Canonical Functions and Constraint Satisfaction

Antoine Mottet

Workshop {Symmetry, Logic, Computation}

- Finding general conditions for tractability of infinite-domain CSPs, akin to the finite case

What is this about?

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- If possible find decidable conditions.

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Theorem

Assume that the finite-domain tractability conjecture holds. If the relations of $\mathbb{A}$ are definable in a unary language, then $\mathrm{CSP}(\mathbb{A})$ is in P or NP-complete.

Outline

Computation: Constraint Satisfaction

Symmetry: Canonical Functions

~~Logic~~ Computation

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Let $\mathbb{A}$ be a relational structure, in a fixed finite signature τ.

Definition (CSP($\mathbb{A}$))

Input: a finite τ-structure $\mathbb{B}$
Question: $\exists$ homomorphism $h\colon \mathbb{B} \to \mathbb{A}$?

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Complexity: linear time

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Input: a hypergraph with vertices V and hyperedges $E_+(x, y, z)$ and $E_\times(x, y, z)$
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$$\begin{cases} s(x) + s(y) = s(z) & (x, y, z) \in E_+ \\ s(x) \times s(y) = s(z) & (x, y, z) \in E_\times \end{cases}$$

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Complexity: undecidable.

Theorem (Matiyasevich-Davis-Robinson-Putnam)

Every recursively enumerable set $S \subseteq \mathbb{Z}$ is the projection on one variable of the set of solutions of some instance of CSP($\mathbb{Z}, +, \times$).

Conjecture (Feder-Vardi, '93)

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Transition to infinite domains:

- Find a reasonable class $\mathcal{A}$ of infinite structures,
- Classify the complexity of $\mathrm{CSP}(\mathbb{A})$ for all $\mathbb{A} \in \mathcal{A}$, assuming the tractability conjecture.

Definition

$\mathbb{B}$ is finitely bounded if there exists a finite family $\mathcal{F}$ of finite structures such that for all finite $\mathbb{C}$,

$$\mathbb{C} \text{ substructure of } \mathbb{B} \Leftrightarrow \forall F \in \mathcal{F}, F \text{ not a substructure of } \mathbb{C}$$

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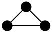
- Universal triangle-free graph: $\mathcal{F} = \{$$\}$.

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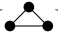
- Universal triangle-free graph: $\mathcal{F} = \{$$\}$.

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$\mathbb{B}$ is homogeneous if every partial isomorphism with finite domain can be extended to an automorphism.

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Question: How to prove it, assuming the finite-domain conjecture?

BP conjecture is confirmed for:

- Reducts of $(\mathbb{N}, =)$ (Bodirsky, Kára, '06)
- Reducts of the Rado graph (Bodirsky, Pinsker, JACM'15)
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In the first 3 cases, the classification is of the form:

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Not true for $(\mathbb{Q}, <)$.

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Oligomorphic groups, clones

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- $\phi\colon \mathscr{C} \to \mathscr{P}$ is continuous iff for every $g \in \mathscr{C}$, there is a finite set $X' \subset X$ such that $g|_{X'} = h|_{X'} \Rightarrow \phi(g) = \phi(h)$.

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Remark: G-canonical functions form a clone.

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- non-example: the maximum function

Definition (Mash-up)

G-canonical functions g, h, $\mathcal{O}, \mathcal{O}'$ G-orbits of m-tuples.
ω is a mash-up of g, h if it is G-canonical and

$$\omega(\mathcal{O}, \mathcal{O}') = g(\mathcal{O}, \mathcal{O}')$$
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Definition

$\mathscr{C}, \mathscr{D}$ clones. $\phi\colon \mathscr{C} \to \mathscr{D}$ is a clone homomorphism if $\phi(pr_i^n) = pr_i^n$ and $\phi(f \circ (g_1, \dots, g_n)) = \phi(f) \circ (\phi(g_1), \dots, \phi(g_n))$.

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Definition (Barto, Opršal, Pinsker)

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h1 homomorphisms preserve equations of height 1.

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4. *there is no pseudo-cyclic operation in $\mathscr{C}$;*

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Algebraic Dichotomy

Theorem

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Any of these properties is decidable!

Outline

Computation: Constraint Satisfaction

Symmetry: Canonical Functions

~~Logic~~ Computation

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Assume the tractability conjecture, and let $\mathbb{A}$ be a reduct of a finitely bounded homogeneous structure $\mathbb{B}$. If $\mathbb{A}$ has a pseudo-cyclic polymorphism that is $\mathrm{Aut}(\mathbb{B})$*-canonical, then* $\mathrm{CSP}(\mathbb{A})$ *is in P.*

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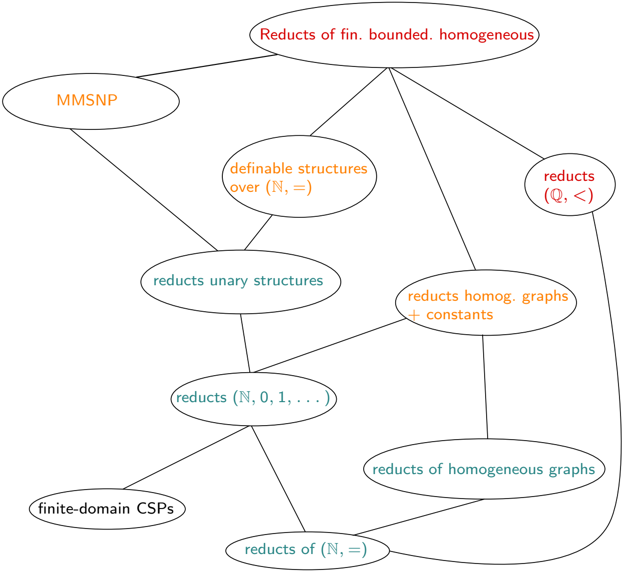
Consequence: membership in P is decidable!

if P=NP or the tractability conjecture is true

Where are canonical functions enough?



MMSNP

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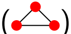
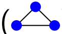
$\exists Red\ \exists Blue\ \forall x, y, z\ (\neg($  $) \wedge \neg($  $))$

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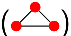
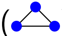
$\exists Red\ \exists Blue\ \forall x, y, z\ (\neg($  $) \wedge \neg($  $))$

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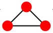
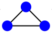
$\exists Red\ \exists Blue\ \forall x, y, z\ (\neg($  $) \wedge \neg($  $))$

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Theorem

If all obstructions are monochromatic, then tractability is witnessed by canonical functions.