Modelling interconnected systems with decorated corelations

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All hypergraph categories are decorated corelation categories.
David (yesterday): Introduced hypergraph categories.

John (this morning): Introduced decorated cospans.

Me (now): All hypergraph categories are decorated corelation categories.

Dan (next): Hypergraph categories via relations.

Ross (tomorrow): Hypergraph categories in categorical quantum mechanics.
Hypergraph categories model network compositionality
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Hypergraph category is a symmetric monoidal category in which each object is equipped with a special commutative Frobenius monoid in a way coherent with the monoidal product.
Hypergraph categories model network compositionality

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Recall from John Baez’s talk . . .

In many areas of science and engineering, people use networks, drawn as boxes connected by wires:

We need a good general theory of these!
Let $C$ have finite colimits. Then $\text{Cospan}(C)$ is a hypergraph category.

The monoidal product is the coproduct $+$ in $C$.

The Frobenius maps are given by the codiagonal map $\nabla: X + X \to X$ and the initial map $!: \emptyset \to X$.

Decorated cospan categories inherit this hypergraph structure via the embedding $\text{Cospan}(C) \to F\text{Cospan}$. 
Decorated cospans build hypergraph categories

Also recall...

Say we start with a category $\mathbf{C}$ with finite colimits: in our example, $\mathbf{C} = \mathbf{FinSet}$. We can build a bicategory where morphisms are cospans in $\mathbf{C}$:

$$
\begin{array}{ccc}
N & \overset{o}{\xleftarrow{i}} & N' \\
X & \overset{\text{id}}{\swarrow} & Y \\
\end{array}
$$

and composition is done by pushout:

$$
\begin{array}{ccc}
N +_Y N' & \overset{o}{\xleftarrow{i}} & N' \\
X & \overset{\text{id}}{\swarrow} & Y & \overset{\text{id}}{\swarrow} & Z \\
\end{array}
$$
Decorated cospans build hypergraph categories

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Decorated cospan categories are good for syntax.

But when composing decorated cospans, the morphism grows.
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But when composing decorated cospans, the morphism grows, and grows, and grows...
What about hypergraph categories for semantics?
Decorated corelations are better for semantics

Consider the pair of decorated cospans

\[
\begin{array}{cccc}
X & \rightarrow & N & \rightarrow & Y \\
\downarrow & & \uparrow & & \downarrow \\
N & & 1\Omega & & N'
\end{array}
\]

Their composite is

\[
1\Omega
\]

But this is, in an extensional sense, the same as

\[
2\Omega
\]

To construct a category which does not see the difference between these two circuits, we use decorated corelations. The key idea is that we only want the part of a decoration that lives on the boundary.
Decorated corelations are better for semantics

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Their composite is

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\bullet & \rightarrow & \bullet \\
\downarrow & & \downarrow \\
\bullet & \rightarrow & \bullet \\
\end{array} \]
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Consider the pair of decorated cospans

\[
\begin{align*}
X & \to N & 1\Omega & \to N' & \to Y & \to Z \\
& \downarrow & & & \downarrow & \downarrow \\
& & 1\Omega & & 1\Omega & & 1\Omega \\
& & N & & N' & & Z
\end{align*}
\]

Their composite is

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Their composite is

\[
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\end{align*}
\]

But this is, in an extensional sense, the same as

\[
\begin{align*}
\rightarrow & \rightarrow & 2\Omega & \rightarrow & Z & \rightarrow
\end{align*}
\]

To construct a category which does not see the difference between these two circuits, we use decorated corelations.
Decorated corelations are better for semantics

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\[ X \xrightarrow{1\Omega} N \xleftarrow{1\Omega} Y \xrightarrow{1\Omega} N' \xleftarrow{} Z \]

Their composite is

\[ \cdot \xrightarrow{1\Omega} \cdot \xrightarrow{1\Omega} \cdot \xleftarrow{} \cdot \]

But this is, in an extensional sense, the same as

\[ \cdot \xrightarrow{2\Omega} \cdot \xleftarrow{} \cdot \]

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\[
\begin{align*}
\{ (x, y, z, x - y, 2y - (x + z), z - y) \} 
\subseteq \mathbb{R}^{N + Y N'} \oplus \mathbb{R}^{N + Y N'} \\
\{ (x, y, x - y, y - x) \} \subseteq \mathbb{R}^N \oplus \mathbb{R}^N
\end{align*}
\]
Decorated corelations are better for semantics

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Decorated corelations are better for semantics

\[
\left\{ \left( x, z, \frac{1}{2}(x - z), \frac{1}{2}(z - x) \right) \right\} 
\subseteq \mathbb{R}^{\text{Im} f} \oplus \mathbb{R}^{\text{Im} f}
\]

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\]
Decorated correlations are better for semantics

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\left\{ \left( x, z, \frac{1}{2}(x - z), \frac{1}{2}(z - x) \right) \right\} \\
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\]
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\[ \{ (x, z, \frac{1}{2}(x - z), \frac{1}{2}(z - x)) \} \subseteq \mathbb{R}^{\text{Im} f} \oplus \mathbb{R}^{\text{Im} f} \]
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\end{align*}
\]

\[
\begin{align*}
\{ (y', z, y' - z, z - y') \} \\ \subseteq \mathbb{R}^{N'} \oplus \mathbb{R}^{N'}
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\begin{align*}
\{ (x, z, \frac{1}{2}(x - z), \frac{1}{2}(z - x)) \} 
\subseteq \mathbb{R}^{\text{Im}f} \oplus \mathbb{R}^{\text{Im}f} \\
\{ (x, y, z, x - y, 2y - (x + z), z - y) \} 
\subseteq \mathbb{R}^{N+Y \ N'} \oplus \mathbb{R}^{N+Y \ N'} \\
\{ (x, y, x - y, y - x) \} \quad \{ (y', z, y' - z, z - y') \} 
\subseteq \mathbb{R}^{N} \oplus \mathbb{R}^{N} \\
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\[ \{(x, z, \frac{1}{2}(x - z), \frac{1}{2}(z - x))\} \subseteq \mathbb{R}^{\text{Im}f} \oplus \mathbb{R}^{\text{Im}f} \]

\[ \{(x, y, z, x - y, 2y - (x + z), z - y)\} \subseteq \mathbb{R}^{N + Y N'} \oplus \mathbb{R}^{N + Y N'} \]

\[ \{(x, y, x - y, y - x)\} \subseteq \mathbb{R}^{N} \oplus \mathbb{R}^{N} \]

\[ \{(y', z, y' - z, z - y')\} \subseteq \mathbb{R}^{N'} \oplus \mathbb{R}^{N'} \]
To recap:

- Write cospan $f : X + Z \to N + \gamma N'$.
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- Write cospan $f: X + Z \rightarrow N + Y N'$.
- Factor $f = m \circ e$, where $m \in \text{Inj}$ and $e \in \text{Sur}$. 
Decorated corelation categories

To recap:

- Write cospan \( f : X + Z \to N +_Y N' \).
- Factor \( f = m \circ e \), where \( m \in \text{Inj} \) and \( e \in \text{Sur} \).
- Transfer decoration along \( \xrightarrow{c} \xleftarrow{m} \).
Decorated corelation categories

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- Transfer decoration along \( \xymatrix{ c & m \ar[l] \ar[r] } \).

More generally, we need a **costable factorisation system** \((\mathcal{E}, \mathcal{M})\) on \(\mathcal{C}\).
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Here \( \mathcal{C} = \text{FinSet} \), \((\mathcal{E}, \mathcal{M}) = (\text{Sur}, \text{Inj})\).
Decorated corelation categories

To recap:

- Write cospan $f : X + Z \to N + \text{Im } Y N'$.
- Factor $f = m \circ e$, where $m \in \text{Inj}$ and $e \in \text{Sur}$.
- Transfer decoration along $\frac{c}{\to} \frac{m}{\leftarrow}$.

More generally, we need a costable factorisation system $(\mathcal{E}, \mathcal{M})$ on $\mathcal{C}$.

Here $\mathcal{C} = \text{FinSet}$, $(\mathcal{E}, \mathcal{M}) = (\text{Sur}, \text{Inj})$.

We write $\mathcal{C}; \mathcal{M}^{\text{op}}$ for the category with $\frac{c}{\to} \frac{m}{\leftarrow}$ as morphisms.
Decorated corelation categories

**Theorem**

Suppose that $C$ has finite colimits and a costable factorisation system $(\mathcal{E}, \mathcal{M})$, and

$$F: (C; \mathcal{M}^{\text{op}}, +) \longrightarrow (\text{Set}, \times)$$

is a lax symmetric monoidal functor.
Theorem

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$$F : (C; \mathcal{M}^{\text{op}}, +) \rightarrow (\text{Set}, \times)$$

is a lax symmetric monoidal functor. Then there is a hypergraph category of $F$-decorated corelations, $F\text{Corel}$ where

- an object is an object of $C$
- a morphism from $X$ to $Y$ is a cospan

such that $[f, g] : X + Y \rightarrow N$ lies in $\mathcal{E}$, together with a decoration $d \in F(N)$. (Actually, an isomorphism class of these!)
Decorated corelation functors

A hypergraph functor is a strong monoidal functor that preserves the Frobenius maps.
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**Theorem**

Suppose that $C$ has finite colimits and a costable factorisation system $(\mathcal{E}, \mathcal{M})$, and $F, G : (C; \mathcal{M}^{\text{op}}) \to (\text{Set}, \times)$ are lax symmetric monoidal functors, and

$$\theta : F \Rightarrow G$$

is a monoidal natural transformation.
Decorated corelation functors

A hypergraph functor is a strong monoidal functor that preserves the Frobenius maps.

**Theorem**

Suppose that $\mathcal{C}$ has finite colimits and a costable factorisation system $(\mathcal{E}, \mathcal{M})$, and $F, G : (\mathcal{C}; \mathcal{M}^{\text{op}}) \rightarrow (\text{Set}, \times)$ are lax symmetric monoidal functors, and

$$\theta : F \Rightarrow G$$

is a monoidal natural transformation. Then we obtain a hypergraph functor

$$T_\theta : F\text{Corel} \rightarrow G\text{Corel}.$$
Theorem
Every hypergraph category is equivalent, as a hypergraph category, to a decorated corelation category.
Theorem
Every hypergraph category is equivalent, as a hypergraph category, to a decorated correlation category.

In fact, allowing changes of the base category $\mathcal{C}$ and factorisation system $(\mathcal{E}, \mathcal{M})$, we can define a category of decorated correlation categories. This category is equivalent to the category of hypergraph categories.
Summary

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For coarser, ‘black box’ semantics, we can use decorated corelations.
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Decorated cospans give hypergraph categories, but ‘freely so’.

For coarser, ‘black box’ semantics, we can use decorated corelations.

This solution is general:

All hypergraph categories are decorated corelation categories.
Thanks for listening.

For more
John Baez's network theory program: http://math.ucr.edu/baez/networks/
These slides are available at: http://www.brendanfong.com/fcorel.pdf/