Trading Information Complexity for Error

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April 10, 2017

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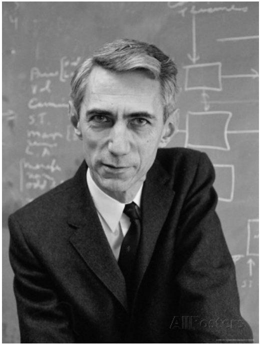
- Main question: How much one can save in information complexity by allowing an error of ϵ?
- How does $\mathrm{IC}_\mu(f, 0) - \mathrm{IC}_\mu(f, \epsilon)$ behave?
- How does $\mathrm{IC}(\mathrm{AND}, 0) - \mathrm{IC}(\mathrm{AND}, \epsilon)$ behave?

  [BGPW13]: We know $\mathrm{IC}(\mathrm{AND}, 0) \approx 1.4923$

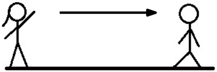
Information complexity

Continuation of Shannon's information theory.

Shannon (1916-2001)



Shannon's setting



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- She wants to transmit them to Bob.

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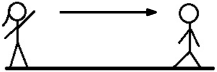

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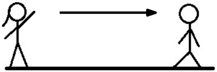

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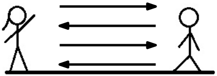
- She wants to transmit them to Bob.
- On average how many bits does she need to send?
- $$\lim_{n\to\infty} \frac{C_n(X)}{n} = \mathbb{H}(X),$$

  where $\mathbb{H}(X)$ is the entropy of X and it captures the amount of information in X.

Communication complexity



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- Alice receives X and Bob receives Y.

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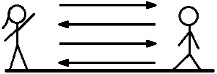

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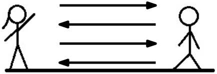

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- How many bits of communication is necessary?

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Theorem (Yao's minimax theorem)

$$\mathrm{CC}[f, \epsilon] = \max_\mu \mathrm{CC}[f, \mu, \epsilon].$$

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Information cost of a protocol is the amount of information Alice and Bob learn from the communicated bits Π about each other's inputs.

$$\mathrm{IC}_\mu(\pi) = I(X;\Pi|Y) + I(Y;\Pi|X)$$

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$$\mathrm{IC}_\mu(T) := \inf_\pi \mathrm{IC}_\mu(\pi) \qquad \text{where } \pi \text{ performs } T$$

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Theorem (Braverman-Rao'10: Amortized communication = Information complexity)

For $\epsilon > 0$,

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$$\mathrm{IC}(f,\epsilon) := \max_{\mu,\nu} \mathrm{IC}_\mu[f,\nu,\epsilon]$$

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and for every $0 < \alpha < 1$,

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[Braverman'12]: $\mathrm{IC}^D(f,\epsilon) = \mathrm{IC}(f,\epsilon)$?

Set Disjointness

$$S, T \subseteq \{1, \ldots, n\}, \qquad \mathrm{DISJ}_n(S, T) = \begin{cases} 1 & S \cap T = \emptyset \\ 0 & S \cap T \neq \emptyset \end{cases}$$

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Theorem (KS92, Raz92)

For $\epsilon < 1/2$,

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An idea: Computing $\mathrm{DISJ}_n(X, Y)$ is essentially equivalent to computing all $X_i \wedge Y_i$ (amortized).

Set Disjointness

Theorem (BGPW13)

$$\lim_{\epsilon \to 0} \mathrm{CC}[\mathrm{DISJ}_n, \epsilon] \approx \mathrm{IC}(\mathrm{DISJ}_n, 0) \approx 0.4827n + o(n)$$

where

$$0.4827... = \mathrm{IC}^0(\mathrm{AND}, 0) := \max_{\mu} \{\mathrm{IC}_{\mu}(\mathrm{AND}, 0) : \mu(1,1) = 0\}.$$

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[BGPW13]'s questions

- What is the behaviour of $\mathrm{IC}(\mathrm{AND}, 0) - \mathrm{IC}(\mathrm{AND}, \epsilon)$ as $\epsilon \to 0$?
- What is the behaviour of $\mathrm{IC}[\mathrm{DISJ}_n, 0] - \mathrm{IC}[\mathrm{DISJ}_n, \epsilon]$ as $\epsilon \to 0$?

Theorem (Dagan-Filmus-H-Li)

$$\mathrm{IC}(\mathrm{AND}, 0) - \mathrm{IC}(\mathrm{AND}, \epsilon) = \Theta(h(\epsilon)),$$

$$\mathrm{IC}[\mathrm{DISJ}_n, \epsilon] \text{ and } \mathrm{CC}[\mathrm{DISJ}_n, \epsilon] = 0.4827n - \Theta(h(\epsilon)n).$$

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- The lower bound $\mathrm{IC}_\mu(\mathrm{AND}, \epsilon) \geq \mathrm{IC}_\mu(\mathrm{AND}, 0) - \Theta(h(\epsilon))$ is difficult. We do not know what the optimal protocol is for $\mathrm{IC}_\mu(\mathrm{AND}, \epsilon)$.

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- The approach of [BGPW13] based on verifying some local convexity conditions does not seem to work.

Theorem (Dagan-Filmus-H-Li)

$$\mathrm{IC}^D[\mathrm{DISJ}_n, \epsilon] := \max_\mu \mathrm{IC}_\mu[\mathrm{DISJ}_n, \mu, \epsilon] =$$

$$n[\mathrm{IC}^0(\mathrm{AND}) - \Theta(\sqrt{h(\epsilon)})] + O(\log n).$$

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This with the previous theorem separates distributional prior-free and non-distributional prior-free information complexity and shows that Braverman's analysis is tight

$$\mathrm{IC}^D(\mathrm{DISJ}_n, \epsilon) \neq \mathrm{IC}(\mathrm{DISJ}_n, \epsilon).$$

Theorem (Dagan-Filmus-H-Li)

$$\mathrm{IC}_\mu[\mathrm{DISJ}_n, \mu, \epsilon] \le n[\mathrm{IC}^0(\mathrm{AND}) - \Theta(\sqrt{h(\epsilon)})] + O(\log n).$$

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- Alice and Bob try to compute $(X_i \wedge Y_i)$ for all $i = \sigma_1, \sigma_2, \ldots$ for a random permutation σ of $\{1, 2, \ldots, n\}$.
- To compute $(X_i \wedge Y_i)$, they run an (almost) optimal for $\mathrm{IC}_{\nu_{\sigma_i}}(\mathrm{AND}, \epsilon/2p, 1 \to 0)$, where
    - $1 \to 0$ means one-sided error.
    - $p = \Pr_\mu[\mathrm{DISJ}_n(X, Y) = 1]$.
    - ν_{σ_i} is the corresponding marginal conditioned on the event that the protocol has not yet terminated.

Open Problems

We proved

$$\mathrm{IC}^0(\mathrm{AND}, \epsilon) \leq \frac{\mathrm{IC}(\mathrm{DISJ}_n, \epsilon)}{n} \leq \frac{\mathrm{IC}(\mathrm{DISJ}_n, \epsilon, 1 \to 0)}{n} \leq \mathrm{IC}^0(\mathrm{AND}, \epsilon, 1 \to 0)$$

- Here $1 \to 0$ denote one-sided error (if output is 0 we are always correct).
- IC^0 means we only consider μ with $\mu(1,1) = 0$.

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Conjecture

$$\frac{\mathrm{IC}(\mathrm{DISJ}_n,\epsilon)}{n} = \mathrm{IC}^0(\mathrm{AND},\epsilon,1\to 0) \pm o(1).$$

- We prove that for every function
$\mathrm{IC}_\mu(f,0) - \Theta(\sqrt{h(\epsilon)}) \leq \mathrm{IC}_\mu(f,\epsilon) \leq \mathrm{IC}_\mu(f,0) - \Theta(h(\epsilon))$.

  Is the upper-bound always the truth?