The Mathematics of Networks

John Baez, U. C. Riverside

Simons Institute of Computing, workshop on Compositionality, 6 December 2016
In many areas of science and engineering, people use *networks*, drawn as boxes connected by wires:

We need a good general theory of these!
Networks of some particular kind, with specified inputs and outputs, can be seen as morphisms in some symmetric monoidal category:

\[
\begin{array}{c}
X \\
\bullet \\
\bullet \\
\bullet \\
\end{array}
\rightarrow
\begin{array}{c}
\bullet \\
\bullet \\
\bullet \\
\end{array}
\rightarrow
\begin{array}{c}
\bullet \\
\bullet \\
\bullet \\
\end{array}
\rightarrow
\begin{array}{c}
\bullet \\
\bullet \\
\bullet \\
\end{array}
\rightarrow
\begin{array}{c}
Y \\
\bullet \\
\bullet \\
\end{array}
\]
Networks of some particular kind, with specified inputs and outputs, can be seen as morphisms in some symmetric monoidal category:

\[
\begin{array}{c}
\bullet \\
X & \rightarrow & \bullet \\
\end{array}
\]

Such networks let us describe open systems, meaning systems where:

- stuff can flow in or out;
- we can combine systems to form larger systems by composition and tensoring.
To use networks as a ‘syntax’ for open systems, we follow the ideas of ‘functorial semantics’:

- Networks of some kind, with specified input and outputs, will be morphisms in some symmetric monoidal category $\mathbf{X}$.
- To ‘interpret’ these networks we use a symmetric monoidal functor $F : \mathbf{X} \to \mathbf{Y}$, where $\mathbf{Y}$ is a symmetric monoidal category good for semantics, e.g. $\textbf{Set}$ or $\textbf{Rel}$. 
How can we construct symmetric monoidal categories with networks as morphisms?
How can we construct symmetric monoidal categories with networks as morphisms?

One way is to use generators and relations.
How can we construct symmetric monoidal categories with networks as morphisms?

One way is to use generators and relations.

We can present a (strict) symmetric monoidal category by specifying:

- a set of generating objects;
- a set of generating morphisms;
- a set of relations between morphisms.

We can then specify a (strict) symmetric monoidal functor by sending generators to generators in such a way that relations are preserved.
How can we construct symmetric monoidal categories with networks as morphisms?

One way is to use generators and relations.

We can present a (strict) symmetric monoidal category by specifying:

- a set of generating objects;
- a set of generating morphisms;
- a set of relations between morphisms.

We can then specify a (strict) symmetric monoidal functor by sending generators to generators in such a way that relations are preserved.
Another way, pioneered by Brendan Fong, is to use decorated cospans. For example, this:

![Diagram of a cospan of finite sets]

is really a cospan of finite sets:

![Diagram of a cospan]

where $S$ is ‘decorated’ with extra structure making it into the set of vertices of a graph: $E \xymatrix{ \ar[r]^s & S \ar[l]^t }$. 
Let’s look at a more interesting example: Petri nets.

A **Petri net** is a bipartite graph. The two kinds of vertices are called **places** and **transitions**.
In computer science, Petri nets became popular as models of concurrency starting in the 1970s. But they were invented for chemistry in 1939:

\[ C + O_2 \rightarrow CO_2 \]

\[ CO_2 + NaOH \rightarrow NaHCO_3 \]

\[ NaHCO_3 + HCl \rightarrow H_2O + NaCl + CO_2 \]

as an alternative to the more familiar reaction networks:
Now they’re used in epidemiology...
...systems biology...

... and many other fields.
In a Petri net with rates, each transition is assigned a rate constant: a positive real number. We can then write down a rate equation describing dynamics. For example, this Petri net with rates:

\[
\begin{align*}
\frac{dA_1}{dt} &= -r_1 A_1 A_2 \\
\frac{dA_2}{dt} &= -r_1 A_1 A_2 + 2r_2 A_3 \\
\frac{dA_3}{dt} &= r_1 A_1 A_2 - r_2 A_3
\end{align*}
\]
In a Petri net with rates, each transition is assigned a rate constant: a positive real number. We can then write down a rate equation describing dynamics. For example, this Petri net with rates:

\[ \begin{align*}
\frac{dA_1}{dt} &= -r_1 A_1 A_2 \\
\frac{dA_2}{dt} &= -r_1 A_1 A_2 + 2r_2 A_3 \\
\frac{dA_3}{dt} &= r_1 A_1 A_2 - r_2 A_3
\end{align*} \]
So far these Petri nets describe \textit{closed} systems.
So far these Petri nets describe *closed* systems.

But there’s a symmetric monoidal category of *open Petri nets with rates*, called *Petri*, where:

- an object is a finite set;
- a morphism $f : X \rightarrow Y$ is a Petri net with rates together with functions from $X$ and $Y$ to its set of places:
To compose morphisms $f : X \to Y$ and $g : Y \to Z$: we put them in series, identifying outputs of $f$ with inputs of $g$: 

we put them in parallel.
To compose morphisms $f : X \to Y$ and $g : Y \to Z$:

we put them in series, identifying outputs of $f$ with inputs of $g$:

To tensor morphisms, we put them in parallel.
An open Petri net with rates $f: X \rightarrow Y$ gives an **open rate equation** involving flows in and out, which can be arbitrary smooth functions of time. For example this:
An open Petri net with rates \( f: X \rightarrow Y \) gives an **open rate equation** involving flows in and out, which can be arbitrary smooth functions of time. For example this:

\[
\begin{align*}
\frac{dA_1}{dt} & = -r_1 A_1 A_2 + l_1(t) \\
\frac{dA_2}{dt} & = -r_1 A_1 A_2 + l_2(t) + l_3(t) \\
\frac{dA_3}{dt} & = 2r_1 A_1 A_2 - O_1(t)
\end{align*}
\]
So: open Petri nets with rates serve as a ‘syntax’, with open dynamical systems providing one possible ‘semantics’.

Let’s understand this using functorial semantics! We’ll get a symmetric monoidal functor

\[ \Box : \text{Petri} \to \text{Dynam} \]

Other choices of semantics correspond to other symmetric monoidal functors.
There is a symmetric monoidal category \textbf{Dynam} where:

- an object is a finite set;
- a morphism $f : X \rightarrow Y$ is an \textbf{open dynamical system}, meaning a cospan of finite sets

\begin{tikzpicture}
  \node (X) at (0,0) {$X$};
  \node (S) at (1,1) {$S$};
  \node (Y) at (2,0) {$Y$};
  \draw[->] (X) -- (S) node[above] {$i$};
  \draw[->] (S) -- (Y) node[above] {$o$};
\end{tikzpicture}

equipped with a smooth vector field $v$ on $\mathbb{R}^S$. 

Given input and output flows $I(t) \in \mathbb{R}^X$, $O(t) \in \mathbb{R}^Y$, an open dynamical system describes how a point $A(t) \in \mathbb{R}^S$ changes with time:

$$
\frac{d}{dt} A(t) = v(A(t)) + i^*(I(t)) - o^*(O(t))
$$

where $i^*$, $o^*$ push forward $\mathbb{R}$-valued functions from $X$, $Y$ to $S$. 
There is a symmetric monoidal category $\textbf{Dynam}$ where:

- an object is a finite set;
- a morphism $f : X \to Y$ is an **open dynamical system**, meaning a cospan of finite sets

\[
\begin{array}{ccc}
X & \leftarrow & S & \rightarrow & Y \\
i & & o & & \\
& \downarrow & & \downarrow & \\
& X & & Y & \\
\end{array}
\]

equipped with a smooth vector field $v$ on $\mathbb{R}^S$.

Given input and output flows $I(t) \in \mathbb{R}^X$, $O(t) \in \mathbb{R}^Y$, an open dynamical system describes how a point $A(t) \in \mathbb{R}^S$ changes with time:

\[
\frac{d}{dt} A(t) = v(A(t)) + i_*(I(t)) - o_*(O(t))
\]

where $i_*, o_*$ push forward $\mathbb{R}$-valued functions from $X, Y$ to $S$. 
Theorem (JB–Blake Pollard)

There is a symmetric monoidal functor \( \square : \text{Petri} \rightarrow \text{Dynam} \) sending any open Petri net with rates to its open dynamical system.

This is a statement of compositionality: we can determine the rate equation of a Petri net with rates by breaking it down into a composite and/or tensor product of simpler open Petri nets with rates, and repeatedly using:

\[
\square(fg) = \square(f) \square(g)
\]

\[
\square(f \otimes g) = \square(f) \otimes \square(g). 
\]
How do we prove this theorem? We use Fong’s theory of decorated cospans.

Say we start with a category $\mathbf{C}$ with finite colimits: in our example, $\mathbf{C} = \mathbf{FinSet}$. We can build a bicategory where morphisms are cospans in $\mathbf{C}$:

$$
\begin{array}{c}
\begin{array}{c}
\overset{N}{X} \\
\overset{i}{\nearrow} \quad \overset{N'}{\nearrow} \\
\overset{o}{\nwedge} \quad \overset{o'}{\nwedge}
\end{array}
\end{array}
$$

and composition is done by pushout:

$$
\begin{array}{c}
\begin{array}{c}
\overset{N + Y}{X} \\
\overset{i}{\nearrow} \quad \overset{o}{\nwedge} \\
\overset{i'}{\nwedge} \quad \overset{o'}{\nwedge} \\
\overset{N'}{Y} \quad \overset{Z}{\nearrow}
\end{array}
\end{array}
$$
Pushouts are defined only up to isomorphism, which is why we get a bicategory. But there is a category $\text{Cospan}(C)$ where morphisms are *isomorphism classes* of cospans in $C$.
Pushouts are defined only up to isomorphism, which is why we get a bicategory. But there is a category $\text{Cospan}(C)$ where morphisms are isomorphism classes of cospans in $C$.

Next, if we choose a functor $F: C \to \text{Set}$, we can try to build a category where a morphism is an isomorphism class of cospans

$$
\begin{array}{c}
S \\
i \\
o \\
X \leftrightarrow Y
\end{array}
$$

with $S$ ‘decorated’ by an element of $F(S)$.

In the case of Petri, $F(S)$ is the set of all ways of making $S$ into the set of places in some Petri net with rates.
Pushouts are defined only up to isomorphism, which is why we get a bicategory. But there is a category \textbf{Cospan}(C) where morphisms are *isomorphism classes* of cospans in \( C \).

Next, if we choose a functor \( F: C \to \text{Set} \), we can try to build a category where a morphism is an isomorphism class of cospans

![Coshap Diagram](attachment:image.png)

with \( S \) ‘decorated’ by an element of \( F(S) \).

In the case of \textbf{Petri}, \( F(S) \) is the set of all ways of making \( S \) into the set of places in some Petri net with rates.

But how do we ‘compose the decorations’ when we compose cospans?
Given composable morphisms

\[
\begin{array}{ccc}
S & \xrightarrow{d} & F(S) \\
\downarrow{i} & & \downarrow{o} \\
X & & Y \\
\downarrow{i} & & \downarrow{o} \\
S' & \xrightarrow{d'} & F(S') \\
\end{array}
\]

we compose the cospans by taking a pushout. We compose the decorations by taking \((d, d') \in F(S) \times F(S')\) and applying the composite function

\[
F(S) \times F(S') \rightarrow F(S + S') \rightarrow F(S + Y S')
\]

where the first step comes from \(F\) being a \textit{lax} monoidal functor.
Theorem (Brendan Fong)

Suppose that $\mathbf{C}$ has finite colimits and

$$F: (\mathbf{C}, +) \longrightarrow (\mathbf{Set}, \times)$$

is a lax symmetric monoidal functor. Then there is a symmetric monoidal category of $\mathbf{F}$-decorated cospans, $\mathbf{FCosp}$, where:

- an object is an object of $\mathbf{C}$;
- a morphism from $X$ to $Y$ is a cospan

$$\begin{array}{ccc}
S & \xrightarrow{d} & F(S) \\
\downarrow{i} & & \downarrow{o} \\
X & & Y
\end{array}$$


together with a decoration $d \in F(S)$. 
Theorem (Brendan Fong)

Suppose that $\mathbf{C}$ has finite colimits and

$$F : (\mathbf{C}, +) \longrightarrow (\mathbf{Set}, \times)$$

is a lax symmetric monoidal functor. Then there is a symmetric monoidal category of $F$-decorated cospans, $\mathbf{FCospan}$, where:

▶ an object is an object of $\mathbf{C}$;
▶ a morphism from $X$ to $Y$ is a cospan

\[
\begin{array}{c}
  S \\
  i \downarrow \quad \downarrow o \\
X \quad \quad \quad \quad \quad \quad \quad Y
\end{array}
\]

together with a decoration $d \in F(S)$. (Just kidding: actually, a morphism is an isomorphism class of these!)
Corollary (JB–Blake Pollard)

There is a symmetric monoidal category $\text{Petri}$ where:

- an object is a finite set;
- a morphism $f : X \to Y$ is a cospan of finite sets

\[
\begin{array}{c}
S \\
i \quad o \\
X \quad Y
\end{array}
\]

$\text{together with a Petri net with rates having } S \text{ as its set of places.}$
Corollary (JB–Blake Pollard)

There is a symmetric monoidal category $\text{Petri}$ where:

- an object is a finite set;
- a morphism $f : X \to Y$ is a cospan of finite sets

\[
\begin{array}{ccc}
  & S & \\
  i & & o \\
  X & \leftarrow & Y
\end{array}
\]

Togetheer with a Petri net with rates having $S$ as its set of places. (Just kidding: actually an isomorphism class of such cospans!)

So, a morphism looks like this:
Corollary (JB–Blake Pollard)

There is a symmetric monoidal category \textbf{Dynam} where:

\begin{itemize}
  \item an object is a finite set;
  \item a morphism \( f : X \rightarrow Y \) is a cospan of finite sets
\end{itemize}

\[ S \xymatrix{ & X \ar[ld]_i \ar[rd]^o & \ar[dd] \cr & & \cr Y & } \]

\textit{together with a smooth vector field on } \( \mathbb{R}^S \).
Corollary (JB–Blake Pollard)

There is a symmetric monoidal category $\text{Dynam}$ where:

- an object is a finite set;
- a morphism $f : X \to Y$ is a cospan of finite sets

\[
\begin{array}{ccc}
S & \ni & \mathcal{K} \\
| & i & | \\
X & \to & Y \\
| & o & | \\
Y & \to & X \\
\end{array}
\]

Togeter with a smooth vector field on $\mathbb{R}^S$. (Just kidding: actually an isomorphism class of these!)
Next, how do we get our symmetric monoidal functor

□: Petri → Dynam ?
Next, how do we get our symmetric monoidal functor

\[ \Box : \text{Petri} \rightarrow \text{Dynam} \]?

**Theorem (Brendan Fong)**

Suppose \( \mathbf{C} \) has finite colimits, \( F, G : (\mathbf{C}, +) \rightarrow (\text{Set}, \times) \) are lax symmetric monoidal functors, and \( \theta : F \Rightarrow G \) is a monoidal natural transformation. Then we obtain a symmetric monoidal functor

\[ T_\theta : F\text{Cospan} \rightarrow G\text{Cospan}. \]
Corollary (JB–Blake Pollard)

There is a symmetric monoidal functor □: Petri $\rightarrow$ Dynam sending any open Petri net with rates to the corresponding open dynamical system.
The same methods, and also the ‘generators and relations’ approach, let us study many categories of networks — and how they’re connected by functors.
The same methods, and also the ‘generators and relations’ approach, let us study many categories of networks — and how they’re connected by functors.

- **Electrical circuits**: JB and B. Fong, *A compositional framework for passive linear networks*. 

![Diagram of electrical circuit](attachment:image.png)
Signal-flow graphs in control theory:
Jason Erbele, *Categories in Control: Applied PROPs*.
B. Fong, *The Algebra of Open and Interconnected Systems*. 
We find a ‘network of network languages’: interesting symmetric monoidal categories connected by symmetric monoidal functors:
We find a ‘network of network languages’: interesting symmetric monoidal categories connected by symmetric monoidal functors:
We find a ‘network of network languages’: interesting symmetric monoidal categories connected by symmetric monoidal functors:
We find a ‘network of network languages’: interesting symmetric monoidal categories connected by symmetric monoidal functors:

```
ResCirc → Markov → Petri
```

```
□
```

```
Dynam
```

```
steady states
```

```
Rel
```
We find a ‘network of network languages’: interesting symmetric monoidal categories connected by symmetric monoidal functors:

\[
\text{ResCirc} \xrightarrow{} \text{Markov} \xrightarrow{} \text{Petri} \\
\text{Circ} \xrightarrow{} \text{Dynam} \xrightarrow{} \text{Rel} \\
\quad \square \xrightarrow{} \text{steady states}
\]
We find a ‘network of network languages’: interesting symmetric monoidal categories connected by symmetric monoidal functors:
We find a ‘network of network languages’: interesting symmetric monoidal categories connected by symmetric monoidal functors:

\[
\begin{align*}
\text{ResCirc} & \rightarrow \text{Markov} \rightarrow \text{Petri} \\
\text{Circ} & \rightarrow \text{Dynam} \\
\text{SigFlow} & \rightarrow \text{LinRel}(\mathbb{C}(z)) \rightarrow \text{Rel}
\end{align*}
\]
We find a ‘network of network languages’: interesting symmetric monoidal categories connected by symmetric monoidal functors:
There is also more to say about decorated cospans! For example:

**Theorem (Courser)**

Suppose that $\mathcal{C}$ has finite colimits and $F : (\mathcal{C}, +) \longrightarrow (\text{Set}, \times)$ is a lax symmetric monoidal functor. Then there is a symmetric monoidal bicategory where:

- an object is an object of $\mathcal{C}$;
- a morphism from $X$ to $Y$ is a cospan $X \xleftarrow{i} S \xrightarrow{o} Y$ together with a decoration $d \in F(S)$;
- a 2-morphism is a map of cospans $X \xleftarrow{i} S \xrightarrow{o} Y$ such that $F(f)$ maps $d \in F(S)$ to the decoration $d' \in F(S')$. 