

Symmetry-Preserving Finite Synthesis & Amalgamation

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- amalgamation: overlap specifications
- local vs. global consistency
- local vs. global symmetry

finite model constructions with a touch of
combinatorial geometry and group theory

amalgamation?

- **synthesis & amalgamation:**

hypergraphs & relational structures: (de)composition patterns
model theory & algebra: homogeneity, Fraïssé limits
geometry: atlases of charts & changes of co-ordinates

- **local/partial symmetries:**

logic & model theory: partial isomorphisms
geometry & algebra: groupoids, inverse semigroups

- **local consistency:**

logic & combinatorics: constraint satisfaction
logic & databases: constraints, dependencies
logic & computation: e.g. quantum information

... calling for global solutions, closures, completions

a generic free amalgamation construction

from finite families

$$\left. \begin{array}{l} (\mathcal{A}_s)_{s \in S} \quad \text{of relational structures} \\ (\rho_e : \mathcal{A}_s \xrightarrow{\text{part}} \mathcal{A}_{s'})_{e \in E[s, s']} \quad \text{of partial isomorphisms} \end{array} \right\} (*)$$

find natural free amalgam of disjoint copies

$(\mathcal{A}_s, w) \simeq \mathcal{A}_s$ tagged by walks w in $I = (S, (E[s, s'])),$
with (\mathcal{A}_s, w) and $(\mathcal{A}_{s'}, w \cdot e)$ overlapping according to ρ_e

$$(((\mathcal{A}_s), (\rho_e)) \otimes I^*) / \approx \quad \text{based on multi-sorted monoid } I^* \text{ of walks in } I$$

- realises the amalgamation pattern $(*)$
- is free in a universal algebraic sense
- is generic but infinite (unless I is acyclic)

realisation of $H = ((\mathcal{A}_s), (\rho_e))$ in general:

a relational structure \mathcal{A} with an atlas given by superimposed hypergraph structure (A, \tilde{S}) of charts $\pi_{\tilde{s}}: \mathcal{A} \upharpoonright \tilde{s} \simeq \mathcal{A}_{\pi(\tilde{s})}$ s.t.

- **locally, all ρ_e -overlaps are realised:**

each $\pi_{\tilde{s}}^{-1}(\mathcal{A}_s)$ overlaps with some $\pi_{\tilde{s}'}^{-1}(\mathcal{A}_{s'})$ according to ρ_e

- **globally, no incidental overlaps occur:**

if $\tilde{s} \cap \tilde{t} \neq \emptyset$, then this is due to a composition of ρ_{e_i}
for some *single* walk $w = e_1 \cdots e_m$ from $\pi(\tilde{s})$ to $\pi(\tilde{t})$

NB: the second, “no-nonsense” condition avoids potential relational inconsistencies for amalgams:

compositions of partial isomorphisms are safe while unions are not!

aside: the rôle of groupoids/inverse semigroups

composition structure of partial bijections:

- with partial composition (as a total operation)
 \rightsquigarrow inverse semigroups
- with exact composition (as a partial operation)
 \rightsquigarrow groupoids

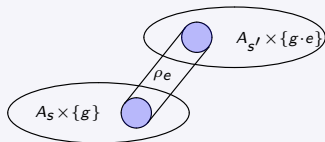
groupoids capture local/partial symmetries

... just as groups capture global symmetries

(II) reduced products with groupoids

I-groupoid: $\mathbb{G} = (G, (G_{st})_{s,t \in S}, \cdot, (1_s)_{s \in S})$ with
associative compositions $G_{st} \times G_{tu} \rightarrow G_{su}$,
neutral elements $1_s \in G_{ss}$, inverses, ...
designated generators $(g_e)_{e \in E}$

\rightsquigarrow **reduced products** $\mathbf{H} \otimes \mathbb{G}$
as candidate realisations

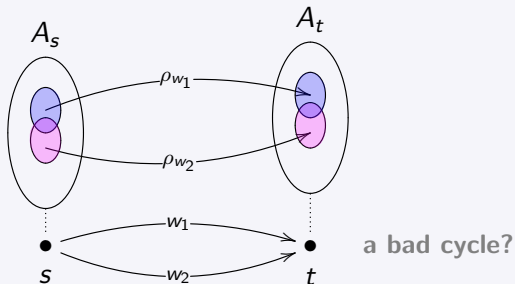


when is this a realisation of \mathbf{H} ?

obstructions: wrong kind of cycles (1)

violation of path independence for elements

- conflicting identifications may collapse individual A_s

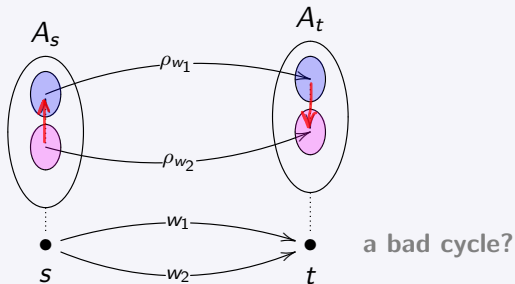


can be overcome by relatively simple pre-processing:
replace I by finite covering \hat{I} that unfolds bad cycles

obstructions: wrong kind of cycles (2)

violations of path independence for tuples

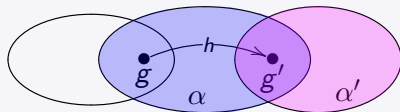
- incidental parallel identifications may cause conflicts at the relational level



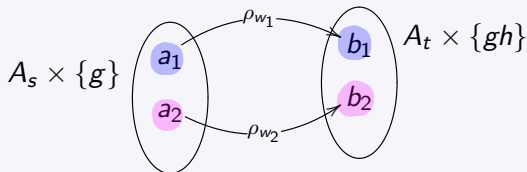
⇒ substantial acyclicity conditions on suitable groupoids

an appropriate notion of acyclicity: control coset cycles

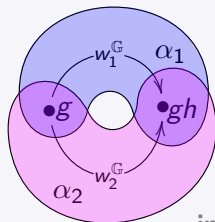
- not just short generator cycles in the Cayley graph of \mathbb{G} , but short cycles of cosets $g\mathbb{G}[\alpha]$ generated by subsets $\alpha \subseteq E$



- in particular, need to avoid certain coset cycles of length 2



$$w_1^{\mathbb{G}} = h = w_2^{\mathbb{G}}$$

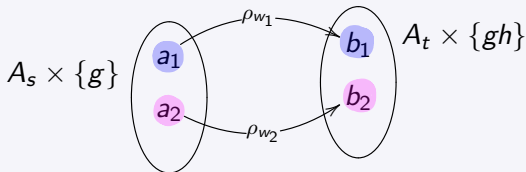


in reduced product

in \mathbb{G}

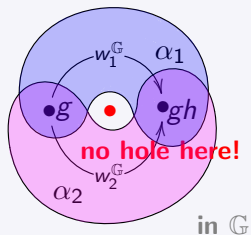
bad coset cycles & obstruction no. (2)

... in particular, need to avoid certain coset cycles of length 2:



$$w_1^G = h = w_2^G$$

in reduced product



in G

α_1 : generators carrying a_1 }
 α_2 : generators carrying a_2 }

want $h = w^G$ for $w \in \langle \alpha_1 \cap \alpha_2 \rangle$
 so that w carries both a_1 and a_2

$$\langle \alpha_1 \rangle \cap \langle \alpha_2 \rangle = \langle \alpha_1 \cap \alpha_2 \rangle$$

avoiding a coset 2-cycle

any degree of acyclicity in finite groupoids

theorem (O_13)

for every $N \in \mathbb{N}$ and incidence pattern $I = (S, E)$ there are finite I -groupoids \mathbb{G} without coset cycles of length up to N

corollary

for any $H = ((\mathcal{A}_s), (\rho_e))$ obtain realisations $H \otimes \mathbb{G}$ that

- respect all symmetries of H (genericity)
- have any desired degree of (local) acyclicity

with an atlas of charts onto the (\mathcal{A}_s)
forming an N -acyclic hypergraph
with changes of co-ordinates generated by the (ρ_e)

**generic & locally free finite realisations
of any finite amalgamation pattern**

applications (1): hypergraph coverings

corollary

every finite hypergraph admits, for $N \in \mathbb{N}$, finite coverings that

- are N -acyclic in the sense that every induced sub-hypergraph on up to N vertices is acyclic (tree decomposable, α -acyclic)
- possess a fibre-transitive automorphism group that lifts all symmetries of the given hypergraph

idea: generic N -acyclic finite realisations of the overlap pattern corresponding to the “exploded view” of the given hypergraph

applications (2): lifting local to global symmetries

EPPA: extension properties for partial automorphisms

Hrushovski 92 (graphs)

Herwig 98 (relational structures)

Herwig–Lascar 00 (fmp w.r.t. forbidden homomorphisms)

here, as a corollary

for finite \mathcal{A} and collection of partial automorphisms (ρ_p) find finite EPPA extension $\mathcal{B} \supseteq \mathcal{A}$ as generic realisation of pattern of self-overlaps of \mathcal{A} induced by the ρ_p

- with atlas of \mathcal{A} -charts forming N -acyclic hypergraph on B
- hence, locally free & universal w.r.t. homomorphisms

which implies full Herwig–Lascar result

applications (3): various uses in modal/guarded logics

- **finite model properties & finite controllability for guarded logics and constraints**
using finite coverings of controlled acyclicity and/or Herwig–Lascar extension properties
- **van Benthem style characterisation thms (fmt & classical) for the guarded fragment GF and relatives like, e.g., GNF**
using finite coverings of controlled acyclicity
- **van Benthem style characterisation thms (fmt & classical) for (modal) common knowledge logic**
new, with Felix Canavoi, using (finite) $S5$ -frames over Cayley groups with controlled coset-acyclicity

some related references

Bárány–Gottlob–O (LMCS 2014, arXiv:1309.5822): Querying the guarded fragment

Bárány–ten Cate–O (VLDB 2012, arXiv:1203.0077): Queries with guarded negation

Grädel–O (2014): The freedoms of (guarded) bisimulation

Hodkinson–O (BSL 2003): Finite conformal hypergraph covers and Gaifman cliques in finite structures

Herwig–Lascar (Transactions of the AMS 2000): Extending partial isomorphisms and the profinite topology on free groups

O (Journal of the ACM 2012): Highly acyclic groups, hypergraph covers and the guarded fragment

O (arXiv:1404.4599): **Finite groupoids, finite coverings and symmetries in finite structures**, see current version at:

→ <http://www.mathematik.tu-darmstadt.de/~otto/>