Data Structures for Semistrict Higher Categories

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arXiv:1610.06908

Compositionality Workshop
Simons Institute, University of California, Berkeley, USA
5 December 2016

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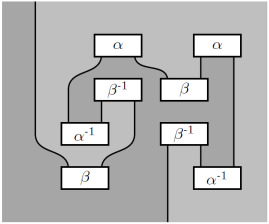
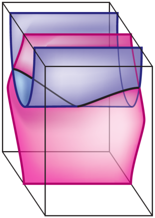
Higher-dimensional algebra lets us compose in the plane, or in higher dimensions:





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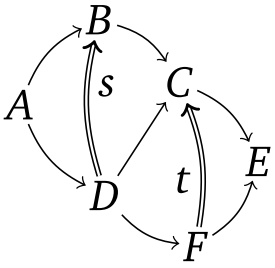

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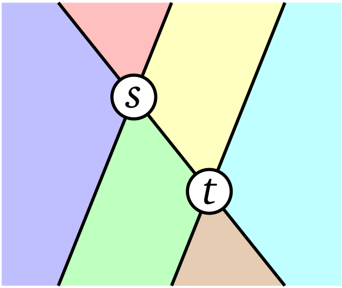

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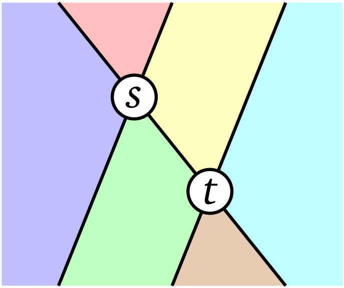

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We expect n-categories to have an n-dimensional graphical calculus.

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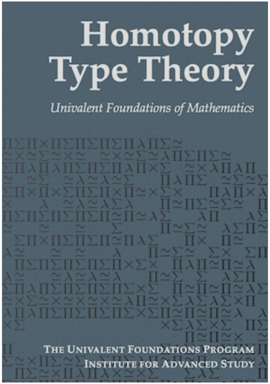

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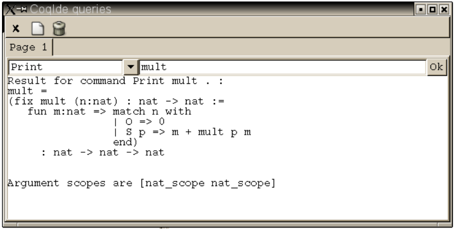
Proof assistants like Coq and Agda can’t always help, because they use 1-dimensional algebra.



We need an alternative that brings out higher category theory’s geometrical essence.

Globular

Our solution is *Globular*, a proof assistant for higher category theory, available now at this address:

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- Supports proofs up to the level of semistrict 4-categories.

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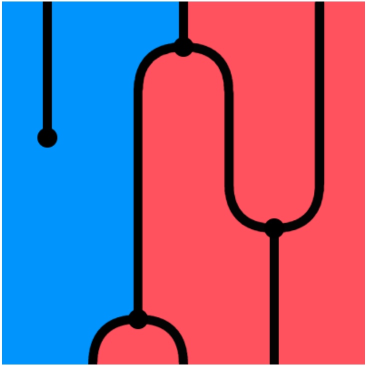

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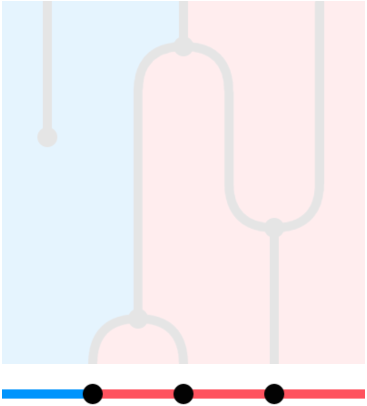

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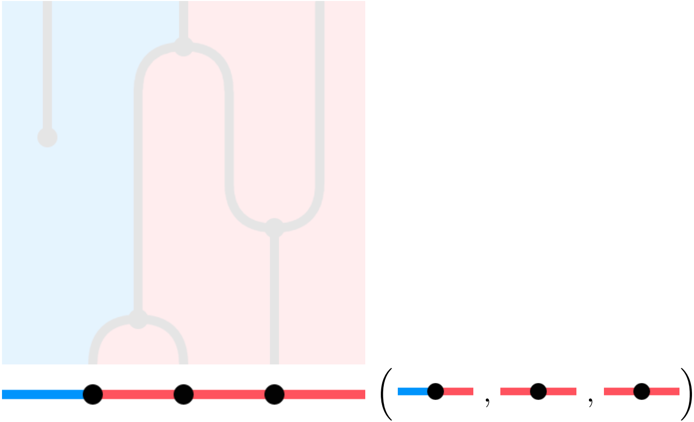

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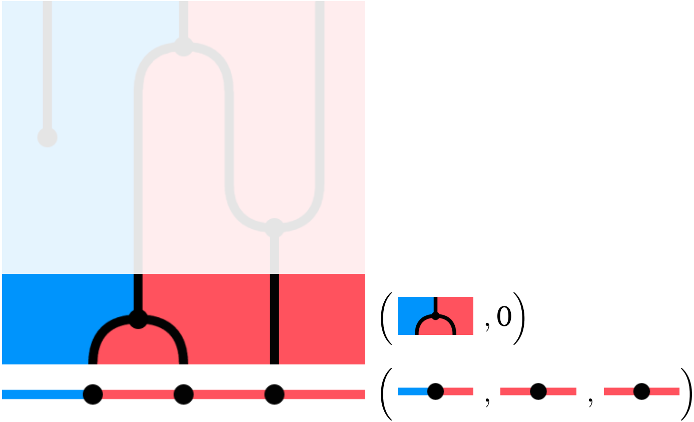

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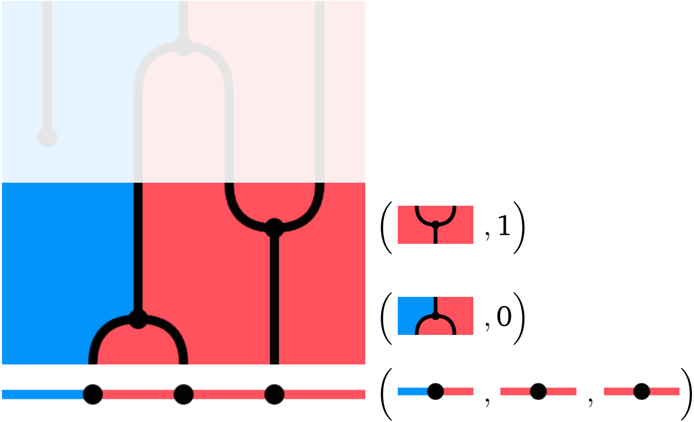

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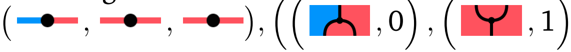

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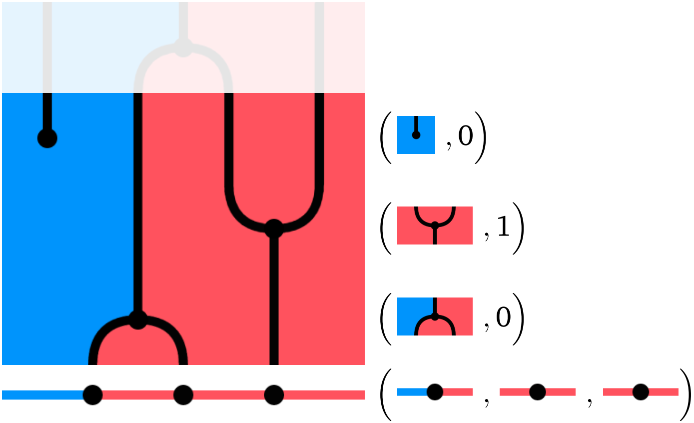

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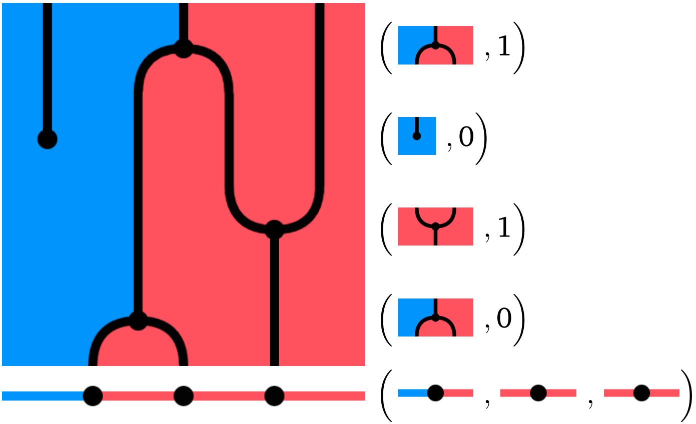

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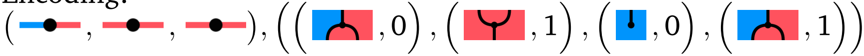

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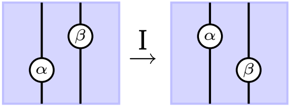

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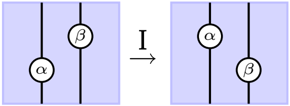

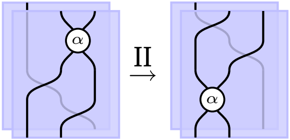

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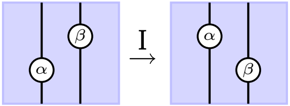

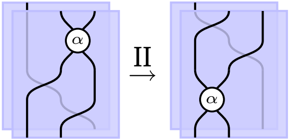

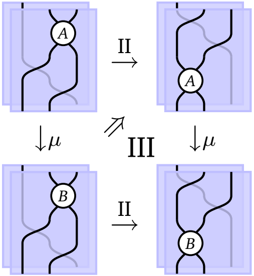

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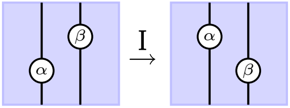

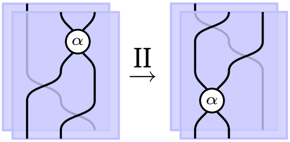

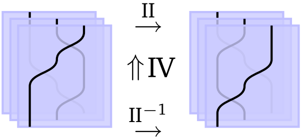

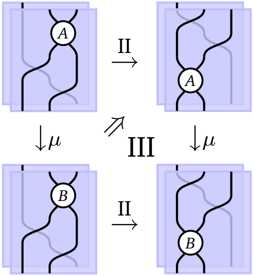

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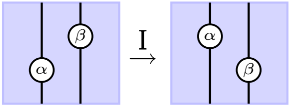

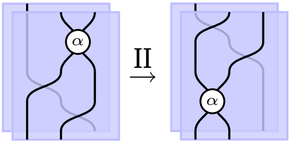

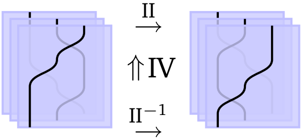

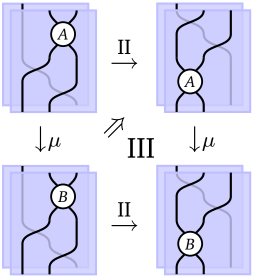

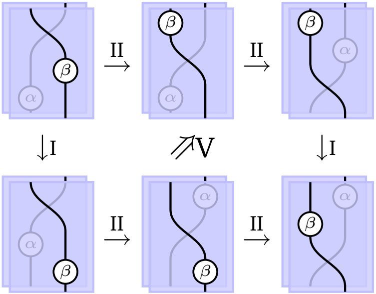

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