A topological approach to compositionality in complex systems

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Workshop on Compositionality at Simons Institute, 8th December 2016
Complex systems are composed of many non-identical elements, entangled in loops of nonlinear interactions, and characterized by the characteristic 'emergence' behaviours.

M. Rasetti, E. Merelli Topological field theory of data: mining data beyond complex networks, Cambridge University Press, 2016
The TFTD is based on

1. embedding data space into a combinatorial topological object, a **simplicial complex**;

2. considering the **complex as base space** of a (block) **fiber bundle**

3. assuming a **field action** (which has a free part, the combinatorial Laplacian over the simplicial complex, and an interaction part depending on the process algebra)

4. constructing the **gauge group** (semi-direct product of the group generated by the algebra of processes (the fibers) and the group of (simplecio-morphisms modulo isotopy) of the data space.

Emergent features of data-represented complex systems were shown to be expressed by the correlation functions of the field theory.”

M. Rasetti, E. Merelli Topological field theory of data: mining data beyond complex networks, Cambridge University Press, 2016
Persistent Homology

Persistent homology is an algebraic method for discerning topological features of data, e.g., components, graph structure, holes. A graph captures connectivity, but ignores higher-order features, such as holes.

movie by Matthew L. Wright
Topological Invariants

1. a path in a space S is a continuous map
2. homotopy is an equivalence relation on paths in space S

\[ G = \langle a, b \mid aba^{-1}b^{-1} \rangle \]

\[ ab \simeq ba \]

Each equivalence class of paths might be called behavior of its members
Topological Interpretation of Dynamics of a System

- $S$ is the space of states
- Each state is defined by a vector that moves over $S$ driven by a dynamical system
- If the dynamics moves the vector towards a boundary, we can say that there is a deadlock.
- This happens because $S$ has not been defined globally. In fact the boundary breaks the translational symmetry.
- If we allow the boundary to disappear by adding an extra-relation, global in nature, we obtain a global topology that is not trivial
- For example we can add two relations among the generators of the manifold
The trivial case of torus:
for any $g \neq 0$, (e.g. $g=1$ torus) there are three irreducible classes behaviors

i) the set of closed paths homotopy to 0. In this case, we are given a local interpretation and we are not aware that at global level the genus can be different that 0.

ii) the set of closed paths homotopy to the first generator $a$ of the topological space (the homology group) whose basic cycle fixed on point $P_B$ can be used to reduce any path going around the neck of the torus $a$ by a continuous deformation;

iii) similar to the previous set, but the paths are homotopy to the second generator $b$

to each point $|\psi>$ of the states space is associated a path, represented by the arrow $|\psi> \rightarrow |\psi'>$ that is nothing but an operator $G$ that determines the dynamic of the system moving $|\psi>$ to $G(|\psi>)=|\psi'>$

the set of all possible $G$ represents a group, i.e the group of transformations
The principle of non-linear composition states that the emergent behaviour of a complex systems can not be fully determined by the behaviour of its constituents and the rules to combine them, because is the global context that induces the non-linearity interactions among them.
Frege's principle: in mathematics, semantics, and philosophy of language, the principle of compositionality is the principle for which the meaning of a complex expression is determined by the meanings of its constituent expressions and the rules used to combine them; 

\[(a+b)^* \simeq a^*(b+a^*)^*\]

under bisimulation relation, but cannot be proven in any axioms system.

Theorem [Milner 1984]: not every X-behaviour is a star behaviour.
The ‘process interpretation’ scheme of $\mathcal{P}$ in $\mathcal{P}$ is indeed nothing but a quiver $\mathcal{Q}$ (or, more generally, a set of quivers, over some arbitrary ring $\kappa$). Associate to quiver $\mathcal{Q}$ its ‘natural’ path algebra $\mathcal{A} \equiv \mathcal{P}_k\mathcal{Q}$, i.e., the path algebra of which $\mathcal{Q}$ is the basis.

The structure is simpler and elegant because space $\mathcal{P}$ has an underlying natural formal language (that generates in general a subgroup of the much wilder group of all possible homeomorphisms of $\mathcal{P}(\mathcal{P})$)
1. define a processes as Quivers $Q$

A quiver $Q$ is a direct graph. $Q = (Q_0, Q_1, s, e)$, where $Q_0$ is a set of vertices (states) and $Q_1$ a set of arrows (transitions) and $s, e : Q_1 \rightarrow Q_0$, are maps.

Given an arrow $a \in Q_1$ with $a : i \rightarrow j$ for $i, j \in Q_0$
When $s(a) = e(a)$, arrow $a$ is said to be a loop.

An path in a quiver $Q$ is either an ordered composition of arrows $p = a_1 a_2 \ldots a_n$ with $e(a_t) = s(a_{t+1})$ for $1 \leq t < n$
or the symbol $v_i$ for $i \in Q_0$

A path $p$ that starts and ends at the same vertex is a cycle. Loops are cycles.

The process quiver $Q$ represents a system behavior.

Behavioural equivalences: the paths in process quivers are distinguished by some homotopic equivalence.

paths: $v_1, v_2, v_3, a, b, a b$

paths: $v_1, a^*, b^*, (a+b)^*$

paths: $v_1, v_2, a^*, b^*, (a+b)^*$

A quiver with relations is a pair $(Q, R)$,
2. associate a natural path algebra $\mathcal{P}_{kQ}$ to given $Q$

- Let $k$ be a field. The path algebra $\mathcal{P}_{kQ}$ of the quiver $Q$ is defined to be the $k$-vector space generated by all paths in $Q$. The composition (product) of two paths is induced by simple concatenation of paths if it exists, and zero otherwise. $Q$ is the basis of $\mathcal{P}_{kQ}$.

- $\mathcal{P}_{kQ}$ is unitary if $Q_0$ is finite

$\mathcal{P}_{QR}$ is a path co-algebra of quivers with relations
3. identify the Lie algebra $\mathcal{L}$ given by $\mathcal{P}$

Identify the Lie algebra $\mathcal{L}$ arising from $\mathcal{P}_{kQ}$

4. define $\mathcal{L}$ in Hopf algebra $\mathcal{H}$

Turn $\mathcal{L}$ into a Hopf algebra $\mathcal{H}$, equipping it in particular with a coalgebra

5. redefine the theorem G in $\mathcal{H}$ and prove that hold in $\mathcal{H}$ and not in $\mathcal{P}$

Theorem G:
The SHC [Star Height Conjecture] is a topological application to the space $\mathcal{P}$, generated by the formal representation of a (any) process $\mathcal{P}$.

Corradini's Star Heigh conjecture:
the set of regular expressions (without 0) with $\text{hnewp}$ is the largest language for which bisimulation admits a finite equational axiomatization.

Def. $\text{nhewp}$ structural property:
1. each *-behaviour must avoid to enter in a pure cycle,
2. each cycle must be of the form $E^* = 1 + EE^*$
   $E^*F \rightarrow X = EX + F$
3. in *-behaviour $a^*a \neq aa^*$
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After the recent earthquakes in Central Italy, the research and historical heritage of the University of Camerino, one of the world’s oldest research institutions, is in danger.

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#ilfuturononcrolla
thanks!