Compositionality
Simons Institute, 8 December 2016
Fix a *finite relational vocabulary*: $\tau = (R_1, \ldots, R_m)$. and consider finite $\tau$-structures

$$\mathbb{A} = (A, R_1^A, \ldots, R_m^A)$$

$$\mathbb{B} = (B, R_1^B, \ldots, R_m^B)$$

As a special case, we have graphs, where $\tau$ consists of a single binary relation $E$. 
Homomorphism and Isomorphism

\( A \xrightarrow{\text{hom}} B \): there is \( h : A \rightarrow B \) s.t. for any \( a \):

\[
R^A(a) \Rightarrow R^B(h(a)).
\]

\( A \cong B \): there is a \textit{bijection} \( h : A \rightarrow B \) s.t. for any \( a \):

\[
R^A(a) \Leftrightarrow R^B(h(a)).
\]

Or, equivalently \( A \cong B \) if there are \( h : A \xrightarrow{\text{hom}} B \) and \( g : B \xrightarrow{\text{hom}} A \) such that

\[
h \circ g = \text{id}_B \quad \text{and} \quad g \circ h = \text{id}_A
\]
The problem of deciding, given \( A \) and \( B \), whether \( A \xrightarrow{\text{hom}} B \) is NP-complete.

The problem of deciding, given \( A \) and \( B \), whether \( A \cong B \) is

- not known to be NP-complete;
- not known to be in \( P \);
- known to be in quasi-polynomial time \( (\text{Babai 2016}) \)

The \textit{k-local consistency} test gives an algorithm, running in time \( n^{O(k)} \) that gives an \textit{approximate} test for \( A \xrightarrow{\text{hom}} B \).
Finite Variable Logic

$\exists^+ k\text{FO}$: existential, positive formulas of first-order logic, using no more than $k$ distinct variables.

$$\exists x_1 \cdots \exists x_k \bigwedge_{i \neq j} E(x_i, x_j)$$

In $\exists^+ k\text{FO}$ we can express the existence of a $k$-clique, but not a $(k + 1)$-clique.

$$\exists x_1 \exists x_2 E(x_1, x_2) \land (\exists x_1 E(x_2, x_1) \land \cdots)$$

In $\exists^+ 2\text{FO}$, we can express the existence of a path of length $n$ for any $n$. 

Anuj Dawar
December 2016
$k$-local Consistency

Write $A \equiv^k B$ to denote that for any sentence $\varphi$ of $\exists^+ \cdot k \text{FO}$

$$\text{if } A \models \varphi \text{ then } B \models \varphi.$$  

The $k$-local consistency test determines whether $A \equiv^k B$

$$A \overset{\text{hom}}{\rightarrow} B \iff A \equiv^n B \Rightarrow A \equiv^k B$$

where $|A| = n$ and $n > k$. 

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December 2016
The relation $A \equiv^k B$ has a pebble game characterization due to (Kolaitis-Vardi 1992).
The game is played by two players—Spoiler and Duplicator—using $k$ pairs of pebbles $\{(a_1, b_1), \ldots, (a_k, b_k)\}$.

**Spoiler** moves by picking a pebble $a_i$ and placing it on an element of $A$.

**Duplicator** responds by placing $b_i$ on an element of $B$.

**Spoiler** wins at any stage if the partial map from $A$ to $B$ defined by the pebble pairs is not a partial homomorphism.

If **Duplicator** has a strategy to play forever without losing, then $A \equiv^k B$. 
Duplicator can compose strategies witnessing \( A \equiv^k B \) and \( B \equiv^k C \) to get one for \( A \equiv^k C \).
Strategies more formally

A *strategy* for \( A \equiv \rangle^k B \) is a set \( H \) of pairs \((a, b)\) where \( a \) and \( b \) are \( l \)-tuples of elements from \( A \) and \( B \) respectively for some \( 0 \leq l \leq k \), such that:

1. for each \((a, b) \in H\), the map \( a \mapsto b \) is a partial homomorphism;
2. if \((a, b) \in H\), then \((a', b') \in H\) whenever \( a' \) and \( b' \) are obtained by deleting corresponding elements of \( a \) and \( b \); and
3. if \((a, b) \in H\) and \(|a| = |b| = l < k\), then there is a function \( f : A \rightarrow B \) so that for each \( a \in A\), \((aa, bf(a)) \in H\).

\[ \text{id}_A : A \equiv \rangle^k A \] is the strategy consisting of all pairs \((a, a)\).

Say that a strategy \( H : A \equiv \rangle^k B \) is *injective* if the function \( f \) in (2) can always be chosen to be injective.
The following are equivalent for any $A$ and $B$:

1. There are strategies $H : A \equiv^k B$ and $I : B \equiv^k A$ such that $I \circ H = \text{id}_A$ and $H \circ I = \text{id}_B$.

2. There are injective strategies $H : A \equiv^k B$ and $I : B \equiv^k A$.

3. There is a \textit{bijective} strategy $H : A \equiv^k B$.

The last condition amounts to saying the \textit{Duplicator} has a winning strategy in the \textit{bijection game}. \hfill (Hella 1996)
Hella’s bijection game characterizes the equivalence $A \equiv^k B$, which says that the two structures cannot be distinguished by any sentence of $C^k$—$k$-variable first-order logic with counting quantifiers.

This equivalence relation has many independent characterizations. $G \equiv^k H$ for a pair of graphs $G, H$ iff they cannot be distinguished by the $(k - 1)$-dimensional Weisfeiler-Leman method. This is a much studied approximation of graph isomorphism.
A structure $\mathcal{A}$ is a core if there is no proper substructure $\mathcal{A}' \subseteq \mathcal{A}$ such that $\mathcal{A} \xrightarrow{\text{hom}} \mathcal{A}'$.

Every structure $\mathcal{A}$ has a core $\mathcal{A}' \subseteq \mathcal{A}$ such that $\mathcal{A} \xrightarrow{\text{hom}} \mathcal{A}'$. Moreover, $\mathcal{A}'$ is unique up to isomorphism.

Say $\mathcal{A}'$ is a $k$-core of $\mathcal{A}$ if:

1. $\mathcal{A} \equiv^k \mathcal{A}'$;
2. $\mathcal{A}' \equiv^k_{\text{inj}} \mathcal{A}$;
3. for any $\mathcal{B}$, if $\mathcal{A} \equiv^k \mathcal{B}$ and $\mathcal{B} \equiv^k_{\text{inj}} \mathcal{A}$ then $\mathcal{A}' \equiv^k_{\text{inj}} \mathcal{B}$.

Every structure $\mathcal{A}$ has a $k$-core and it is unique up to $\equiv^k$. 
Some Questions

If $C$ is a class of structures closed under $\equiv^k$ and \textit{homomorphisms}, is it closed under $\equiv^{k'}$; or $\equiv^{k'}$ for some $k'$?

Can we extract suitable \textit{isomorphism tests} from other approximations of homomorphism given by algebraic constraint satisfaction algorithms? Conversely, what homomorphism approximations do we get from group-theoretic methods for testing isomorphism?