

# Packing degenerate graphs using pseudorandomness

Julia Böttcher

London School of Economics

Simons Institute, "Proving and Using Pseudorandomness" March 2017

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 but we need more: parallel classes which are themselves partitions of the vertices into disjoint blocks (resolvable)

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#### Theorem

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Ray-Chaudhuri and Wilson show an analogous result for S(2, k, n) for all k.

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For every  $1 \le t \le k$ ,  $\varepsilon > 0$ , and *n* large: There is a partition of  $K_n^{(t)}$  into edge-disjoint  $K_k^{(t)}$  and a leftover edge-set of size  $\le \varepsilon n^t$ .

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- in a first round choose few K<sub>k</sub><sup>(t)</sup>-copies randomly and of these select only those without overlaps
- delete the edges in the selected copies
- continue with a second round in the remainder, and so on

### Breakthrough:

#### Theorem

For large *n*, if the obvious divisibility conditions are satisfied, then a Steiner system S(t, k, n) exists.

Divisibility conditions:  $\binom{k-i}{t-i}$  should divide  $\binom{n-i}{t-i}$  for  $1 \le i \le t-1$ 

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with a random process we will have a leftover

 so let's be prepared for this: before starting the process, find some clever structure that can absorb any leftover

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Recently: alternative proof and more

# **Graph Packings**

## Definition

PACKING

A family  $H_1, \ldots, H_k$  of graphs packs into a graph G, if there are pairwise edge-disjoint copies of  $H_1, \ldots, H_k$  in G.

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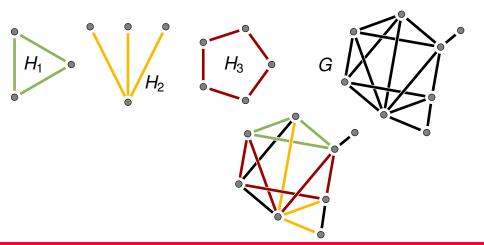


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## Conjecture

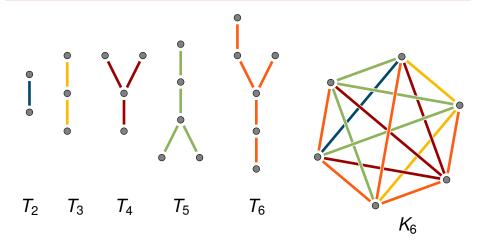
GYÁRFÁS & LEHEL '76

#### Any family $T_1, T_2, \ldots, T_n$ of trees with $v(T_i) = i$ packs into $K_n$ .

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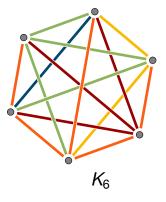
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Magic:

1. perfect packing: 
$$\sum_{i=1}^{n} e(T_i) = {n \choose 2}$$



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into  $K_{2n+1}$ .

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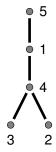
For every tree *T* on n + 1 vertices, there is a packing of 2n + 1 copies of *T* into  $K_{2n+1}$ .

- also gives a perfect packing
- bipartite versions of these packing conjectures exist

#### Definition

GRACEFUL LABELLING

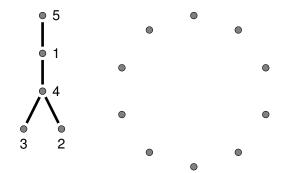
An injection  $f: V(H) \rightarrow \{1, \dots, e(H) + 1\}$  is graceful if the induced edge labels |f(x) - f(y)| for  $xy \in E(H)$  are distinct.



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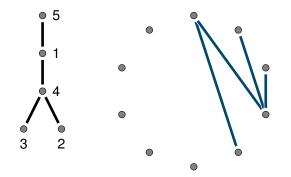
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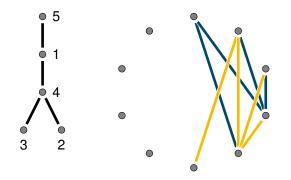
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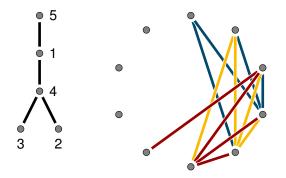
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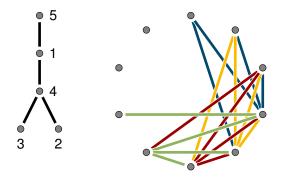
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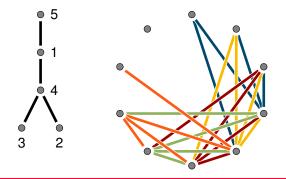
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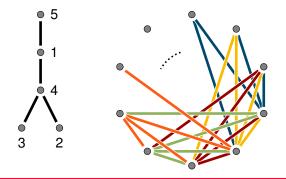
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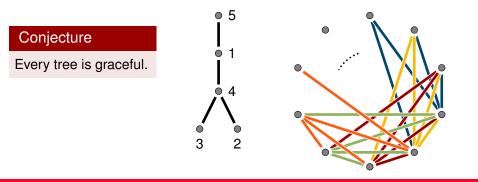
An injection  $f: V(H) \rightarrow \{1, ..., e(H) + 1\}$  is graceful if the induced edge labels |f(x) - f(y)| for  $xy \in E(H)$  are distinct.



#### Definition

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# What was known (until a few years ago)?

#### Conjecture

GYÁRFÁS & LEHEL '76

#### Any family $T_1, T_2, \ldots, T_n$ of trees with $v(T_i) = i$ packs into $K_n$ .

paths & stars; all but two trees are stars	Gyárfás & Lehel '76
$T_{n-2}, T_{n-1}, T_n$	Hobbs, Bourgeois & Kasiraj '87
all but three trees are stars	Roditty '88
• $T_1, \ldots, T_s$ with $s < \lfloor n/\sqrt{2} \rfloor$	Bollobás '83
trees of small diameter which have	
a vertex with many leaf-children	DOBSON '97,'02,'07
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#### Conjecture

Rosa '67

Every tree is graceful.

paths and caterpillars, firecrackers, banana trees, olive trees, ...
 trees of diameter at most 7

#### A near-perfect version of the Tree Packing Conjecture

near-perfect packing: uses all but a small proportion of the host graph.

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#### Theorem

B, HLADKÝ, PIGUET, TARAZ '16

For all  $\varepsilon > 0$ ,  $\Delta \in \mathbb{N}$  there is  $n_0 \in \mathbb{N}$  such that for  $n \ge n_0$ : Let  $T_1, \ldots, T_t$  be a family of trees with

$$v(T_i) \le n,$$

$$\sum_{i=1}^t e(T_i) \le {n \choose 2}$$

$$\Delta(T_i) \le \Delta.$$

Then  $T_1, \ldots, T_t$  pack into  $K_{(1+\varepsilon)n}$ .

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Also gives near-perfect version for the conjecture of Ringel:

• 2n + 1 copies of a tree *T* with v(T) = n + 1 pack into  $K_{2n+1}$ .

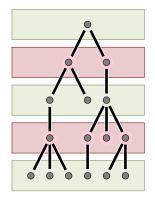
# Idea of near-perfect tree packing

Let T be a tree, G a host graph.

- even layers of T: primary vertices
- odd layers of T: secondary vertices

Random process:

- 1. map primary vertices randomly to V(G),
- 2. map secondary vertices randomly into neighbourhoods



Near-perfect packing results:

 almost spanning bounded degree graphs from any nontrivial minor-closed family
 MESSUTI, RÖDL AND SCHACHT '16

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#### Breakthrough:

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JOOS, KIM, KÜHN AND OSTHUS

For  $\Delta$  fixed and *n* sufficiently large, the Tree Packing Conjecture holds for trees  $T_2, \ldots, T_n$  of maximum degree at most  $\Delta$ .

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Near-graceful labelings: for trees with maximum degree  $O(n/\log n)$ 

ADAMASZEK, ALLEN, GROSU AND HLADKÝ

Ferber. Lee and Mousset

KIM. KÜHN, OSTHUS AND TYOMKYN

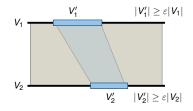
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## The idea of perfect tree packing with bounded degrees

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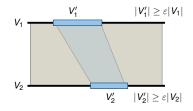
Blow-Up Lemma

Komlós, Sárközy, Szemerédi, '97

For  $\Delta$  fixed, in an ( $\varepsilon$ , d)-superregular pair ( $V_1$ ,  $V_2$ ), we can embed any bipartite H with classes  $X_1$  and  $X_2$  with  $|X_i| = |V_i|$  and max. degree  $\leq \Delta$ .

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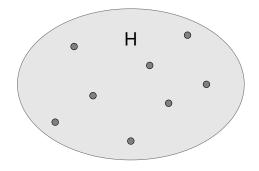


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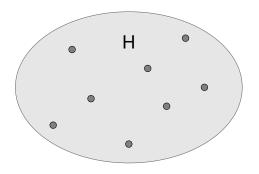
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Packing version: analogous result for near-perfect packing of such H



For packing  $G_1, \ldots, G_t$  in H:

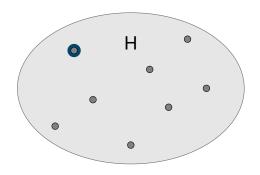
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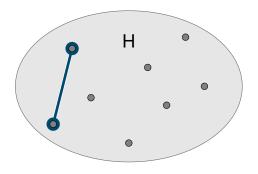
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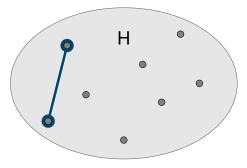
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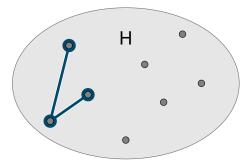
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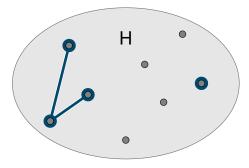
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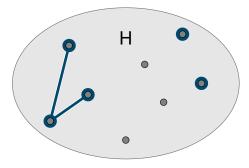
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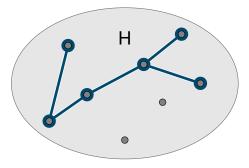
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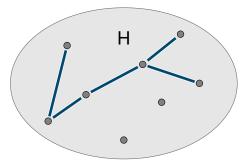
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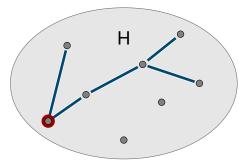
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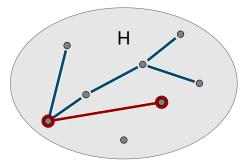
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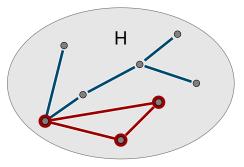
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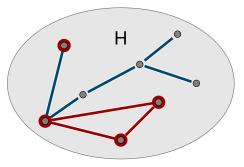
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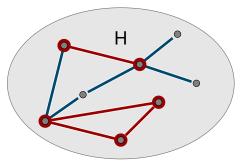
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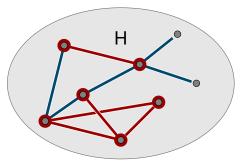
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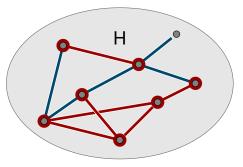
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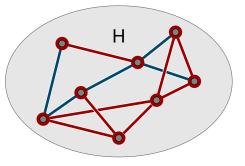
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Hope: after embedding some  $G_i$ , remainder of *H* is quasirandom



## A near-perfect packing of spanning degenerate graphs

*G* is *D*-degenerate if its vertices can be ordered  $x_1, \ldots, x_n$  such that  $x_i$  has at most *D* neighbours among  $x_1, \ldots, x_{i-1}$  for all *i*.

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#### Theorem

Allen, B, Hladký, Piguet

For all  $\varepsilon > 0$ ,  $D \in \mathbb{N}$  there are c > 0 and  $n_0 \in \mathbb{N}$  such that for  $n \ge n_0$ : Let  $G_1, \ldots, G_t$  be a family of <u>D</u>-degenerate graphs with

$$v(G_i) \le n,$$

$$\sum_{i=1}^t e(G_i) \le (1-\varepsilon) {n \choose 2},$$

$$\Delta(G_i) \le cn/\log n.$$

Then  $G_1, \ldots, G_t$  pack into  $K_n$ .

covers more general graph class than all previous near-perfect packing results

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  - we need to control sequential dependencies
  - for extending each  $G'_i$  to a copy of  $G_i$ :
    - before starting the process, reserve  $\frac{1}{2}\varepsilon\binom{n}{2}$  random edges  $H^*$  of  $K_n$
    - choose  $G'_i \setminus G_i$  as independent set
    - use a matching argument to show  $G'_i$  can be completed in  $H^*$

# Sequential dependencies

#### Lemma

#### Let

- $\Omega$  be a finite probability space,
- $(\mathcal{F}_0, \mathcal{F}_1, \ldots, \mathcal{F}_n)$  be partitions of  $\Omega$ , with  $\mathcal{F}_i$  refining  $\mathcal{F}_{i-1}$ .
- $Y_i$  be nonnegative random variables, constant on each part of  $\mathcal{F}_i$ .
- $\mathcal{E}$  be an event.

Suppose that almost surely, either

 $\mathcal{E}$  does not occur, or

 $\sum_{i=1}^{n} \mathbb{E} [Y_i | \mathcal{F}_{i-1}] = \mu \pm \nu, \sum_{i=1}^{n} \operatorname{Var} [Y_i | \mathcal{F}_{i-1}] \leq \sigma^2, \text{ and } 0 \leq Y_i \leq R$ 

Then

$$\mathbb{P}\left[\mathcal{E} \text{ and } \sum_{i=1}^{n} Y_{i} \neq \mu \pm (\nu + \varrho)\right] \leq 2\exp\left(-\frac{\varrho^{2}}{2\sigma^{2} + 2R\varrho}\right).$$

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#### Many thanks!