Approximating Boolean functions with depth-2 circuits

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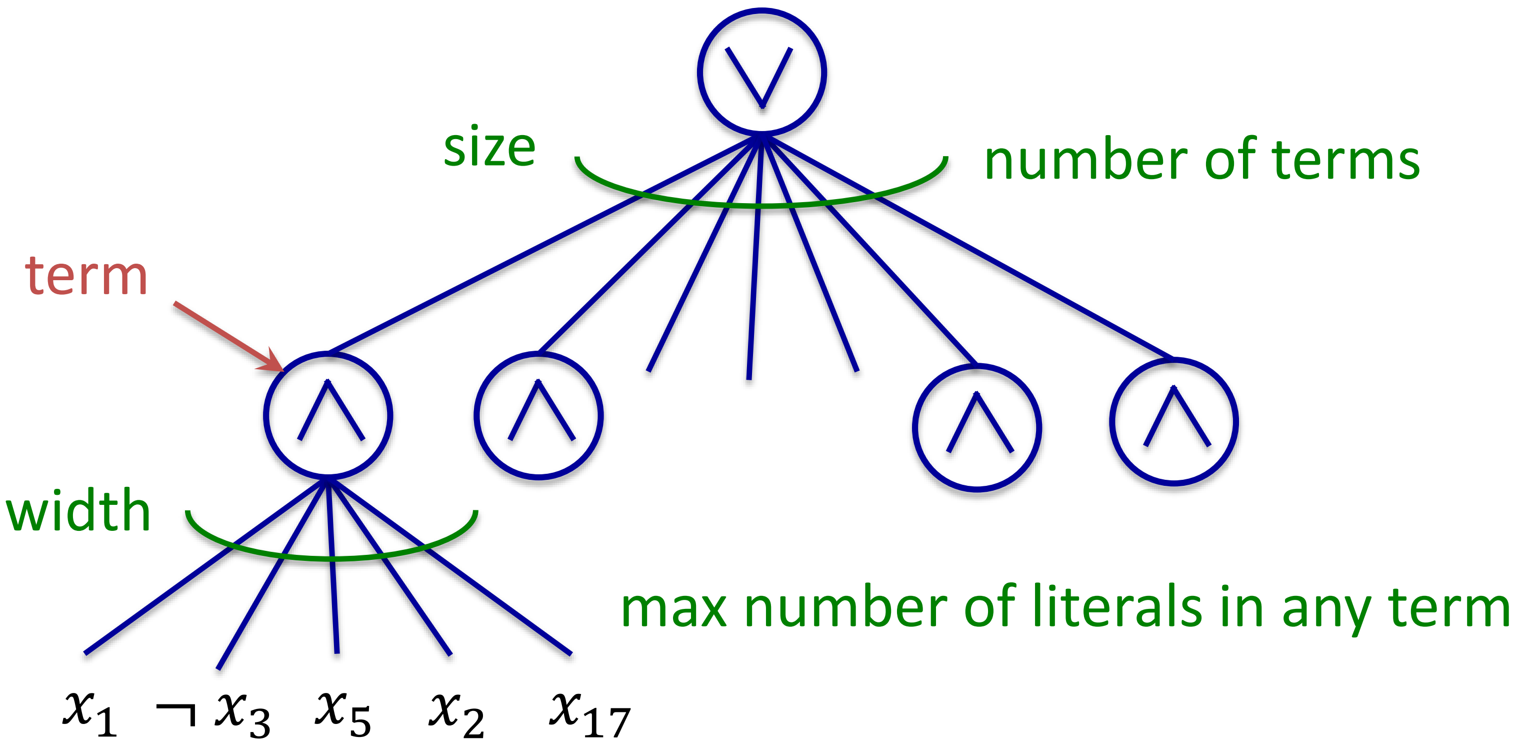
28 August 2013, Berkeley

DNFs

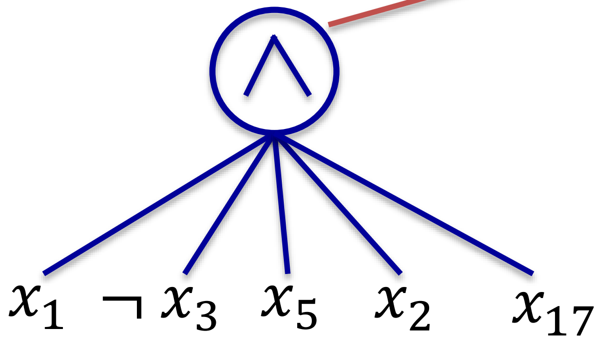


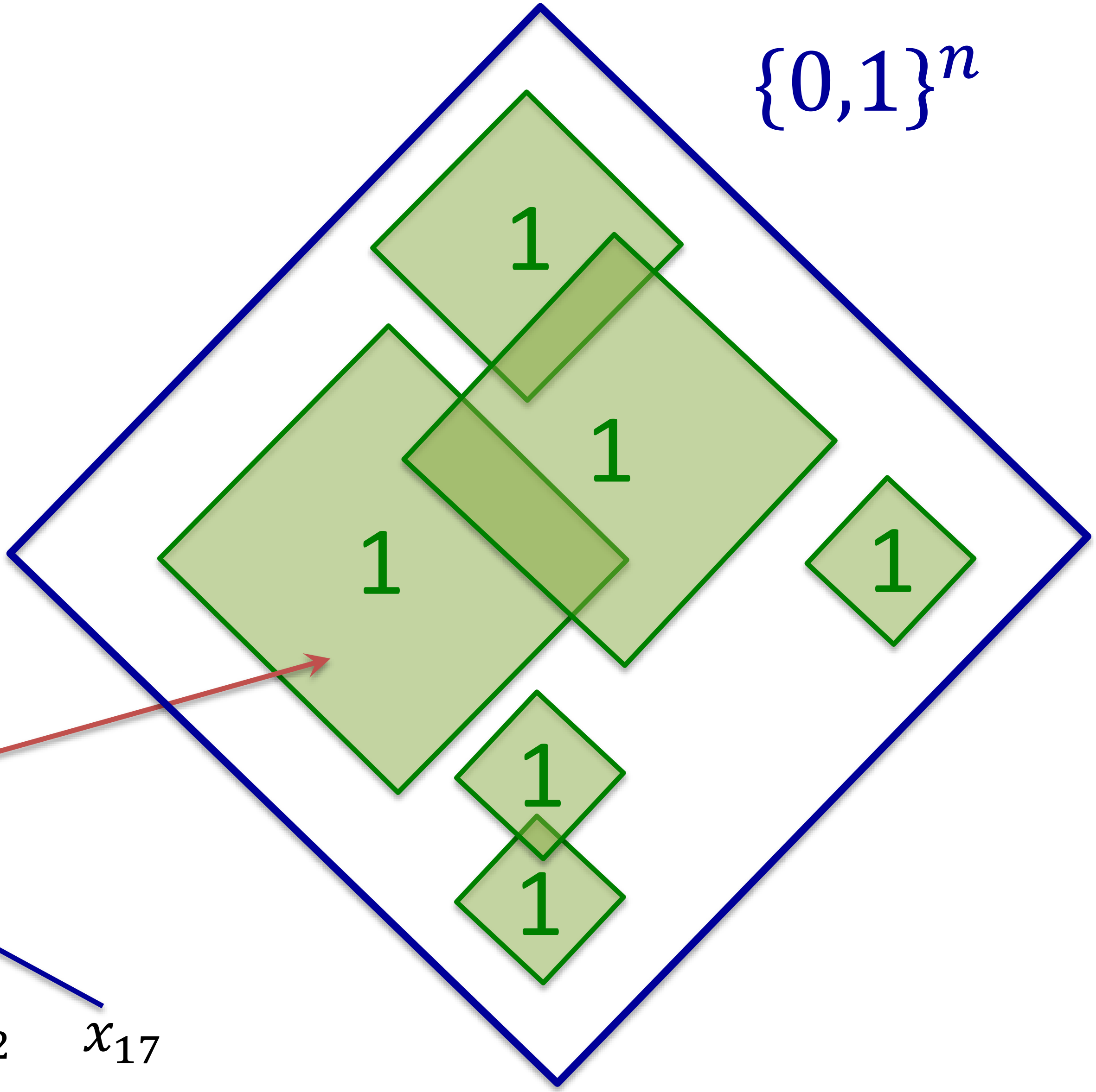
size
number of terms
term
width
max number of literals in any term
x_1 $\neg x_3$ x_5 x_2 x_{17}

Size s, width w DNF

$\equiv$

Union of s subcubes of dimension $\geq n - w$





DNFs and PARITY

- A simple exercise often used to introduce complexity theory:

  Any DNF computing PARITY has size $\geq 2^{n-1}$ and width $\geq n$.

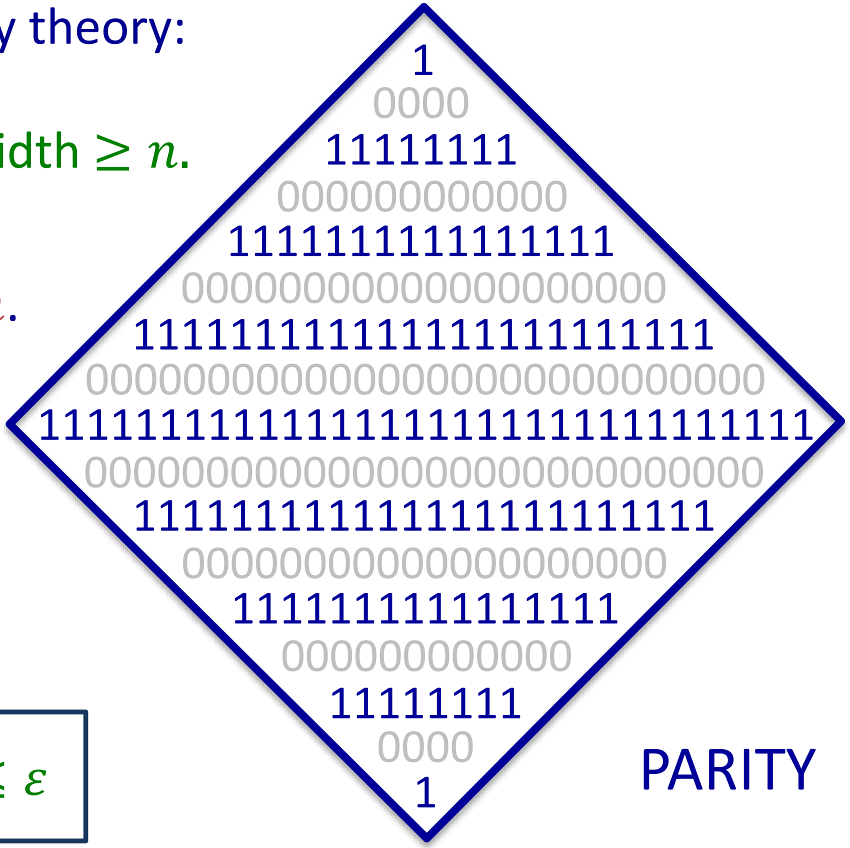
- *Every* Boolean function: DNF size $\leq 2^{n-1}$, width $\leq n$.

  $\Longrightarrow$ PARITY = hardest function

But what about *approximation*?

DNF only has to be correct on 0.99-fraction of inputs $\{0,1\}^n$

Definition: f is an ε-approximator for g if $\Pr[f(x) \neq g(x)] \leq \varepsilon$



PARITY

Starting point of this research

1. Is *approximating* PARITY asymptotically easier than computing it *exactly*?
2. Is PARITY also the hardest function to approximate?
3. Universal bounds on approximability of every Boolean function?

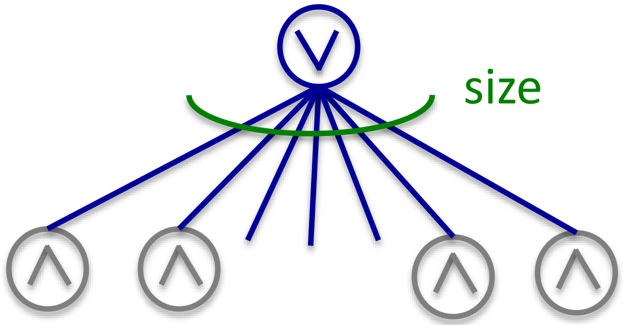
Tradeoffs between accuracy and efficiency in circuit complexity

Basic, seemingly simple, problems open even for DNFs!

Approximating PARITY with DNFs

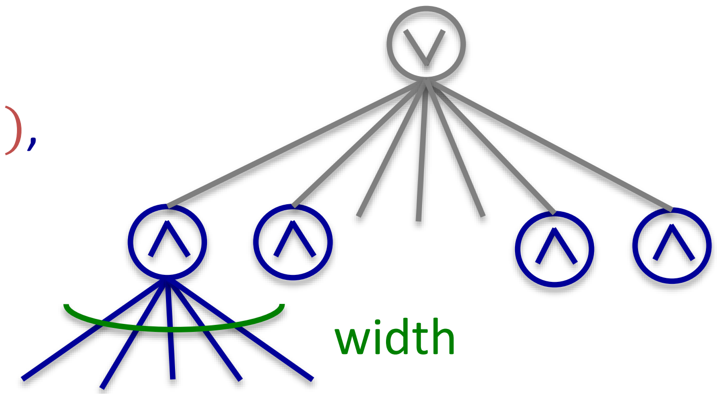
Theorem [Lupanov 61]:
Any DNF computing PARITY has size $\geq 2^{n-1}$ and width $\geq n$.

Does 0.1-approximating PAR require DNF size $\Omega(2^n)$,
or can we 0.1-approximate PAR with size $o(2^n)$?



Approximation not much easier: $\Omega(2^n)$ *vs.*
Approximation a lot easier: $\leq 2^n/\exp(n)$

Does 0.1-approximating PAR require DNF width $n - O(1)$,
or can we 0.1-approximate PAR with width $n - \omega(1)$?



Approximation not much easier: $n - O(1)$ *vs.*
Approximation a lot easier: $n - \Omega(n)$

Previous work: correlation bounds between PAR and AC^0

- Long and fruitful line of research.
- Started in the 80's [FSS 84, Ajtai 83, Håstad 86], remains active today.

A small AC^0 circuit agrees with PAR on at most $\frac{1}{2}$ + *tiny* fraction of inputs.

- [Håstad 12]: correlation of size-s DNF with PARITY $2^{-\Omega(n/\log(s))}$.

$\Longrightarrow$ any DNF that agrees with PAR on 99% of inputs has size $2^{\Omega(n)}$.

But still leaves open exponential gap of $\Omega(2^n)$ *vs.* $\leq 2^n/\exp(n)$.

our results in this work

Approximating PARITY with DNFs

Theorem [Lupanov 61]:
Any DNF computing PARITY has size $\geq 2^{n-1}$ and width $\geq n$.

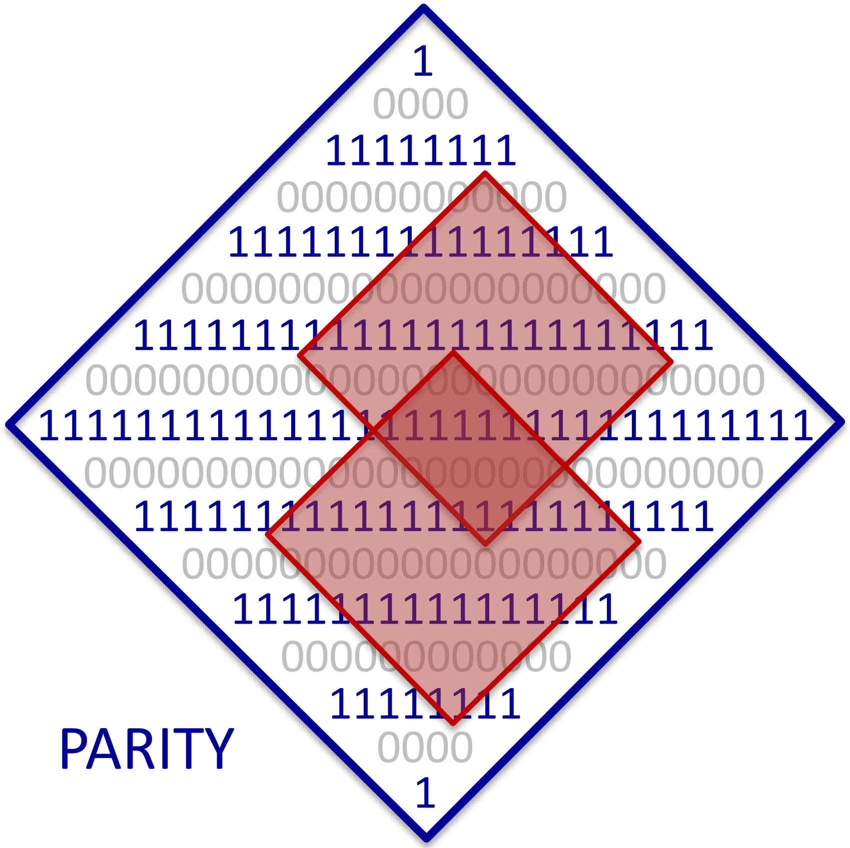
Theorem [Blais-T.]:
PAR can be ε-approximated by a DNF of size $2^{(1-2\varepsilon)n}$ and width $(1-2\varepsilon)n$.

- Exponential savings on size, linear savings on width.
- (Almost) matching lower bounds:

Theorem [Blais-T.]:
Any DNF that ε-approximates PARITY has size $2^{(1-4\varepsilon)n}$ and width $(1-2\varepsilon)n$.

Theorem [Blais-T.]:
PAR can be ε-approximated by a DNF of size $2^{(1-2\varepsilon)n}$ and width $(1-2\varepsilon)n$.

- Parity can be 0.01-approximated by the union of $\Omega(n)$ dimensional subcubes.
- Each covers *exponentially* many points.
- Incurs error 50% within each subcube
  - Yet overall error only 1%!
- Solution: overlap heavily over 0-inputs, essentially disjoint over 1-inputs.



PARITY

Universal bounds on DNF size

- PARITY = hardest function to compute exactly. Same true for approximation?

Theorem [Blais-T.]: No!
Any DNF that 0.1-approximates a random function has size $\geq 2^n/n$.

Theorem [Blais-T.]:
PAR can be ε-approximated by a DNF of size $\leq 2^{(1-2\varepsilon)n}$.

- PARITY *exponentially* easier to approximate than *almost all* functions!

Is there a function that requires size $\Omega(2^n)$ to approximate
or can we prove $o(2^n)$ upper bound for *all* functions?

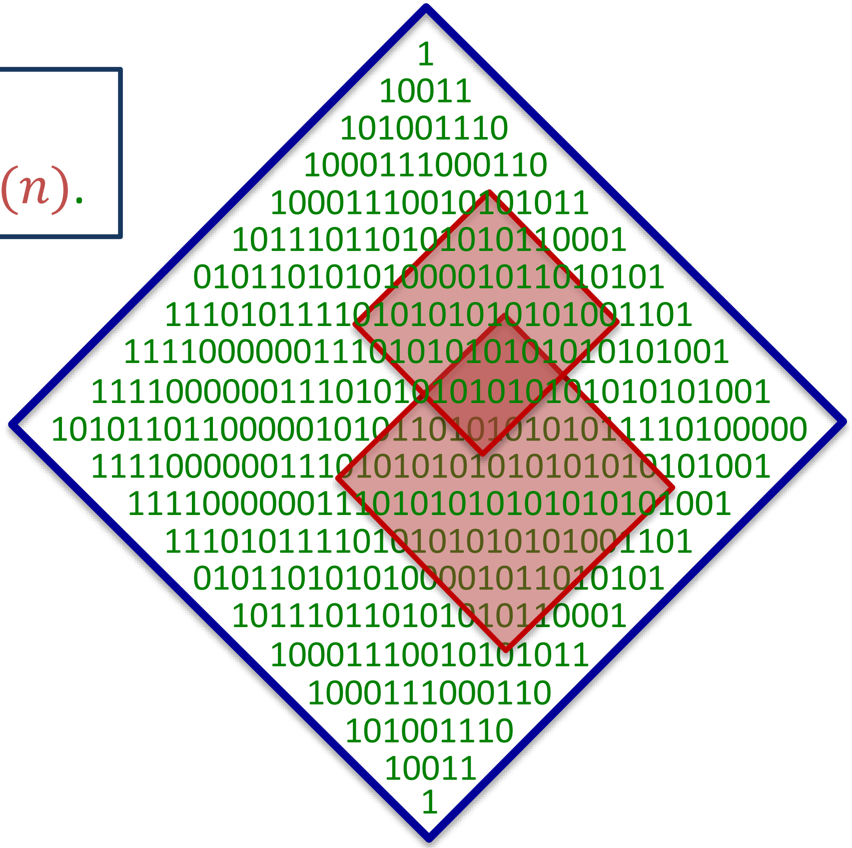
Theorem [Blais-T.]:
Every function can be 0.1-approximated by a DNF of size $\leq 2^n/\log(n)$.

Universal bounds on DNF width

- Parity can be 0.1-approximated by union of $\Omega(n)$-dimensional subcubes.
- Same true for *any* function?

Theorem [Blais-T.]: Yes!
Every function can be 0.1-approximated by a DNF of width $\leq n - \Omega(n)$.

- Random function: every 1-monochromatic subcube has dimension $\leq \log(n)$.
- [Blais-T.] All cubes can be made exponentially larger at the cost of small constant error.



The rest of this talk

1. Universal upper bound on DNF size.
2. Universal upper bound on DNF width.
3. DNF approximator for PARITY.
4. Open problems.

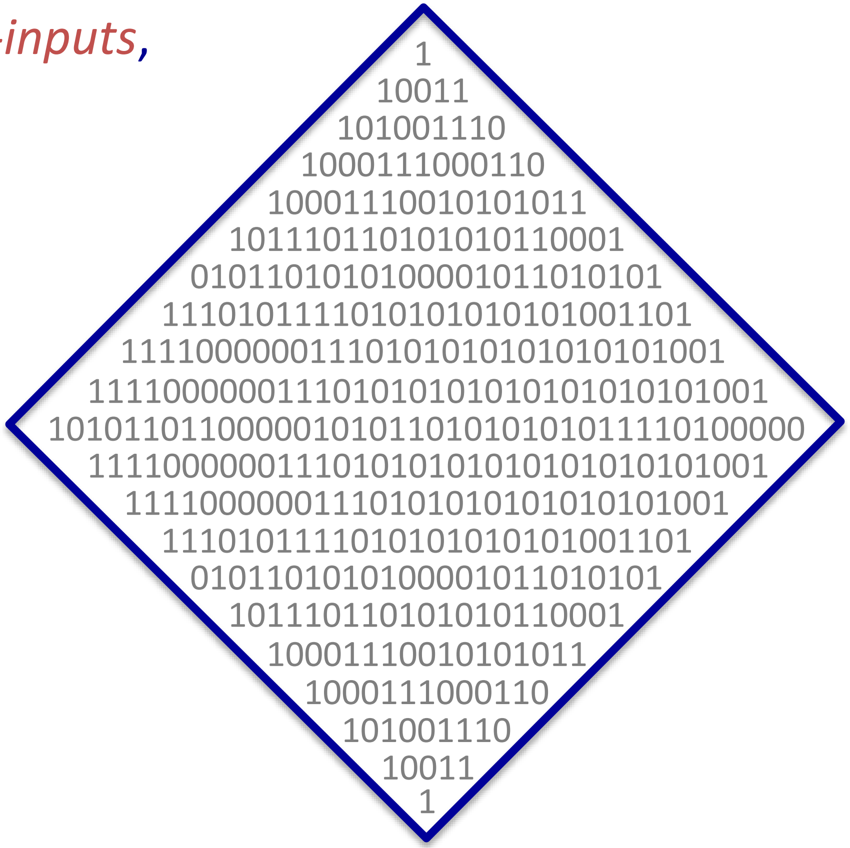
* Unfortunately, will not have time for lower bounds.

Theorem:
Every function can be 0.1-approximated by a DNF of size $\leq 2^n/\log(n)$.

Goal: *small* family of subcubes, covers *almost all 1-inputs*, but *almost none of 0-inputs*. Seems tough!

First try:

1. Randomly flip tiny fraction of 0's to 1's.
2. Include all "large" 1-monochromatic subcubes.
   - Error on 0-inputs
   - Error on 1-inputs
   - DNF size

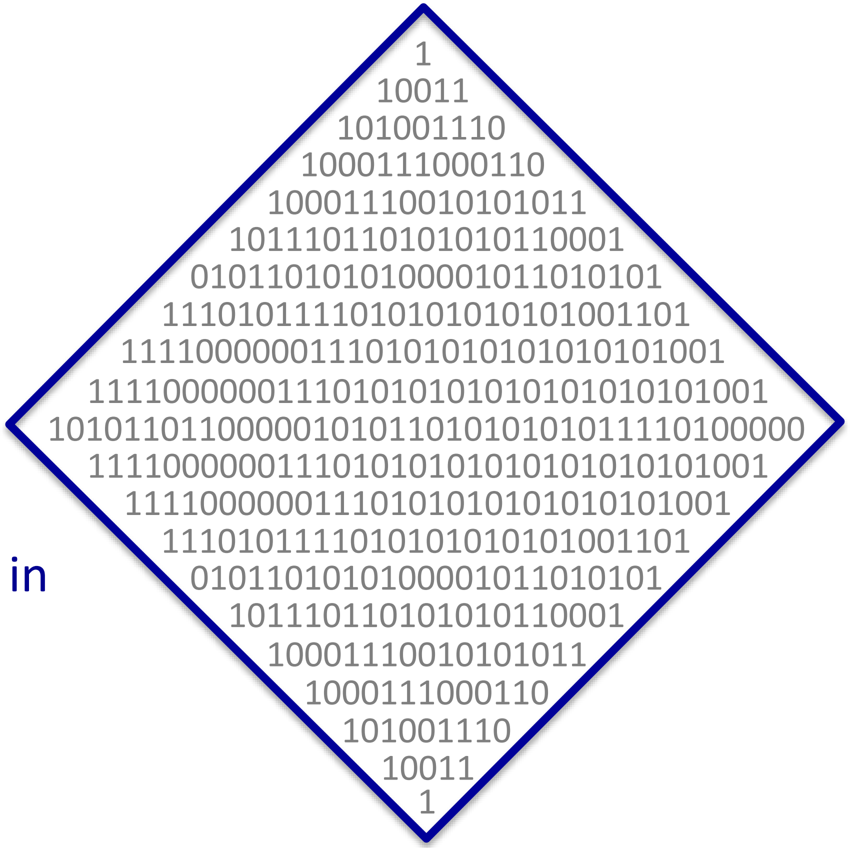


Theorem:
Every function f can be 0.1-approximated by a DNF of size $\leq 2^n/\log(n)$.

1. Flip each 0-input to 1 with tiny probability.
   Tiny fraction of 0's covered.

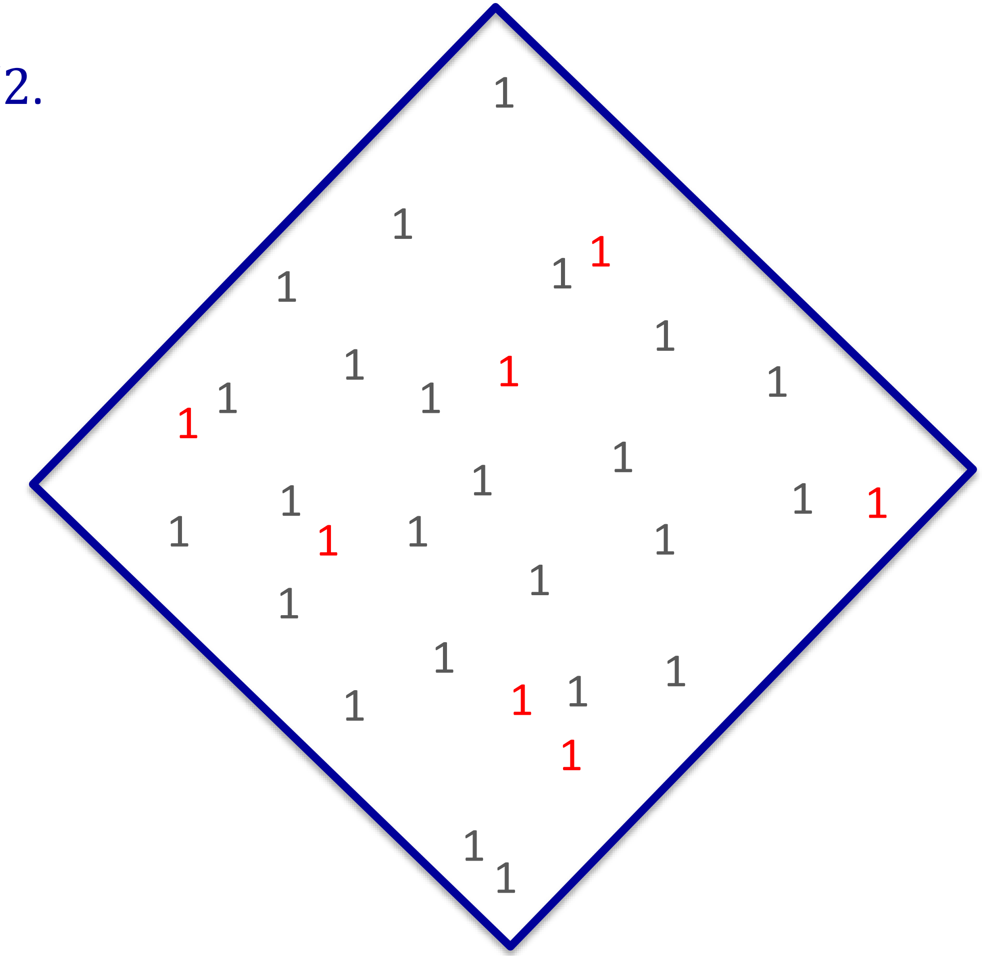
2. Define *special* subcubes, every x is contained in "many" special subcubes.
   Any 1-input x likely to be covered.

3. Include each 1-monochromatic special subcube in approximator with small probability.
   DNF approximator has small size.



1. Flip each 0-input to 1 independently with probability $\varepsilon/2$.

- w.p. 1-$o(1)$ at most ε fraction 0-inputs flipped.
- Conditioned on this, error on 0-inputs $\leq \varepsilon$.
- Remains to consider error on 1-inputs and DNF size:
  - w.p. $\geq 3/4$, error on 1-inputs $\leq \varepsilon$.
  - w.p. $\geq 3/4$, DNF size $O(2^n/\log(n))$.



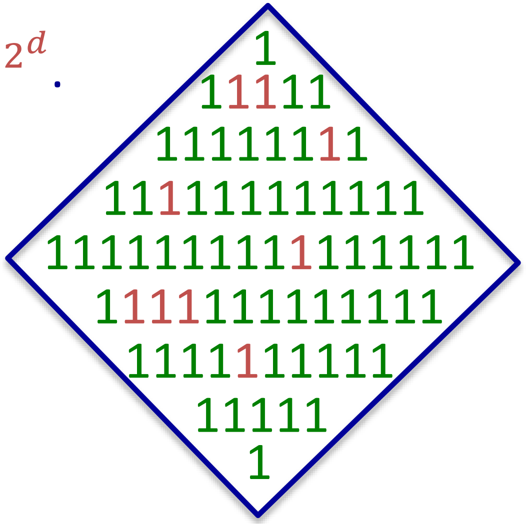
2. Let $d \sim \log\log(n)$, partition $[n]$ into n/d blocks of size d.

"Special" subcube := $\underbrace{\boxed{100011}}_{d}\underbrace{\boxed{******}}_{d}\boxed{110101}\boxed{111101}\boxed{000101}\boxed{110101}$

- All *'s in exactly one block.
- Every x is contained in n/d special subcubes.

$$\Pr[x\ not\ covered] = \left(1 - \varepsilon^{2^d}\right)^{n/d} \leq \varepsilon/4.$$

3. Each special subcube included with probability exactly ε^{2^d}.



$$E[\#\ subcubes\ included] = \varepsilon^{2^d} \cdot \frac{n}{d} \cdot 2^{n-d}$$

$$\sim 2^n/\log(n).$$

Theorem:

Every function can be 0.1-approximated by a DNF of size $\leq 2^n/\log(n)$.

The rest of this talk

✓ Universal upper bound on DNF size.
2. Universal upper bound on DNF width.
3. DNF approximator for PARITY.
4. Open problems.

Theorem:
Every function can be 0.1-approximated by a DNF of width $\leq n - \Omega(n)$.

1. Fix $d = 0.001n$. Approximator will have width $n - d = n - \Omega(n)$.

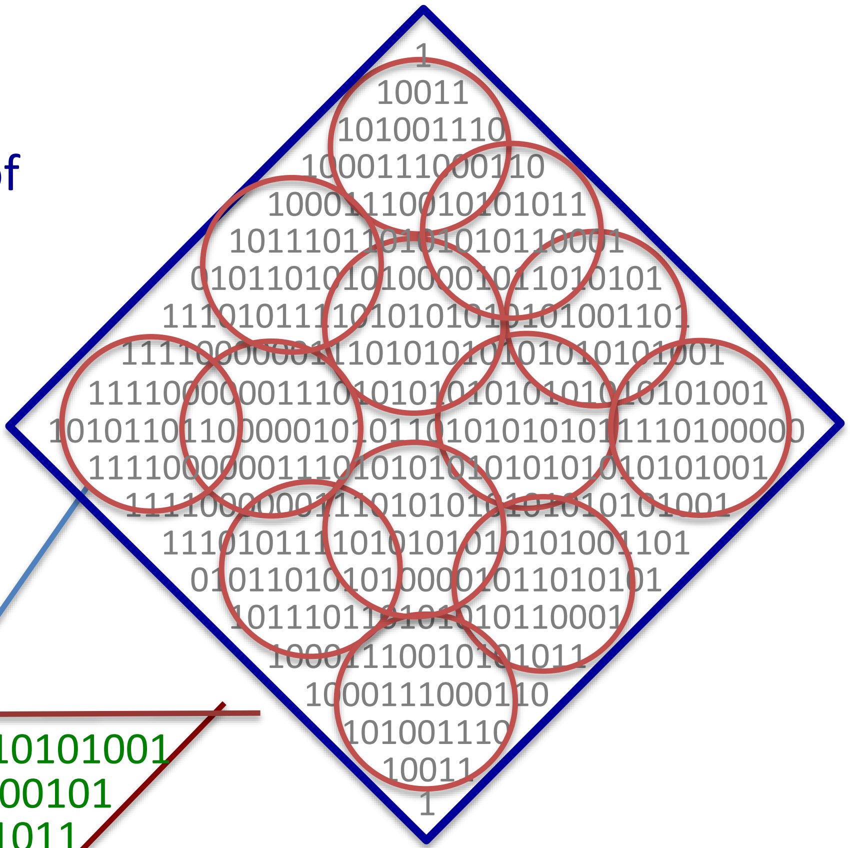
2. Cover 99.9% of $\{0,1\}^n$ with $\sim 10 \cdot 2^n / vol(d)$ balls of radius d. Essentially a partition.

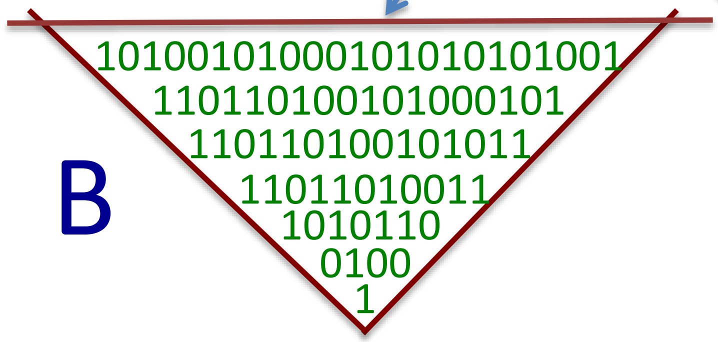
2. Construct width n-d $=$ n-$\Omega(n)$ DNF for each ball B satisfying:

   99.9% correct within B, always 0 outside B.

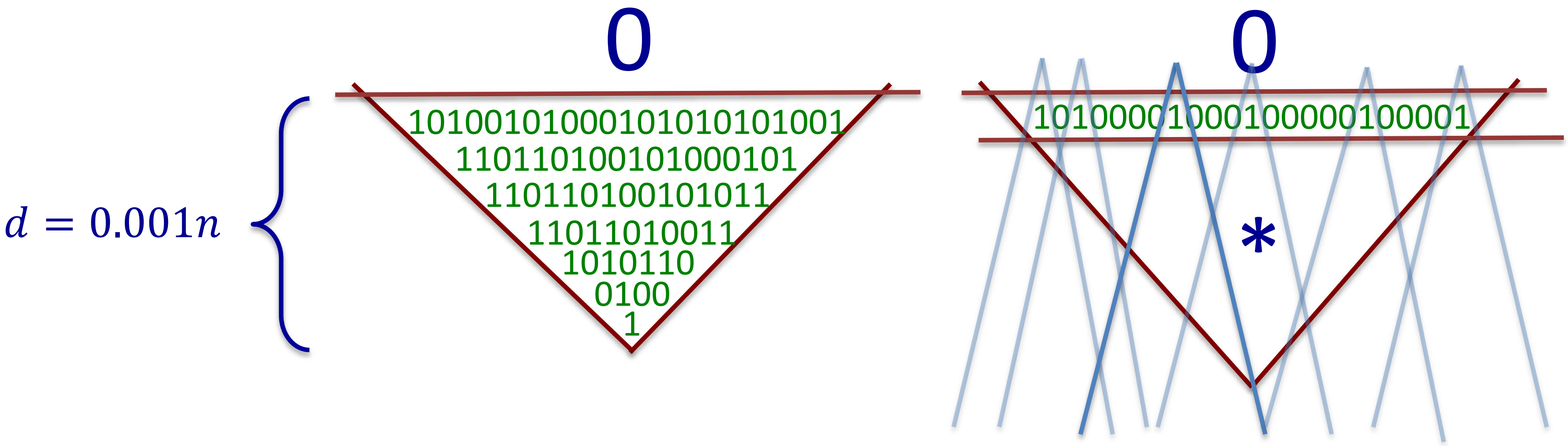
3. Final approximator: OR of sub-approximators.

   (OR of DNFs = DNF)





Small-width approximators for Hamming balls



- 99.99% of points lie on surface
- Suffices to be 100% correct on surface
- One width n-d term for each point

Theorem:

Every function can be 0.1-approximated by a DNF of width $\leq n - \Omega(n)$.

The rest of this talk

✓ Universal upper bound on DNF size.

✓ Universal upper bound on DNF width.

3. DNF approximator for PARITY.
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Theorem:
PAR can be ε-approximated by a DNF of size $2^{(1-2\varepsilon)n}$ and width $(1-2\varepsilon)n$.

$\varepsilon = 1/4$ $\quad$ $x = \underbrace{100110101010010}_{y}\underbrace{1000101010101001}_{z}$

- PAR(x) = PAR(y) $\oplus$ PAR(z)
- Consider $F(x)$ = PAR(y) $\vee$ PAR(z):
  - PAR(x) = 1 $\Longrightarrow$ $F(x)$ = 1
  - PAR(x) = 0 $\Longrightarrow$ $F(x)$ = 0 half the time.

$\Pr[F(x) = \text{PAR}(x)] = 3/4.$

PAR(y) and PAR(z) have trivial DNFs of size $2^{(n/2)-1}$ and width $n/2$.

$\Longrightarrow$ $(1/4)$-approximate PAR with size $2^{n/2}$ and width $n/2$.

The rest of this talk

- ✓ Universal upper bound on DNF size.
- ✓ Universal upper bound on DNF width.
- ✓ DNF approximator for PARITY.

4. Open problems.

Open problems

Every function can be 0.1-approximated by a DNF of size $\leq 2^n/\log(n)$.

Any DNF that 0.1-approximates a random function has size $\geq 2^n/n$.

1. Close this gap.
2. Explicit hard function showing $\geq 2^n/\mathrm{poly}(n)$.


thank you!