Data Structures for Semistrict Higher Categories

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Higher-dimensional algebra

Ordinary algebra lets us compose along a line:

\[ xy^2 y x^3 z \]
Higher-dimensional algebra

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*Higher-dimensional* algebra lets us compose in the plane, or in higher dimensions:
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The graphical calculus representation is the dual diagram.

We expect $n$-categories to have an $n$-dimensional graphical calculus.
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Proof assistants like Coq and Agda can’t always help, because they use 1-dimensional algebra.

We need an alternative that brings out higher category theory’s geometrical essence.
Globular

Our solution is *Globular*, a proof assistant for higher category theory, available now at this address:

http://globular.science
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- Supports proofs up to the level of semistrict 4-categories.
Signature and diagram structures

A signature is a list of allowed moves, given as source-target pairs.
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Encoding:

\[
(\text{---}, \text{---}, \text{---})
\]
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Encoding: 

\[ \left( \begin{array}{c} \text{\includegraphics[width=2cm]{signature1.png}} \\ 0 \end{array} \right), \left( \begin{array}{c} \text{\includegraphics[width=2cm]{signature2.png}} \\ \text{\includegraphics[width=2cm]{signature3.png}} \\ \text{\includegraphics[width=2cm]{signature4.png}} \end{array} \right) \]
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![Diagram](image)

**Encoding:**

\[(\text{blue, 1), (red, 0), (black, black, black)}\]

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Encoding:

\[
\begin{align*}
&(\text{\includegraphics[width=0.05\textwidth]{signature_diagram1.png}}, 0) \\
&(\text{\includegraphics[width=0.05\textwidth]{signature_diagram2.png}}, 1) \\
&(\text{\includegraphics[width=0.05\textwidth]{signature_diagram3.png}}, 0) \\
&(\text{\includegraphics[width=0.05\textwidth]{signature_diagram4.png}}, 0) \\
\end{align*}
\]
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Encoding:
\[(\text{fig 1, 0}, \text{fig 2, 1}), (\text{fig 3, 0}), (\text{fig 4, 1}), (\text{fig 5, 0}), (\text{fig 6, 1})\]
Homotopies

We also need *homotopy moves*:
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\[ \alpha \rightarrow \beta \quad \text{I} \quad \alpha \rightarrow \beta \]

\[ \alpha \rightarrow \beta \quad \text{II} \quad \alpha \]
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\[
\begin{align*}
\alpha & \beta \\
\alpha & \beta \\
\alpha & \beta \\
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We also need homotopy moves:

1. $\alpha \beta \xrightarrow{\text{I}} \alpha \beta$
2. $\alpha \xrightarrow{\text{II}} \alpha$
3. $\alpha \xrightarrow{\text{III}} \alpha$
4. $\xrightarrow{\text{IV}}$

$\Rightarrow$

$\xrightarrow{\text{II}}$

$\xrightarrow{\text{III}}$

$\xrightarrow{\text{II}}$

$\downarrow \mu$

$\xrightarrow{\Rightarrow}$

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\xrightarrow{II} \alpha \beta 
\xrightarrow{IV} \alpha \beta 
\xrightarrow{II^{-1}} 

2. $A 
\xrightarrow{II} A$
\xrightarrow{III} $A$

3. $B 
\xrightarrow{II} B$
\xrightarrow{II} $B$

4. $\alpha \beta 
\xrightarrow{V} \alpha \beta 
\xrightarrow{I} \alpha \beta 
\xrightarrow{II} \alpha \beta 
\xrightarrow{II} \alpha \beta$
Semistrict $n$-categories

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Thank you!