Towards a resource theory of contextuality

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Introduction

- **Contextuality**: a fundamental non-classical phenomenon of QM

  - Raussendorf (2013) – MBQC
  - “Contextuality in measurement-based quantum computation”
  - Howard, Wallman, Veith, & Emerson (2014) – MSD
  - “Contextuality supplies the ‘magic’ for quantum computation”
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- **Contextuality** as a **resource** for QC:
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- Abramsky–Brandenburger: unified framework for non-locality and contextuality in general measurement scenarios

(NB: there may be more than one useful measure)
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- A–B: qualitative hierarchy of contextuality for empirical models
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- A–B: qualitative hierarchy of contextuality for empirical models

- quantitative grading – measure of contextuality (NB: there may be more than one useful measure)
Overview

We introduce the **contextual fraction**
(generalising the idea of non-local fraction)

It satisfies a number of desirable properties:
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  ⇔ **resource theory**
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  \( \rightsquigarrow \) resource theory
- Relates to quantifiable **advantages** in QC and QIP tasks
Contextuality
Empirical data

<table>
<thead>
<tr>
<th>A</th>
<th>B</th>
<th>(0, 0)</th>
<th>(0, 1)</th>
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<td>$\frac{1}{8}$</td>
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</tbody>
</table>

$\sigma_A \in \{0, 1\}$

$\sigma_B \in \{0, 1\}$

$m_A \in \{a_1, a_2\}$

$m_B \in \{b_1, b_2\}$

preparation

measurement device

measurement device
Abramsky–Brandenburger framework

Measurement scenario $\langle X, M, O \rangle$:

- $X$ is a finite set of measurements or variables
- $O$ is a finite set of outcomes or values
- $M$ is a cover of $X$, indicating **joint measurability** (contexts)

**Example**: $(2,2,2)$ Bell scenario

- The set of variables is $X = \{a_1, a_2, b_1, b_2\}$.
- The outcomes are $O = \{0, 1\}$.
- The measurement contexts are: $\{\{a_1, b_1\}, \{a_1, b_2\}, \{a_2, b_1\}, \{a_2, b_2\}\}$.

A joint outcome or event in a context $C$ is $s \in O^C$, e.g. $s = [a_1 \mapsto 0, b_1 \mapsto 1]$. (These correspond to the cells of our probability tables.)
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\{ \{ a_1, b_1 \}, \{ a_1, b_2 \}, \{ a_2, b_1 \}, \{ a_2, b_2 \} \}.
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Another example: 18-vector Kochen–Specker

- A set of 18 variables, $X = \{A, \ldots, O\}$
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- A set of 18 variables, $X = \{A, \ldots, O\}$
- A set of outcomes $O = \{0, 1\}$
- A measurement cover $M = \{C_1, \ldots, C_9\}$, whose contexts $C_i$ correspond to the columns in the following table:

<table>
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<th></th>
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<th>$U_3$</th>
<th>$U_4$</th>
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<tr>
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<td>L</td>
<td>N</td>
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<td>J</td>
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</table>
Empirical Models

Fix a measurement scenario \( \langle X, M, O \rangle \).
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**Empirical model:** family $\{e_C\}_{C \in M}$ where $e_C \in \text{Prob}(O^C)$ for $C \in M$.

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**Compatibility** condition: these distributions “agree on overlaps”, i.e.

\[
\forall C, C' \in M \cdot e_C|_{C \cap C'} = e_{C'}|_{C \cap C'}.
\]

where marginalisation of distributions: if \( D \subseteq C \), \( d \in \text{Prob}(O^C) \),

\[
d|_D(s) := \sum_{t \in O^C, t|_D = s} d(t).
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For multipartite scenarios, compatibility = the **no-signalling** principle.
Contextuality

A (compatible) empirical model is non-contextual if there exists a global distribution \( d \in \text{Prob}(O^X) \) (on the joint assignments of outcomes to all measurements) that marginalises to all the \( e_C \):

\[
\exists d \in \text{Prob}(O^X) \cdot \forall C \in \mathcal{M} \cdot d|_C = e_C. \]

That is, we can glue all the local information together into a global consistent description from which the local information can be recovered.

Contextuality: family of data which is locally consistent but globally inconsistent.

The import of results such as Bell's and Bell–Kochen–Specker's theorems is that there are empirical models arising from quantum mechanics that are contextual.
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family of data which is **locally consistent** but **globally inconsistent**.
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Strong Contextuality:

**no** event can be extended to a global assignment.
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Strong Contextuality: no event can be extended to a global assignment.

E.g. K–S models, GHZ, the PR box:

<table>
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<tr>
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<td>$b_1$</td>
<td>✓</td>
<td>×</td>
<td>×</td>
<td>✓</td>
</tr>
<tr>
<td>$a_2$</td>
<td>$b_2$</td>
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<td>✓</td>
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The contextual fraction
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Non-contextuality: global distribution $d \in \text{Prob}(O^X)$ such that:

$$\forall C \in \mathcal{M}. \ d|_C = e_C.$$

Non-contetual fraction: maximum weigth of such a subdistribution.

Equivalently, maximum weight $\lambda$ over all convex decompositions $e = \lambda e_{NC} + (1 - \lambda) e'$ where $e_{NC}$ is a non-contextual model.

$NCF(e) = \lambda = \text{CF}(e) = 1 - \lambda$
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Which fraction of a model admits a non-contextual explanation?
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Consider **subdistributions** $c \in \text{SubProb}(O^X)$ such that:

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Equivalently, maximum weight \( \lambda \) over all convex decompositions

\[
e = \lambda e^{NC} + (1 - \lambda)e'
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where \( e^{NC} \) is a non-contextual model.
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$$\text{NCF}(e) = \lambda \quad \text{CF}(e) = 1 - \lambda$$
Checking contextuality of \( e \) corresponds to solving

\[
\text{Find } \quad d \in \mathbb{R}^n \\
\text{such that } \quad M d = v^e \\
\text{and } \quad d \geq 0
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(Non-)contextual fraction via linear programming

Checking contextuality of $e$ corresponds to solving

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Computing the non-contextual fraction corresponds to solving the following linear program:

Find $c \in \mathbb{R}^n$
maximising $1 \cdot c$
subject to $M c \leq v^e$
and $c \geq 0$. 
E.g. Equatorial measurements on GHZ($n$)

Figure: Non-contextual fraction of empirical models obtained with equatorial measurements at $\phi_1$ and $\phi_2$ on each qubit of $|\psi_{\text{GHZ}(n)}\rangle$ with: (a) $n = 3$; (b) $n = 4$. 
Violations of Bell inequalities
Generalised Bell inequalities

An **inequality** for a scenario $\langle X, M, O \rangle$ is given by:

- a set of coefficients $\alpha = \{ \alpha(C, s) \}_{C \in M, s \in E(C)}$
- a bound $R$

For a model $e$, the inequality reads as $B_{\alpha}(e) \leq R$, where $B_{\alpha}(e) := \sum_{C \in M, s \in E(C)} \alpha(C, s) e_C(s)$.

Wlog we can take $R$ non-negative (in fact, we can take $R = 0$).

It is called a Bell inequality if it is satisfied by every NC model. If it is saturated by some NC model, the Bell inequality is said to be **tight**.
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Violation of a Bell inequality

A Bell inequality establishes a bound for the value of $B_\alpha(e)$ amongst NC models.
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For a general (no-signalling) model $e$, the quantity is limited only by

$$\|\alpha\| := \sum_{C \in \mathcal{M}} \max \{ \alpha(C, s) \mid s \in \mathcal{E}(C) \}$$
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The normalised violation of a Bell inequality $\langle \alpha, R \rangle$ by an empirical model $e$ is the value

$$\frac{\max\{0, B_\alpha(e) - R\}}{\|\alpha\| - R}.$$
Proposition
Let $e$ be an empirical model.
Bell inequality violation and the contextual fraction

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- This is attained: there exists a Bell inequality whose normalised violation by $e$ is exactly $\text{CF}(e)$.
**Proposition**

Let $e$ be an empirical model.

- The normalised violation by $e$ of any Bell inequality is at most $\text{CF}(e)$.

- This is attained: there exists a Bell inequality whose normalised violation by $e$ is exactly $\text{CF}(e)$.

- Moreover, this Bell inequality is tight at “the” non-contextual model $e^{\text{NC}}$ and maximally violated by “the” strongly contextual model $e^{\text{SC}}$:

$$e = \text{NCF}(e)e^{\text{NC}} + \text{CF}(e)e^{\text{SC}}.$$
Bell inequality violation and the contextual fraction

Quantifying Contextuality LP:

Find \( c \in \mathbb{R}^n \) maximising \( 1 \cdot c \)
subject to \( M c \leq v^e \)
and \( c \geq 0 \).

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e = \lambda e^{NC} + (1 - \lambda)e^{SC} \text{ with } \lambda = 1 \cdot x^*.
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Bell inequality violation and the contextual fraction

Quantifying Contextuality LP:

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Dual LP:

Find $y \in \mathbb{R}^m$

minimising $y \cdot v^e$

subject to $M^T y \geq 1$

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$a := 1 - |M|y$

Find $a \in \mathbb{R}^m$
maximising $a \cdot v^e$
subject to $M^T a \leq 0$
and $a \leq 1$. 
Bell inequality violation and the contextual fraction

Quantifying Contextuality LP:

Find \( c \in \mathbb{R}^n \)
maximising \( 1 \cdot c \)
subject to \( Mc \leq v^e \)
and \( c \geq 0 \).

\[ e = \lambda e^{NC} + (1 - \lambda)e^{SC} \text{ with } \lambda = 1 \cdot x^* \].

Dual LP:

Find \( y \in \mathbb{R}^m \)
minimising \( y \cdot v^e \)
subject to \( M^T y \geq 1 \)
and \( y \geq 0 \).

\[ a := 1 - |M|y \]

Find \( a \in \mathbb{R}^m \)
maximising \( a \cdot v^e \)
subject to \( M^T a \leq 0 \)
and \( a \leq 1 \).

computes tight Bell inequality (separating hyperplane)
Operations on empirical models
Contextuality as a resource

▶ More than one possible measure of contextuality.
▶ What properties should a good measure satisfy?
▶ Monotonicity wrt operations that do not introduce contextuality

Towards a resource theory as for entanglement (e.g. LOCC), non-locality, ...
Contextuality as a resource

More than one possible measure of contextuality.
Contextuality as a resource

- More than one possible measure of contextuality.
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- Towards a resource theory as for entanglement (e.g. LOCC), non-locality, ...
Consider operations on empirical models.

- These operations should not increase contextuality.
- Write type statements $e: \langle X, M, O \rangle$ to mean that $e$ is a (compatible) empirical model on the scenario $\langle X, M, O \rangle$.
- The operations remind one of process algebras.
Algebra of empirical models

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Algebra of empirical models

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The operations remind one of process algebras.
Operations

\[\langle X, M, O \rangle, \alpha : (X, M) \sim (X', M') \Rightarrow e [\alpha] : \langle X', M', O \rangle\]

For \(C \in M\), s.t. \(C' \subseteq C\):

\[\alpha(C) \rightarrow O, e[C'](s) := e_{C'}(s \circ \alpha^{-1})\]

\[\langle X, M, O \rangle, (X', M') \leq (X, M) \Rightarrow e|_{M'} : \langle X', M', O \rangle\]

For \(C' \in M'\):

\[C' \rightarrow O, (e|_{M'})_{C'}(s) := e_{C'}(s)\]

\[\langle X, M, O \rangle, f : O \rightarrow O' \Rightarrow e/f : \langle X, M, O' \rangle\]

For \(C \in M\), s.t. \(C' \subseteq C\):

\[C \rightarrow O', (e/f)_{C}(s) := \sum t : C \rightarrow O, f \circ t = s e_{C}(t)\]
Operations

▶ relabelling

\[ e : \langle X, M, O \rangle, \ \alpha : (X, M) \cong (X', M') \quad \leadsto \quad e[\alpha] : \langle X', M', O \rangle \]
Operations

- relabelling
  \[ e : \langle X, M, O \rangle, \ \alpha : (X, M) \cong (X', M') \implies e[\alpha] : \langle X', M', O \rangle \]

For \( C \in M \), \( s : \alpha(C) \rightarrow O \), \( e[\alpha]_{\alpha(C)}(s) := e_C(s \circ \alpha^{-1}) \)
Operations

- **relabelling**
  \[ e : \langle X, M, O \rangle, \; \alpha : (X, M) \cong (X', M') \leadsto e[\alpha] : \langle X', M', O \rangle \]
  
  For \( C \in M \), \( s : \alpha(C) \rightarrow O \), \( e[\alpha]_{\alpha(C)}(s) := e_C(s \circ \alpha^{-1}) \)

- **restriction**
  \[ e : \langle X, M, O \rangle, \; (X', M') \leq (X, M) \leadsto e \upharpoonright M' : \langle X', M', O \rangle \]
Operations

- **relabeling**
  \[ e : \langle X, \mathcal{M}, O \rangle, \ \alpha : (X, \mathcal{M}) \cong (X', \mathcal{M}') \rightsquigarrow e[\alpha] : \langle X', \mathcal{M}', O \rangle \]

  For \( C \in \mathcal{M}, s : \alpha(C) \rightarrow O \), \( e[\alpha]_{\alpha(C)}(s) := e_C(s \circ \alpha^{-1}) \)

- **restriction**
  \[ e : \langle X, \mathcal{M}, O \rangle, (X', \mathcal{M}') \leq (X, \mathcal{M}) \rightsquigarrow e \restriction \mathcal{M}' : \langle X', \mathcal{M}', O \rangle \]

  For \( C' \in \mathcal{M}', s : C' \rightarrow O \), \( (e \restriction \mathcal{M}')_{C'}(s) := e_C|_{C'}(s) \)
  with any \( C \in \mathcal{M} \) s.t. \( C' \subseteq C \)
Operations

- **relabelling**
  \[ e : \langle X, \mathcal{M}, O \rangle, \ \alpha : (X, \mathcal{M}) \cong (X', \mathcal{M}') \leadsto e[\alpha] : \langle X', \mathcal{M}', O \rangle \]

  For \( C \in \mathcal{M}, s : \alpha(C) \to O, e[\alpha]_{\alpha(C)}(s) := e_C(s \circ \alpha^{-1}) \)

- **restriction**
  \[ e : \langle X, \mathcal{M}, O \rangle, (X', \mathcal{M}') \leq (X, \mathcal{M}) \leadsto e \upharpoonright \mathcal{M}' : \langle X', \mathcal{M}', O \rangle \]

  For \( C' \in \mathcal{M}', s : C' \to O, (e \upharpoonright \mathcal{M}')_{C'}(s) := e_C|_{C'}(s) \)
  with any \( C \in \mathcal{M} \) s.t. \( C' \subseteq C \)

- **coarse-graining**
  \[ e : \langle X, \mathcal{M}, O \rangle, f : O \to O' \leadsto e/f : \langle X, \mathcal{M}, O' \rangle \]
Operations

- **relabelling**
  \[ e : \langle X, \mathcal{M}, O \rangle, \ \alpha : (X, \mathcal{M}) \cong (X', M') \leadsto e[\alpha] : \langle X', \mathcal{M}', O \rangle \]
  
  For \( C \in \mathcal{M}, s : \alpha(C) \rightarrow O, e[\alpha]_{\alpha(C)}(s) := e_C(s \circ \alpha^{-1}) \)

- **restriction**
  \[ e : \langle X, \mathcal{M}, O \rangle, (X', \mathcal{M}') \leq (X, \mathcal{M}) \leadsto e \upharpoonright \mathcal{M}' : \langle X', \mathcal{M}', O \rangle \]
  
  For \( C' \in \mathcal{M}', s : C' \rightarrow O, (e \upharpoonright \mathcal{M}')_{C'}(s) := e_C|_{C'}(s) \)
  with any \( C \in \mathcal{M} \) s.t. \( C' \subseteq C \)

- **coarse-graining**
  \[ e : \langle X, \mathcal{M}, O \rangle, f : O \rightarrow O' \leadsto e/f : \langle X, \mathcal{M}, O' \rangle \]
  
  For \( C \in \mathcal{M}, s : C \rightarrow O', (e/f)_C(s) := \sum_{t : C \rightarrow O, f \circ t = s} e_C(t) \)
Operations

### Mixing

\[
\langle X, M, O \rangle, e' : \langle X, M, O \rangle, \lambda \in [0, 1] \mapsto e + \lambda e': \langle X, M, O \rangle
\]

For \( C \in M \),

\[
\begin{align*}
C & \rightarrow O', (e + \lambda e') C(s) := \\
(1 - \lambda) e' C(s)
\end{align*}
\]

### Choice

\[
\begin{align*}
e : \langle X, M, O \rangle, e' : \langle X, M, O \rangle & \mapsto e \& e' : \langle X \sqcup X', M \sqcup M', O \rangle \\
C & \in M
\end{align*}
\]

### Tensor

\[
\begin{align*}
e : \langle X, M, O \rangle, e' : \langle X', M', O \rangle & \mapsto e \otimes e' : \langle X \sqcup X', M \star M', O \rangle \\
M \star M' & := \{ C \sqcup D | C \in M, D \in M' \}
\end{align*}
\]

For \( C \in M \), \( D \in M' \),

\[
\begin{align*}
(\otimes C) (s_1, s_2) := e C(s_1) e' D(s_2)
\end{align*}
\]
Operations

- **mixing**
  \[
  e : \langle X, M, O \rangle, \quad e' : \langle X, M, O \rangle, \quad \lambda \in [0, 1] \quad \mapsto \quad e + \lambda \ e' : \langle X, M, O \rangle
  \]
Operations

- **mixing**
  \[ e : \langle X, \mathcal{M}, O \rangle, \quad e' : \langle X, \mathcal{M}, O \rangle, \quad \lambda \in [0, 1] \leadsto e + \lambda e' : \langle X, \mathcal{M}, O \rangle \]

For \( C \in \mathcal{M}, s : C \to O' \),
\[
(e + \lambda e')(s) := \lambda e_C(s) + (1 - \lambda)e'_C(s)
\]
Operations

► **mixing**
\[ e : \langle X, M, O \rangle, \ e' : \langle X, M, O \rangle, \ \lambda \in [0, 1] \rightsquigarrow e + \lambda e' : \langle X, M, O \rangle \]

For \( C \in M, s : C \rightarrow O' \),
\[ (e + \lambda e')(s) := \lambda e_C(s) + (1 - \lambda)e'_C(s) \]

► **choice**
\[ e : \langle X, M, O \rangle, \ e' : \langle X, M, O \rangle \rightsquigarrow e & e' : \langle X \sqcup X', M \sqcup M', O \rangle \]
Operations

- **mixing**
  
  \[ e : \langle X, M, O \rangle, \quad e' : \langle X, M, O \rangle, \lambda \in [0, 1] \quad \leadsto \quad e + \lambda \ e' : \langle X, M, O \rangle \]

  For \( C \in M, s : C \rightarrow O' \),
  
  \[(e + \lambda \ e')(s) := \lambda e_C(s) + (1 - \lambda)e'_C(s)\]

- **choice**
  
  \[ e : \langle X, M, O \rangle, \quad e' : \langle X, M, O \rangle \quad \leadsto \quad e \& e' : \langle X \sqcup X', M \sqcup M', O \rangle \]

  For \( C \in M \), \( (e \& e')_C := e_C \)
  
  For \( D \in M' \), \( (e \& e')_D := e'_D \)
Operations

- **mixing**
  \[ e : \langle X, M, O \rangle, \ e' : \langle X, M, O \rangle, \ \lambda \in [0, 1] \Rightarrow e + \lambda e' : \langle X, M, O \rangle \]

  For \( C \in M, s : C \rightarrow O' \),
  \[(e + \lambda e')_C(s) := \lambda e_C(s) + (1 - \lambda)e'_C(s)\]

- **choice**
  \[ e : \langle X, M, O \rangle, \ e' : \langle X, M, O \rangle \Rightarrow e \& e' : \langle X \sqcup X', M \sqcup M', O \rangle \]

  For \( C \in M, (e \& e')_C := e_C \)
  For \( D \in M', (e \& e')_D := e'_D \)

- **tensor**
  \[ e : \langle X, M, O \rangle, \ e' : \langle X', M', O \rangle \Rightarrow e \otimes e' : \langle X \sqcup X', M \star M', O \rangle \]
Operations

- **mixing**
  \[ e : \langle X, \mathcal{M}, O \rangle, \quad e' : \langle X, \mathcal{M}, O \rangle, \lambda \in [0, 1] \leadsto e + \lambda e' : \langle X, \mathcal{M}, O \rangle \]

  For \( C \in \mathcal{M} \), \( s : C \rightarrow O' \),
  \[(e + \lambda e')_C(s) := \lambda e_C(s) + (1 - \lambda)e'_C(s)\]

- **choice**
  \[ e : \langle X, \mathcal{M}, O \rangle, \quad e' : \langle X, \mathcal{M}, O \rangle \leadsto e \& e' : \langle X \sqcup X', \mathcal{M} \sqcup \mathcal{M}', O \rangle \]

  For \( C \in \mathcal{M} \), \( (e\&e')_C := e_C \)
  For \( D \in \mathcal{M}' \), \( (e\&e')_D := e'_D \)

- **tensor**
  \[ e : \langle X, \mathcal{M}, O \rangle, \quad e' : \langle X, \mathcal{M}', O \rangle \leadsto e \otimes e' : \langle X \sqcup X', \mathcal{M} \star \mathcal{M}', O \rangle \]

  \[ \mathcal{M} \star \mathcal{M}' := \{ C \sqcup D \mid C \in \mathcal{M}, D \in \mathcal{M}' \} \]
Operations

- **mixing**
  \[ e : \langle X, M, O \rangle, \ e' : \langle X, M, O \rangle, \lambda \in [0, 1] \rightsquigarrow e + \lambda e' : \langle X, M, O \rangle \]

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  \[ (e + \lambda e')_C(s) := \lambda e_C(s) + (1 - \lambda)e'_C(s) \]

- **choice**
  \[ e : \langle X, M, O \rangle, \ e' : \langle X, M, O \rangle \rightsquigarrow e \& e' : \langle X \square X', M \square M', O \rangle \]

  For \( C \in M, (e \& e')_C := e_C \)
  For \( D \in M', (e \& e')_D := e'_D \)

- **tensor**
  \[ e : \langle X, M, O \rangle, \ e' : \langle X', M', O \rangle \rightsquigarrow e \otimes e' : \langle X \square X', M \star M', O \rangle \]

  \[ M \star M' := \{ C \square D \mid C \in M, D \in M' \} \]
  For \( C \in M, D \in M', s = \langle s_1, s_2 \rangle : C \square D \rightarrow O, \)
  \[ (e \otimes e')_{C \square D}(s_1, s_2) := e_C(s_1)e'_D(s_2) \]
Operations and the contextual fraction

\[ \text{relabelling} \]
\[ CF(e^\alpha) = CF(e) \]

\[ \text{restriction} \]
\[ CF(e|\sigma') \leq CF(e) \]

\[ \text{coarse-graining} \]
\[ CF(e/f) \leq CF(e) \]

\[ \text{mixing} \]
\[ CF(e + \lambda e') \leq \lambda CF(e) + (1 - \lambda) CF(e') \]

\[ \text{choice} \]
\[ CF(e \& e') = \max\{CF(e), CF(e')\} \]
\[ NCF(e \& e') = \min\{NCF(e), NCF(e')\} \]

\[ \text{tensor} \]
\[ CF(e_1 \otimes e_2) = CF(e_1) + CF(e_2) - CF(e_1) CF(e_2) \]
\[ NCF(e_1 \otimes e_2) = NCF(e_1) NCF(e_2) \]
Operations and the contextual fraction

- relabelling
  \[ CF(e[\alpha]) = CF(e) \]
Operations and the contextual fraction

- **relabelling**
  \[ \text{CF}(e[\alpha]) = \text{CF}(e) \]

- **restriction**
  \[ \text{CF}(e \upharpoonright \sigma') \leq \text{CF}(e) \]
Operations and the contextual fraction

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Operations and the contextual fraction

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Operations and the contextual fraction

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- **choice**
  \[ \text{CF}(e & e') = \max\{\text{CF}(e), \text{CF}(e')\} \]
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- **tensor**
  \[ \text{CF}(e_1 \otimes e_2) = \text{CF}(e_1) + \text{CF}(e_2) - \text{CF}(e_1)\text{CF}(e_2) \]
  \[ \text{NCF}(e_1 \otimes e_2) = \text{NCF}(e_1)\text{NCF}(e_2) \]
Contextual fraction and quantum advantages
Contextual fraction and advantages

- Contextuality has been associated with quantum advantage in information-processing and computational tasks.
Contextual fraction and advantages

- Contextuality has been associated with quantum advantage in information-processing and computational tasks.

- Measure of contextuality to quantify such advantages.
Contextual fraction and MBQC

E.g. Raussendorf (2013) $\ell^2$-MBQC
Contextual fraction and MBQC

E.g. Raussendorf (2013) $\ell^2$-MBQC

- measurement-based quantum computing scheme
  ($m$ input bits, $l$ output bits, $n$ parties)

$\ell^2$-MBQC deterministically computes a non-linear Boolean function $f : 2^m \rightarrow 2^l$ then the resource must be strongly contextual.

Probabilistic version: non-linear function computed with sufficiently large probability of success implies contextuality.
Contextual fraction and MBQC

E.g. Raussendorf (2013) $\ell_2$-MBQC

- measurement-based quantum computing scheme
  ($m$ input bits, $l$ output bits, $n$ parties)

- classical control:
  - pre-processes input
  - determines the flow of measurements
  - post-processes to produce the output

only $\mathbb{Z}_2$-linear computations.
Contextual fraction and MBQC

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- additional power to compute non-linear functions resides in
  certain resource empirical models.
Contextual fraction and MBQC

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only $\mathbb{Z}_2$-linear computations.

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- Raussendorf (2013): If an $\ell^2$-MBQC \textit{deterministically} computes a non-linear Boolean function $f : 2^m \rightarrow 2^l$ then the resource must be \textit{strongly contextual}.
Contextual fraction and MBQC

E.g. Raussendorf (2013) $\ell^2$-MBQC

- measurement-based quantum computing scheme (m input bits, l output bits, n parties)

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average distance between two Boolean functions $f, g : 2^m \rightarrow 2^l$:
$$d(f, g) := 2^{-m} | \{ i \in 2^m | f(i) \neq g(i) \}$$
Contextual fraction and MBQC

- **average distance** between two Boolean functions \( f, g : 2^m \rightarrow 2^l \):
  \( \tilde{d}(f, g) := 2^{-m} \cdot \left| \{ i \in 2^m | f(i) \neq g(i) \} \right| \)

- \( \tilde{\nu}(f) \): average distance between \( f \) and closest \( \mathbb{Z}_2 \)-linear function (how difficult the problem is)
average distance between two Boolean functions

\[ d(f, g) := 2^{-m} \left| \{ i \in 2^m \mid f(i) \neq g(i) \} \right| \]

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\( \ell^2 \)-MBQC computing \( f \) with average probability (over all \( 2^m \) possible inputs) of success \( \bar{p}_S \).
**average distance** between two Boolean functions
\[ f, g : 2^m \rightarrow 2^l : \]
\[ \tilde{d}(f, g) := 2^{-m} \cdot \left| \{ i \in 2^m \mid f(i) \neq g(i) \} \right| \]

\[ \tilde{\nu}(f) : \text{average distance between } f \text{ and closest } \mathbb{Z}_2\text{-linear function} \]
\( \text{ (how difficult the problem is) } \)

\( \ell_2\text{-MBQC computing } f \text{ with average probability (over all } 2^m \text{ possible inputs) of success } \bar{p}_S. \)

Then,
\[ 1 - \bar{p}_S \geq \text{NCF}(e)\tilde{\nu}(f). \]
Contextual fraction and cooperative games

- Game described by \( n \) formulae (one for each possible input).

- These describe the winning condition that the corresponding outputs must satisfy.
Contextual fraction and cooperative games

- Game described by $n$ formulae (one for each possible input).
- These describe the winning condition that the corresponding outputs must satisfy.
- Formulae are $k$-consistent (at most $k$ of them have a joint satisfying assignment)
- cf. Abramsky–Hardy “Logical Bell inequalities”
- Hardness of the game measured by $\frac{n-k}{n}$. 

\[ 1 - \bar{p}_S \leq NCF(e)(n-k) \]
Game described by $n$ formulae (one for each possible input).

These describe the winning condition that the corresponding outputs must satisfy.

Formulae are $k$-consistent (at most $k$ of them have a joint satisfying assignment)

cf. Abramsky–Hardy “Logical Bell inequalities”

Hardness of the game measured by $\frac{n-k}{n}$.

$1 - \bar{p}_S \leq \text{NCF}(e) \cdot \frac{(n-k)}{n}$. 
Further directions

Negative Probabilities Measure

Alternative relaxation of global probability distribution requirement.

Find quasi-probability distribution $q$ on $O_X$ such that $q|_C = e^C$.

. . . with minimal weight $|q| = 1 + 2\epsilon$.

The value $\epsilon$ provides alternative measure of contextuality.

How are these related?

Corresponds to affine decomposition $e = (1 + \epsilon)e_1 - \epsilon e_2$ with $e_1$ and $e_2$ both non-contextual.

Corresponding inequalities $|B_\alpha(e)| \leq R$.

Cyclic measurement scenarios

Resource Theory

Sequencing

What (else) is this resource useful for?
Further directions

- Negative Probabilities Measure

... with minimal weight $|q| = 1 + 2\epsilon$.

The value $\epsilon$ provides an alternative measure of contextuality.

How are these related?

Corresponds to affine decomposition $e = (1 + \epsilon)e_1 - \epsilon e_2$ with $e_1$ and $e_2$ both non-contextual.

Corresponding inequalities $|B_\alpha(e)| \leq R$.

Cyclic measurement scenarios
Further directions

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Further directions

- **Negative Probabilities Measure**
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  - Find quasi-probability distribution $q$ on $O^X$ such that $q|_C = e_C$
  - ...with minimal weight $|q| = 1 + 2\epsilon$.
    The value $\epsilon$ provides alternative measure of contextuality.
  - How are these related?
  - Corresponds to affine decomposition
    $$e = (1 + \epsilon) e_1 - \epsilon e_2$$
    with $e_1$ and $e_2$ both non-contextual.
Further directions

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Further directions

- Negative Probabilities Measure
- Signalling models

Empirical data may sometimes not satisfy no-signalling (compatibility).

Given a signalling table, can we quantify amount of no-signalling and contextuality?

Similarly, we can define no-signalling fraction $\lambda = \frac{\lambda_{eNS} - (1 - \lambda_{eSS})}{eSS}$.

Analysis of real data:

$e_{Delft} \approx 0.0664 e_{SS} + 0.4073 e_{SC} + 0.5263 e_{NC}$

$e_{NIST} \approx 0.0000049 e_{SS} + 0.0000281 e_{SC} + 0.9999670 e_{NC}$

First extract NS fraction, then NC fraction? Or vice-versa? Also: non-uniqueness of witnesses!

Connections with Contextuality-by-Default (Dzhafarov et al.)

Resource Theory

Sequencing

What (else) is this resource useful for?
Further directions

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