstruction attack.
- The upper bound is an efficient algorithm. (Running time polynomial in \(n, m, |U|\).)
- The \((\log |U|)^{1/4}\) gap is unavoidable if using Kolmogorov width. A geometric lower bound based on fingerprinting code attacks may be stronger for constant \(\alpha\) [KN17].
The $K$-norm Mechanism[HT10]

Output $\tilde{x}$ with density $p(z) \propto \exp(\varepsilon \|z - x\|_K)$, where

$$\|y\|_K = \inf\{ t : y \in tK \},$$

is the smallest scaling of $K$ that contains $y$. 
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is the smallest scaling of $K$ that contains $y$.

- The error is controlled by $(\mathbb{E}_{y \sim K}\|y\|_2^2)^{1/2}$.
- Deep results in convex geometry (the *slicing problem*) relate this quantity to the volume of $K$ and packing numbers.
The Projection Algorithm

**Gaussian Noise Algorithm** [DN03, DN04, DKM+06].

1. $x = \frac{1}{n} \sum_{i=1}^{n} x_i$

2. Sample $w \sim \mathcal{N}(0, \frac{m}{n^2} \sigma_{\varepsilon, \delta}^2)^m$, $\sigma_{\varepsilon, \delta} = O\left(\frac{\log(1/\delta)}{\varepsilon^2}\right)$.

3. Output $\tilde{x} = x + w$.

Bad when $n \ll m$!

**Idea:** We know $x \in K$. Use that!
The Projection Algorithm

**Projection Algorithm** $\mathcal{A}_{\text{proj}}$ [NTZ13]

1. $x = \frac{1}{n} \sum_{i=1}^{n} x_i$
2. Compute $\tilde{x} = x + w$, $w \sim N(0, \frac{m}{n^2} \sigma^2_{\varepsilon, \delta})^m$.
3. Output $\hat{x} = \arg\min\{\|x - \tilde{x}\|_2 : x \in K\}$.

![Diagram of the projection algorithm](image)
Bounding the Error

**Support function** of $K \subseteq \mathbb{R}^m$: $h_K(y) = \max\{\langle x, y \rangle : x \in K\}$.
- measures width of $K$ in the direction of $y$
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Main Fact:

$$\|\hat{x} - x\|_2^2 \leq |\langle \hat{x} - x, w \rangle| \leq 2h_K(w).$$

Key observation: $\theta$ is obtuse.
Bounding the Error

Support function of $K \subseteq \mathbb{R}^m$: $h_K(y) = \max \{ \langle x, y \rangle : x \in K \}$.
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Error controlled by mean width $M^*(K) = \mathbb{E}_z h_K(z)$ where $z$ is a random unit vector:

$$sc_{\varepsilon, \delta}(\mathcal{A}_{proj}, K, \alpha) \lesssim \frac{M^*(K) \sqrt{\log(1/\delta)}}{\varepsilon \alpha^2}.$$
Conclusion

*Geometric viewpoint* on differential privacy:

- the sensitivity polytope $K_Q$ gives a geometric picture of the sensitivity of the queries;
- geometric measures of the “size” of $K_Q$ that characterize sample complexity;
- geometric tools to design and analyze *algorithms*.
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Many open problems:
- Close gaps between upper and lower bounds.
- Data dependent results.
- Extend the theory to *non-linear* queries (convex optimization).
- Extend the theory to *interactive* mechanisms.
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Unconditional differentially private mechanisms for linear queries.

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C. Dwork, F. Mcsherry, K. Nissim, and A. Smith.
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Revealing information while preserving privacy.

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Privacy-preserving datamining on vertically partitioned databases.

Moritz Hardt and Kunal Talwar.
On the geometry of differential privacy.

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An improved private mechanism for small databases.