Testing for Affine Invariant Properties of Algebraic Functions

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Based on:

- Bhattacharyya, Fischer, HH, P. Hatami, and Lovett, Every locally characterized affine-invariant property is testable, STOC 2013.
- HH and Lovett, Estimating the distance from testable affine-invariant properties, FOCS 2013.
- HH, P. Hatami, and Lovett, in preparation.
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**Common Theme**

Extending the property testing results in graph theory to the algebraic setting.
Property Testing

- Given a function (e.g. a graph),
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Evaluate it on a small number of points.
Property Testing

- Given a function (e.g. a graph),
- Evaluate it on a small number of points.
- Decide whether
  - it satisfies a given property (e.g. triangle-freeness),
  - or is “far” from satisfying that property.
The field of property testing has emerged from [Blum, Luby, Rubinfeld 93], [Babai, Fortnow, Lund 91], etc.
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Formally defined by [Rubinfeld, Sudan 96], [Goldreich, Goldwasser, Rubinfeld 98].

Closely related to limit theories of combinatorial objects [Lovász-Szegedy 2010].
Our setting

Functions of the form $f : \mathbb{F}_p^n \rightarrow \{0, \ldots, R\}$ where

- $p$ is a fixed prime.
- $R$ is a fixed integer.
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- $p$ is a fixed prime.
- $R$ is a fixed integer.

Two important cases:

- $R = 1$: Functions $f : \mathbb{F}_p^n \rightarrow \{0, 1\}$.
- $R = p - 1$: Functions $f : \mathbb{F}_p^n \rightarrow \mathbb{F}_p$. 

Definition

- \( \text{dist}(f, g) = \Pr[f(x) \neq g(x)] \).
**Definition**

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- \( \text{dist}(f, P) = \min_{g \in P} \text{dist}(f, g) \).
**Definition**

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**Definition**

A (Proximity Oblivious) property tester for $P$ must
- Make a **constant** number of queries to $f$. 

**Notes**

- $P$-far from $P$ accept
  $$\Pr[\text{reject}] \geq \delta(\epsilon)$$

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Definition
A (Proximity Oblivious) property tester for \( P \) must
- Make a constant number of queries to \( f \).
- Accepts if \( f \in P \).
- Rejects with probability \( \geq \delta(\epsilon) > 0 \) if \( \text{dist}(f, P) > \epsilon > 0 \).
Example

Let

\[ P = \{ \text{functions } f : \mathbb{F}_p^n \to \{0, 1\} \text{ where } f \equiv 0 \}. \]
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Test

- Pick \( x \in \mathbb{F}_p^n \) at random.
- If \( f(x) = 0 \) accept
  - otherwise reject.
Example

Let

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Test

- Pick \( x \in \mathbb{F}_p^n \) at random.
- If \( f(x) = 0 \) accept, otherwise reject.

Analysis

- If \( f \equiv 0 \), then \( \Pr[\text{accept}] = 1 \).
- If \( \text{dist}(f, P) > \epsilon \), then \( \Pr[\text{reject}] \geq \epsilon \).
What conditions should we impose on $P$?
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We do not want to treat $\mathbb{F}_p^n$ as a generic set of size $p^n$ and ignore the algebraic structure of $\mathbb{F}_p^n$. 

$\textbf{Example}$: $P = \{\text{Polynomials } f : \mathbb{F}_p^n \to \mathbb{F}_p \text{ of degree } \leq d\}$. 

Kaufman-Sudan $P$ is called affine-invariant if $f \in P \Rightarrow f \circ A \in P$ for any affine transformation $A : \mathbb{F}_p^n \to \mathbb{F}_p^n$. (i.e. $A : x \mapsto Bx + c$).
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**Example**

$$P = \{ \text{Polynomials } f : \mathbb{F}_p^n \rightarrow \mathbb{F}_p \text{ of degree } \leq d \}.$$
Question

Which affine-invariant properties $P$ are testable?

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Example

$$P = \{ \text{Polynomials } f : \mathbb{F}_p^n \to \mathbb{F}_p \text{ of degree } \leq d \}.$$

Local Characterization of $P$

- $f \in P \iff f|_V \in P$ for all affine subspace $V \subseteq \mathbb{F}_p^n$ with $\dim(V) = d + 1$. 
Test for deg ≤ d.

- Pick a $d + 1$-dimensional random affine subspace $V \subseteq \mathbb{F}_p^n$.
- Accept if deg$(f|_V) \leq d$, and reject otherwise.
Test for $\deg \leq d$.

- Pick a $d + 1$-dimensional random affine subspace $V \subseteq \mathbb{F}_p^n$.
- Accept if $\deg(f|_V) \leq d$, and reject otherwise.

We have

- if $f \in P$ then $\Pr[\text{accept}] = 1$.
- if $\text{dist}(f, P) \geq \epsilon$ then $\Pr[\text{reject}] > \delta(\epsilon) > 0$. [Alon, Kaufman, Krivelevich, Litsyn, Ron 2005].
Locally characterizable

$P$ is locally characterizable if there exists $k > 0$ such that

- $f \in P$ ⇐⇒
- $f|_V \in P$ for all affine subspace $V \subseteq \mathbb{F}_p^n$ with $\dim(V) = k$. 

Theorem (Bhattacharyya, Fischer, HH, P. Hatami, and Lovett)
Every locally characterizable property is (PO)-testable.
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Proof Sketch
A classical example

The graph property of triangle-freeness.

The test

Pick three vertices at random.

If they form a triangle reject.

Otherwise accept.

Analysis

If \( \Delta \)-free, we always accept. (trivial)

If \( \epsilon \)-far from \( \Delta \)-free, then \( \Pr[\text{reject}] > \delta(\epsilon) > 0 \). (non-trivial)
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**Clean-up**: Empty non-uniform cells, and the almost-empty cells.
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$\Rightarrow H$ has a $\triangle$ $\Rightarrow H$ has many $\triangle$’s due to its structure.
$\Rightarrow G$ has many $\triangle$’s (we only removed edges from $G$).
A different example

The graph property of **induced** $C_5$-freeness.
A different example

The graph property of induced $C_5$-freeness.

The test

- Pick five vertices at random.
- Reject if they induce a $C_5$.
- Otherwise accept.
A different example

The graph property of induced $C_5$-freeness.

The test

- Pick five vertices at random.
- Reject if they induce a $C_5$.
- Otherwise accept.

Analysis

- If induced-$C_5$-free, we always accept. (trivial)
- If $\epsilon$-far from induced-$C_5$-free, then $\Pr[\text{reject}] > \delta(\epsilon) > 0$. (non-trivial)
- Suppose $G$ is $\epsilon$-far from being induced-$C_5$-free.
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Suppose $G$ is $\epsilon$-far from being induced-$C_5$-free.

**Regularize:** Partition vertices into almost equal parts, so that almost all cells are uniform.

**Clean-up:** Empty non-uniform cells, and the almost-empty cells.

- Might create many $C_5$'s, and so
- $H$ has many $C_5$'s $\nRightarrow$ $G$ has many $C_5$'s.
This can be handled using a stronger regularity lemma.
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For most cells: density $\approx$ subcell density.
The algebraic setting $\mathbb{F}_p^n$
Theorem (Recall)

*Every locally characterizable property is (PO)-testable.*
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The general approach

- Consider $f$ that is $\epsilon$-far from $P$. 
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- Consider $f$ that is $\epsilon$-far from $P$.
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- Clean-up the regularization of $f$ to obtain $g$ close to $f$. 
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- Consider $f$ that is $\epsilon$-far from $P$.
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- Then $g \notin P$ and thus violates some local condition.
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**The general approach**

- Consider $f$ that is $\epsilon$-far from $P$.
- Regularize $f$.
- Clean-up the regularization of $f$ to obtain $g$ close to $f$.
- Then $g \notin P$ and thus violates some local condition.
- Exploit the nice structure of $g$ to show that the test works for $f$. 
Partition $\mathbb{F}_p^n$ such that $f$ is uniform on almost all parts.
Regularization

Partition $\mathbb{F}_p^n$ such that $f$ is uniform on almost all parts.

- Consider polynomials $Q_1, \ldots, Q_c : \mathbb{F}_p^n \to \mathbb{F}_p$ of degree $\leq d$. 
Partition $\mathbb{F}_p^n$ such that $f$ is uniform on almost all parts.

- Consider polynomials $Q_1, \ldots, Q_c : \mathbb{F}_p^n \rightarrow \mathbb{F}_p$ of degree $\leq d$.
- Partition $\mathbb{F}_p^n$ according to $(Q_1(x), \ldots, Q_c(x))$. 
Need an analogue of the AFKS-regularity of graphs for $\mathbb{F}_p^n$. 

BFL Subatoms are chosen by setting $(Q_1(x), \ldots, Q_b(x)) = \vec{c}_0$. 

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**BFL** Subatoms are chosen by setting $(Q_1(x), \ldots, Q_b(x)) = \vec{c}_0$. 

![Diagram](image-url)
clean-up

- Modify $f$ to remove all irregularities:

  - For each big atom $c$, let $t_c$ be the popular value in its subatom.
  - Change the value of $f$ on irregular atoms $c$ to $t_c$.
  - Change the unpopular values on every atom $c$ to $t_c$.

The new function $g$ is not in $P$.

There is a $W$ such that $g|_W \not\in P$.

There are many $W$'s for which $f|_W \not\in P$. 
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Equidistribution for Polynomial factors
\[ f \approx \Gamma(Q_1(x), \ldots, Q_c(x)). \]
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Need to analyze the distribution of \( f|_V \) for a random \( V \).
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Need to analyze the distribution of \( f|_V \) for a random \( V \).

Let \( L_1, \ldots, L_{\rho^k} \) be the points of a random \( V \).
\[ f \simeq \Gamma(Q_1(x), \ldots, Q_c(x)). \]

Need to analyze the distribution of \( f|_V \) for a random \( V \).

Let \( L_1, \ldots, L_{p^k} \) be the points of a random \( V \).

We need to understand the distribution of

\[
\begin{pmatrix}
Q_1(L_1) & \ldots & Q_c(L_1) \\
Q_1(L_2) & \ldots & Q_c(L_2) \\
\vdots & & \ddots \\
Q_1(L_{p^k}) & \ldots & Q_c(L_{p^k})
\end{pmatrix}.
\]
If $Q_1, \ldots, Q_c$ are of "high rank", then $Q_1(X), \ldots, Q_c(X)$ are almost independent (entries in each row are almost independent).

Note that if $\deg(Q_i) = 1$, then

$$Q_1(L_1) + Q_1(L_2) = Q_1(L_3) + Q_1(L_4)$$

if $L_1 + L_2 = L_3 + L_4$.

If $\deg(Q_i) = 2$, then

$$\sum_{S \subseteq \{1, 2, 3\}} (-1)^{|S|} Q_i(e_0 + \sum_{i \in S} e_i) = 0.$$
Green-Tao, Kaufman-Lovett: If $Q_1, \ldots, Q_c$ are of “high rank”, then

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We cannot expect this for all entries

- Note that if $\deg(Q) = 1$, then $Q(L_1) + Q(L_2) = Q(L_3) + Q(L_4)$ if $L_1 + L_2 = L_3 + L_4$. 
Green-Tao, Kaufman-Lovett: If $Q_1, \ldots, Q_c$ are of “high rank”, then

$$Q_1(X), \ldots, Q_c(X),$$

are almost independent (entries in each row are almost independent).

We cannot expect this for all entries

- Note that if $\deg(Q) = 1$, then $Q(L_1) + Q(L_2) = Q(L_3) + Q(L_4)$ if $L_1 + L_2 = L_3 + L_4$.
- If $\deg(Q) = 2$, then $\sum_{S \subseteq \{1,2,3\}} (-1)^{|S|} Q(e_0 + \sum_{i \in S} e_i) = 0$. 
Theorem

If rank is high, these degree related dependencies are the only dependencies (up to a small error).
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- Large values of $p$: [HH, Lovett 2011].
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- Large values of $p$: [HH, Lovett 2011].
- General $p$, but affine systems of linear forms: [Bhattacharyya, Fischer, HH, P. Hatami, and Lovett 2013].
Theorem

*If rank is high, these degree related dependencies are the only dependencies (up to a small error).*

- Large values of \( p \): [HH, Lovett 2011].
- General \( p \), but affine systems of linear forms: [Bhattacharyya, Fischer, HH, P. Hatami, and Lovett 2013].
- General case: [H, P. Hatami, and Lovett in preparation].
Examples of locally characterizable properties
Example
Testable

\[ P = \{ \text{Polynomials } f : \mathbb{F}_p^n \rightarrow \mathbb{F}_p \text{ of degree } \leq d \}. \]
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Definition (Degree structural properties)

- Fix \( d_1, \ldots, d_c \) and \( \Gamma : \mathbb{F}_p^c \rightarrow [R] \).
- The property of being expressible as \( \Gamma(P_1, \ldots, P_c) \) where \( \deg(P_i) \leq d_i \).
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Example

- Polynomials \( f : \mathbb{F}_p^n \to \mathbb{F}_p \) that are products of two quadratics.
- Polynomials \( f : \mathbb{F}_p^n \to \mathbb{F}_p \) that are squares of a quadratics.
- Polynomials \( f : \mathbb{F}_p^n \to \mathbb{F}_p \) of the form \( f = ab + cd \) where \( a, b, c, d \) are cubics.
Theorem (Bhattacharyya, Fischer, HH, P. Hatami, and Lovett)

Every degree structural property is locally characterizable and hence (PO)-testable.
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Every degree structural property is locally characterizable and hence (PO)-testable.

- Our proof uses regularity $f \approx \Gamma(Q_1, \ldots, Q_c)$. 
Theorem (Bhattacharyya, Fischer, HH, P. Hatami, and Lovett)

Every degree structural property is locally characterizable and hence (PO)-testable.

- Our proof uses regularity $f \approx \Gamma(Q_1, \ldots, Q_c)$.
- Consequently does not provide any reasonable bound on the dimension.
A stronger notion of testing
Definition (Recall)

A (Proximity Oblivious) property tester for \( P \) must

- Make a constant number \( q \) of queries.
- Accepts if \( f \in P \).
- Rejects with probability \( \geq \delta(\epsilon) > 0 \) if \( \text{dist}(f, P) > \epsilon > 0 \).
**Definition (Recall)**

A (Proximity Oblivious) property tester for $P$ must

- Make a constant number $q$ of queries.
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**Definition**

A property tester for $P$ must

- Make $q(\epsilon)$ queries.
- Accepts if $f \in P$. (one-sided error).
- Rejects with probability $\geq \delta(\epsilon) > 0$ if $\text{dist}(f, P) > \epsilon > 0$. 

**Theorem (Alon-Shapira 2005)**

Every hereditary graph property is testable with one-sided error.
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Every hereditary graph property is testable with one-sided error.
Definition

An affine-invariant property $P$ is affine subspace hereditary if the restriction of any $f \in P$ to any affine subspace of $\mathbb{F}_p^n$ also satisfies $P$. 
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Conjecture [Bhattacharyya, Grigorescu, Shapira 2010]

Every affine subspace hereditary property is testable with one-sided error.
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Theorem (Bhattacharyya, Fischer, HH, P. Hatami, and Lovett)

Every affine subspace hereditary property of “bounded complexity” is testable with one-sided error.
Estimating the distance from a property
Definition

For a property $P$, and $\alpha > 0$, let $P_\alpha$ be the set of functions with distance at most $\alpha$ from $P$. 

Theorem (Fischer, Newman 2007)
For every testable graph property $P$ and every $\alpha > 0$, the property $P_\alpha$ is testable two-sided error.

Theorem (HH, Lovett 2013)
For every testable affine-invariant property $P$ and every $\alpha > 0$, the property $P_\alpha$ is testable with two-sided error.

One can estimate the distance from every testable property. Was unknown even for simple properties such as cubic polynomials.
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How does the test work?

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How does the test work?

- Let $f : \mathbb{F}_p^n \rightarrow \{0,1,\ldots,\ell\}$ be a given function.
- Let $W$ be a random affine subspace of large dimension.
- With high probability $\operatorname{dist}(f|_W, P) \approx \operatorname{dist}(f, P)$:
  - Completeness: If $f$ is $\alpha$-close to $P$ then $f|_W$ is $(\alpha + \epsilon/2)$-close to $P$.
  - Soundness: If $f$ is $\alpha + \epsilon$-far from $P$ then $f|_W$ is $(\alpha + \epsilon/2)$-far from $P$. 
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- $f|_W$ is $(\alpha + \epsilon/4)$-close to $g|_W$. 
High-level Proof

Completeness: If \( f \) is \( \alpha \)-close to \( P \) then \( f|_V \) is \( (\alpha + \epsilon/2) \)-close to \( P \).

- \( f \) is \( \alpha \)-close to some \( g \) in \( P \).
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- The test cannot distinguish \( g \) from \( g|_W \).
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- $f|_W$ is $(\alpha + \epsilon/2)$-close to $P$. 
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Soundness: If $f$ is $\alpha + \epsilon$-far from $P$ then $f|_W$ is $(\alpha + \epsilon/2)$-far from $P$. 

- If $f|_W$ is $(\alpha + \epsilon/2)$-close to $P$, there is some $h \in P$ that is $(\alpha + \epsilon/2)$-close to $f|_W$.
- Since $f$ and $f|_W$ have similar structures, we can lift $h$ to some $g$:
  - $g$ is $(\alpha + \epsilon/2)$-close to $P$.
  - The test cannot distinguish $g$ and $h$, so $g$ is close to $P$.
- We conclude that $f$ is $(\alpha + \epsilon)$-close to $P$. 

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Is every affine-invariant affine-subspace hereditary property testable?

Find a direct proof (with reasonable bounds) for the fact that degree-structural properties are locally characterizable.

For $f: \mathbb{F}_n^2 \rightarrow \mathbb{F}_2$, the Gowers $U^4$ norm (16 queries) can be used to distinguish:

$\text{Corr}(f, \text{non-classical cubics})$ is non-negligible.

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Thank you!