DERANDOMIZING ISOLATION LEMMA: A GEOMETRIC APPROACH

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Based on joint works with Stephen Fenner and Thomas Thierauf

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For any weight function $w : E \to \mathbb{Z}$, define for any $S \subseteq E$,

$$w(S) = \sum_{e \in S} w(e).$$
For any weight function $w : E \rightarrow \mathbb{Z}$, define for any $S \subseteq E$,

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**Isolation Lemma (Mulmuley, Vazirani, Vazirani 1987)**

Let $B \subseteq 2^E$. 

**Applications:**
- Perfect Matching
- Linear Matroid Intersection in RNC
- Polynomial Identity Testing
- SAT to Unambiguous-SAT [VV86]
- $NL/poly \subseteq UL/poly$ [RA00]
- Disjoint Paths($s_1, t_1, s_2, t_2$) in RP [BH14]
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Let $B \subseteq 2^E$. For each $e \in E$, assign a random weight from $\{1, \ldots, 2|E|\}$. Then with probability $\geq 1/2$ there is a unique minimum weight set in $B$. Applications: Perfect Matching, Linear Matroid Intersection in RNC, Polynomial Identity Testing, SAT to Unambiguous-SAT [VV86], NL/poly $\subseteq$ UL/poly [RA00], Disjoint Paths$(s_1, t_1, s_2, t_2)$ in RP [BH14].
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**Introduction**

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  - The set of perfect matchings of a given graph.
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- Randomized arguments show existence for such families.
Deterministic Isolation is known for

- Sparse families.
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- Perfect matchings in a bipartite graph (quasi-poly) [FGT16].
- Common Independent sets two matroids (quasi-poly) [GT17].
- Minimum vertex covers in a bipartite graph (quasi-poly).
For a set $S \subseteq E$, define $x^S \in \mathbb{R}^E$

$$x_e^S = \begin{cases} 1, & \text{if } e \in S, \\ 0, & \text{otherwise.} \end{cases}$$
Polytope of a Family

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\( w \cdot x^S = w(S) \), for any \( S \subseteq E \).

**Observation**

\( w \) is isolating for \( B \)

\[\iff\]

\( w \cdot x \) has a unique minima over \( P(B) \).
Goal: $w \cdot x$ has a unique minima over $P(B)$ (small weights).
Isolation over the polytope

- **Goal:** \( w \cdot x \) has a unique minima over \( P(B) \) (small weights).
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- For any $w \in \mathbb{R}^E$, points minimizing $w \cdot x$ in $P(B)$ is a face of the polytope $P(B)$.
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In each round, slightly modify the current weight function to get a smaller minimizing face.
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  - points minimizing $w \cdot x$ in $P(B) = a$ face of the polytope $P(B)$.
- In each round, slightly modify the current weight function to get a smaller minimizing face.
- We stop when we reach a zero-dimensional face.
Modifying $w$

- Let $F_w$ be the minimizing face for $w \cdot x$. 

Weights grow as $N^r$ in the $r$-th round.

We will have $\log n$ rounds.
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**Claim**

Let $w_1 = w \times N + w'$, where $\|w'\|_1 < N$.
Then $F_{w_1} \subseteq F_w$. 
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Clearly, $w_0 \cdot v = 0$.

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$v$ is not parallel to $F_1$.

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Constructing $w$ [FKS84]

- $v_1, v_2, \ldots, v_k \in \{- (t - 1), \ldots, 0, 1, \ldots, t - 1\}^m$. 

Easy to construct a function $w$ such that $w \cdot v_i \neq 0$ for each $i \in [k]$. Define $W := (1, t, t^2, \ldots, t^{m - 1})$. Clearly, $W \cdot v_i \neq 0$ for each $i$. Try weight functions $W \mod j$ for $2 \leq j \leq mk \log t$. The construction is blackbox.
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- \( F_{\log m}: \) no length-\( m \) vectors, hence, the face is a corner.
Geometric Approach for Isolation

Sufficient condition for Isolation

Let $F$ be described by $Ax = b, Cx \leq d$.

$L_F = \{x \in \mathbb{Z}^m | Ax = 0\}$.

Let $\lambda_1(L_F)$ be the length of the shortest vector in $L_F$.

Sufficient condition for Isolation

For all faces $F$ of $P(B)$, the number of vectors in $L_F$ of length $\leq 2\lambda_1(L_F)$ is $\text{poly}(m)$. 
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Sufficient condition for Isolation

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Sufficient condition for Isolation

For all faces $F$ of $P(\beta)$, the number of vectors in $L_F$ of length $\leq 2\lambda_1(L_F)$ is poly($m$).
Perfect Matching

Perfect matching polytope

- $\mathcal{B} =$ the set of all perfect matchings in $G(V, E)$.
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**Lemma**

For a graph $H$ with $n$ nodes,

No cycles of length $\leq r$

\[\Downarrow\]

number of cycles of length upto $2r$ is $\leq n^4$. 
Matroid Intersection

- Given two $n \times m$ matrices $A$ and $B$
- $I \subseteq [m]$ is a common base if $\text{rank}(A_I) = \text{rank}(B_I) = n$. 
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- $\mathcal{B}$ = set of common bases.
- $P(\mathcal{B})$ is given by [Edmonds 1970]

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\begin{align*}
\sum_{e \in E} x_e &\geq 0 \quad e \in E, \\
\sum_{e \in S} x_e &\leq \text{rank}(A_S) \quad S \subseteq [m], \\
\sum_{e \in S} x_e &\leq \text{rank}(B_S) \quad S \subseteq [m], \\
\sum_{e \in [m]} x_e &= n.
\end{align*}
\]
For any face $F$ there exist

- $[m] = A_0 \sqcup S_1 \sqcup S_2 \sqcup \cdots \sqcup S_p$
- $[m] = A_0 \sqcup T_1 \sqcup T_2 \sqcup \cdots \sqcup T_q$ and
- positive integers $n_1, n_2, \ldots, n_p$ and $m_1, m_2, \ldots, m_q$
- with $\sum_i n_i = \sum_j m_j = n$

\[
\begin{align*}
x_e &= 0 \quad \forall e \in A_0 \\
\sum_{e \in S_i} x_e &= n_i \quad \forall i \in [p] \\
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Discussion

● For what other polytopes this approach would work?
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Matchings in General graphs.
Discussion

- For what other polytopes this approach would work?
- Matchings in General graphs.
- NP-complete problems?