Lower Bounds for Subgraph Isomorphism and Consequences in First-Order Logic

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Outline

• The Subgraph Isomorphism Problem
• $\text{AC}^0$ and First-Order Logic
• Upper and Lower Bounds for $\text{SUB}(G)$:
  
  $\text{AC}^0$ circuit size $\approx$ FO variable width $\approx$ tree-width($G$)
  
  $\text{AC}^0$ formula size $\approx$ FO quantifier rank $\approx$ tree-depth($G$)

• “Poly-rank” Homomorphism Preservation Theorem
Subgraph Isomorphism Problem
• **k-CLIQUE**
  
Given a graph $X$, does it contain a $k$-clique (complete subgraph of size $k$)?
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  Given a graph X, does it contain a k-clique (complete subgraph of size k)?

• **Time complexity of k-CLIQUE**
  • “Brute-force” upper bound: $O(n^k)$
  • Best known upper bound: $O(n^{0.79k})$
  • Conjectured lower bound: $n^{\Omega(k)}$ (⇒ $P \neq NP$)
• **k-STCONN** ("Distance-k Connectivity")
  Given a directed graph \( X \) with distinguished vertices \( s \) and \( t \), does \( X \) contain a \( st \)-path of length \( k \)?
• **k-STCONN** (“Distance-k Connectivity”)
  Given a directed graph $X$ with distinguished vertices $s$ and $t$, does $X$ contain a $st$-path of length $k$?
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  Given a directed graph $X$ with distinguished vertices $s$ and $t$, does $X$ contain a st-path of length $k$?

• **Space complexity of k-STCONN**
  • Best known upper bound: $O(\log k \cdot \log n)$
  • Conjectured lower bound: $\Omega(\log k \cdot \log n)$ ($\implies L \neq NL$)
• $\text{SUB}_{\text{uncolored}}(G)$

Given a graph $X$, does it contain a subgraph isomorphic to $G$?
• **SUB(G)**

Given a graph $X$ and a coloring $\pi : V(X) \rightarrow V(G)$, does $X$ contain a subgraph $G'$ such that $G' \equiv G$ and $\pi(G') = G$?
• SUB(G)
  Given a graph X and a coloring \( \pi : V(X) \rightarrow V(G) \), does X contain a subgraph \( G' \) such that \( G' \cong G \) and \( \pi(G') = G \)?

• Special cases:
  \[
  \text{SUB}(K_k) = k\text{-CLIQUE} \\
  \text{SUB}(P_k) = k\text{-STCONN}
  \]
Reductions

- $\text{SUB}_{\text{uncolored}}(G) \leq \text{SUB}(G)$
  (by the “color-coding technique” of Alon, Yuster, Zwick)
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  (i.e. every homomorphism \( G \to G \) is one-to-one)
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  (i.e. every homomorphism \( G \to G \) is one-to-one)

• \( \text{SUB}(F) \leq \text{SUB}(G) \) when \( F \) is a minor of \( G \)

Credit: Wikipedia (NikelsonH)
Summary

• SUB(G) are an important and well-structured family of problems.

• (As we will see,) complexity of SUB(G) tied to natural structural parameters of G.

• Determining the complexity of SUB(G) w.r.t. to different computational resources (time, space, ...) would separate various classes (P ≠ NP, L ≠ NL, ...)
Summary

• SUB(G) are an important and well-structured family of problems.

• (As we will see,) complexity of SUB(G) depends on natural structural parameters.

• Determining the complexity of SUB(G) w.r.t. to different computational resources (time, space, ...) would separate various classes (P \neq NP, L \neq NL, ...)

We will focus on circuit size and formula size.
Boolean Circuits and Formulas
Boolean Circuits
P vs. NP

Boolean circuit size =* Turing machine time
  (* up to a polynomial factor, ignoring uniformity)

\( P = \{ \text{problems solvable by polynomial-size circuits} \} \)

\( NP = \{ \text{problems whose solutions are verifiable by polynomial-size circuits} \} \)
P vs. NP

• Holy Grail (P ≠ NP)
  Show that any NP problem (e.g. MAXIMUM CLIQUE) requires super-polynomial circuit size
P vs. NP

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  Show that any NP problem (e.g. MAXIMUM CLIQUE) requires **super-polynomial** circuit size

• **The “parameterized” approach**
  It suffices to show that k-CLIQUE requires circuits of size \( n^{\Omega(k)} \) for any \( k(n) \rightarrow \infty \)
P vs. NP

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- **Circuit lower bounds are hard!**
  Best circuit lower bound for a function in NP:
  
P vs. NP

- Holy Grail (P ≠ NP)
  - Show that any NP problem (e.g., MAXIMUM CLIQUE)
    requires super-polynomial circuit size

- The "parameterized" approach
  - It suffices to show that k-CLIQUE requires circuits of size $n^{\Omega(k)}$ for any $k(n) \rightarrow \infty$

- Circuit lower bounds are hard!
  - Best circuit lower bound for a function in NP:

To prove super-linear lower bounds, need to focus on weaker models of computation (restricted classes of circuits)
Boolean Formulas

- **Formulas** = tree-like circuits
- “Memoryless”: each sub-computation is used once
Boolean Formulas

• Another Holy Grail ($\text{NC}^1 \neq \text{P}$)
  Show that any problem in \text{P} (e.g. STCONN) requires super-polynomial formula size
Boolean Formulas

- **Another Holy Grail** ($\text{NC}^1 \neq \text{P}$)
  Show that any problem in P (e.g. STCONN) requires **super-polynomial** formula size

- **The “parameterized” approach**
  It suffices to show that k-STCONN has formula complexity $n^{\Omega(\log k)}$ for any $k(n) \rightarrow \infty$
Boolean Formulas

• Another Holy Grail ($NC^1 \neq P$)
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• The “parameterized” approach
  It suffices to show that $k$-STCONN has formula complexity $n^{\Omega(\log k)}$ for any $k(n) \rightarrow \infty$

• \textit{Formula lower bounds are hard!}
  Best formula-size lower bound for a function in P:
  
  $n^{1.5}$ (1961), $n^2$ (1966), $n^{2.5}$ (1987), $n^3$ (1998)
Boolean Formulas

- Another Holy Grail (NC \(1 \neq P\))

Show that any problem in P (e.g. STCONN) requires super-polynomial formula size.

- The "parameterized" approach

It suffices to show that k-STCONN has formula complexity \(\Omega(\log k)\) for any \(k(n) \to \infty\).

- Formula lower bounds are hard!

To prove super-polynomial lower bounds, again must focus on restricted classes.

- Formula lower bounds are hard!

Best formula-size lower bound for a function in P:

\[ n^{1.5} \text{ (1961)}, \quad n^2 \text{ (1966)}, \quad n^{2.5} \text{ (1987)}, \quad n^3 \text{ (1998)} \]
AC⁰ Circuit and Formulas

• We restrict attention to circuits and formulas of constant depth (a.k.a. AC⁰ circuits and formulas)
AC^0 & First-Order Logic
Hierarchies Within FO

• **Variable-width** (max # of free vars in a subformula)
  \[ FO^1 \subseteq FO^2 \subseteq FO^3 \subseteq \ldots \]

• **Quantifier-rank** (nesting depth of quantifiers)
  \[ FO_1 \subseteq FO_2 \subseteq FO_3 \subseteq \ldots \]
• **Theorem**

The **model-checking problem** for a FO sentence $\varphi$

*Given a structure $A$ with universe $\{1, \ldots, n\}$, is $A$ a model $\varphi$?*

is solvable by:

- $AC^0$ circuits of size $O(n^{\text{variable-width}(\varphi)})$
Theorem

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- $AC^0$ circuits of size $O(n^{\text{variable-width}(\varphi)})$
  - but only $\text{quantifier-rank}(\varphi)$ layers of fan-in $n$ gates
  
  (formula size $\leq$ depth $\times$ fan-in)
• **Theorem**

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- $\text{AC}^0$ circuits of size $O(n^{\text{variable-width}(\varphi)})$

- $\text{AC}^0$ formulas of size $O(n^{\text{quantifier-rank}(\varphi)})$
Hierarchies Within FO

- Variable-width
  \[ \text{FO}^1 \subseteq \text{FO}^2 \subseteq \text{FO}^3 \subseteq \ldots \]

- Quantifier-rank
  \[ \text{FO}_1 \subseteq \text{FO}_2 \subseteq \text{FO}_3 \subseteq \ldots \]

- Background relations
  \[ \text{FO} \subseteq \text{FO}[<] \subseteq \text{FO}[\text{BIT}] \subseteq \text{FO}[\text{Arb}] \]
Hierarchies Within FO

- Variable-width
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[Barrington-Immerman-Straubing 1990]

uniform-\(\text{AC}^0\)

\(\text{AC}^0\)
Implications

lower bounds for $\text{AC}^0$ circuit size

lower bounds for $\text{FO}[\text{Arb}]$ variable-width

lower bounds for $\text{AC}^0$ formula size

lower bounds for $\text{FO}[\text{Arb}]$ quantifier-rank
Complexity of SUB(G): Upper Bounds
Upper Bounds

• **Theorem (folklore)**

  $\text{SUB}(G)$ is definable in:
  
  o $\text{FO}[ \text{tree-width}(G) + 1 \text{ variables } ]$
  
  o $\text{FO}[ \text{tree-depth}(G) \text{ quantifier rank } ]$
Upper Bounds

• Theorem (folklore)
  SUB(G) is definable in:
  
  o FO[ tree-width(G) + 1 variables ]
  o FO[ tree-depth(G) quantifier rank ]

moreover, existential & positive
Upper Bounds

• Theorem (folklore)
SUB(G) is definable in:
  o FO[$\text{tree-width}(G) + 1$ variables ]
  o FO[$\text{tree-depth}(G)$ quantifier rank ]
SUB(G) is solvable by:
  o $\text{AC}^0$ circuits of size $n^{O(\text{tree-width}(G))}$
  o $\text{AC}^0$ formulas of size $n^{O(\text{tree-depth}(G))}$
Tree-width: $\text{tw}(G)$
Tree-width: $\text{tw}(G)$

- $\text{tw}(\text{any tree}) = 1$, $\text{tw}(K_k) = k - 1$

Credit: Wikipedia (David Eppstein)
Tree-width: \( \text{tw}(G) \)

- **Width-\( k \) tree decomposition** of \( G \): blueprint for a \((k+1)\)-variable first-order sentence defining \( \text{SUB}(G) \)
Tree-depth: $td(G)$
Tree-depth: $td(G)$

- **Def.** The **closure** of a tree $T$ is a graph formed by adding edges between all ancestor-descendant pairs.
Tree-depth: $\text{td}(G)$

- **Def.** The **tree-depth** of a graph $G$ is the minimum height of a tree $T$ such that $G \subseteq \text{closure}(T)$
**Tree-depth: \( td(G) \)**

- **Def.** The **tree-depth** of a graph \( G \) is the minimum height of a tree \( T \) such that \( G \subseteq \text{closure}(T) \)
Tree-depth: \( \text{td}(G) \)

- \( \text{tw}(G) \leq \text{td}(G) \leq \text{tw}(G) \cdot \log |V(G)| \)
- \( \log(\text{longest-path}(G)) \leq \text{td}(G) \leq \text{longest-path}(G) \)

Credit: Wikipedia (David Eppstein)
Tree-depth: \( td(G) \)

- **Height-\( k \) tree** \( T \) with \( G \subseteq \text{closure}(T) \): blueprint for a quantifier rank-\( k \) first-order sentence defining \( \text{SUB}(G) \)
AC⁰ Complexity of SUB(G): Lower Bounds
Lower Bounds

• Theorem [Li-Razborov-R. 2014]
The AC⁰ circuit size of SUB(G) is $n^{\Omega(\text{tw}(G))}$

• Theorem [Kawarabayashi-R. 2016, R. 2016]
The AC⁰ formula size of SUB(G) is $n^{\Omega(\text{td}(G)^{\epsilon})}$
Lower Bounds

• Theorem [Li-Razborov-R. 2014]
The AC\(^0\) circuit size of SUB(G) is \(n^{\Omega(\text{tw}(G))}\)

[R. 2008]
k-CLIQUE has AC\(^0\) circuit size \(n^{\Omega(k)}\)

• Theorem [Kawarabayashi-R. 2016, R. 2016]
The AC\(^0\) formula size of SUB(G) is \(n^{\Omega(\text{td}(G)^{\epsilon})}\)

[R. 2014]
k-STCONN has AC\(^0\) formula size \(n^{\Omega(\log k)}\)
Lower Bounds

• Theorem [Li-Razborov-R. 2014]
The AC\(^0\) circuit size of SUB(G) is \(n^{\Omega(\text{tw}(G))}\)
The FO[Arb] variable-width of SUB(G) is \(\Omega(\text{tw}(G))\)

• Theorem [Kawarabayashi-R. 2016, R. 2016]
The AC\(^0\) formula size of SUB(G) is \(n^{\Omega(\text{td}(G)^\epsilon)}\)
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Lower Bounds

• **Theorem [Li-Razborov-R. 2014]**
The \( AC^0 \) circuit size of \( SUB(G) \) is \( n^{\Omega(\text{tw}(G))} \)
The FO[Arb] variable-width of \( SUB(G) \) is \( \Omega(\text{tw}(G)) \)
“The variable hierarchy is strict over ordered graphs”

• **Theorem [Kawarabayashi-R. 2016, R. 2016]**
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“Poly-rank homomorphism preservation theorem”
Lower Bounds

• Theorem [Li-Razborov-R. 2014]
The $AC^0$ circuit size of $\text{SUB}(G)$ is $n^{\Omega(\text{tw}(G))}$
The $FO[\text{Arb}]$ variable-width of $\text{SUB}(G)$ is $\Omega(\text{tw}(G))$
“The variable hierarchy is strict over ordered graphs”

• The

$k$-CLIQUE is definable in $FO^k$
but not in $FO^{k/4}[\leq]$
“Poly-rank homomorphism preservation theorem”
Lower Bounds

• **Theorem [Li-Razborov-R. 2014]**

  The $\text{AC}^0$ circuit size of $\text{SUB}(G)$ is $n^{\Omega(\text{tw}(G))}$

  The $\text{FO}[\text{Arb}]$ variable-rank of $\text{SUB}(G)$ is $\Omega(\text{td}(G)^\varepsilon)$

  “The variable hierarchy is strict over ordered graphs”

  Proof uses probabilistic method: **average-case** lower bounds w.r.t. particular random input graphs (generalizations of $G(n,p)$)

  “Poly-rank homomorphism preservation theorem”
Hard-On-Average Input Distributions for SUB(G)
Average-Case for $\text{SUB}_{\text{uncolored}}(G)$

- Natural input distribution: $\text{ErdosRenyi}(n,p)$ where $p = p(n)$ is the “threshold” for $G$-subgraphs
Average-Case for $\text{SUB}_{\text{uncolored}}(G)$

- Natural input distribution: $\text{ErdosRenyi}(n,p)$ where $p = p(n)$ is the “threshold” for $G$-subgraphs

$$\Pr[\text{ErdosRenyi}(n,p) \text{ contains a subgraph isomorphic to } G]$$

edge probability $p$
Average-Case for $\text{SUB}_{\text{uncolored}}(G)$

- Natural input distribution: $\text{ErdosRenyi}(n,p)$ where $p = p(n)$ is the “threshold” for $G$-subgraphs

$\Pr[\text{ErdosRenyi}(n,p) \text{ contains a subgraph isomorphic to } G ]$

Diagram:
- $p_{\text{threshold}}$
- $\frac{1}{2}$
- $p$ (edge probability)
Average-Case for $\text{SUB}_{\text{uncolored}}(G)$

- Natural input distribution: $\text{ErdosRenyi}(n,p)$ where $p = p(n)$ is the “threshold” for $G$-subgraphs

Conjectured to be source of hard-on-average instances for many graphs $G$, including $K_k$ [Karp 1976]
Average-Case for SUB(G)

- Natural *family* of input distributions:
  “G-colored Erdos-Renyi random graphs”

\[
\frac{n^{-1}}{n^{-1}}, \frac{n^{-1}}{n^{-1}}, \frac{n^{-1/2}}{n^{-1/2}}, \frac{n^{-3/2}}{n^{-3/2}}
\]
Average-Case for SUB(G)

• Natural *family* of input distributions:
  “G-colored Erdos-Renyi random graphs”

• Different edge density $p_e$ for each $e \in E(G)$ (i.e. each pair of color classes)

• What is a “threshold” family of densities $\{p_e\}_{e \in E(G)}$?
Average-Case for SUB(G)

- **Def:** $\beta : E(G) \to [0, 2]$ is a **threshold weighting** for $G$ if
  1. $\beta(F) := \sum_{e \in E(F)} \beta(e) \leq |V(F)|$ for every $F \subseteq G$
  2. $\beta(G) = |V(G)|$
Average-Case for SUB(G)

• **Def:** \( \beta : E(G) \rightarrow [0,2] \) is a **threshold weighting** for \( G \) if
  1. \( \beta(F) := \sum_{e \in E(F)} \beta(e) \leq |V(F)| \) for every \( F \subseteq G \)
  2. \( \beta(G) = |V(G)| \)

• **Obs:** Every **Markov chain** on \( G \)
  
  \( M : V(G) \times V(G) \rightarrow [0,1] \)

  induces a threshold weighting

  \( \beta_M(\{v,w\}) := M(v,w) + M(w,v) \)
If $G$ has tree-width $k$, then there exists a set of $S \subseteq V(G)$ of size $\Omega(k)$ and a Markov chain $M$ on $G$ that concurrently routes $1 / k \log k$ flow between all pairs of vertices in $S$ [Arora-Rao-Vazirani 2004, Marx 2007]

- **Obs:** Every **Markov chain** on $G$

  $$M : V(G) \times V(G) \rightarrow [0,1]$$

  induces a threshold weighting

  $$\beta_M(\{v,w\}) := M(v,w) + M(w,v)$$
G-colored random graph $X_\beta$
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- $\Pr[ X_\beta \text{ contains a G-subgraph } ]$ bounded away from 0 and 1
- # of G-subgraphs asymptotically Poisson (when G connected...)
For every $F \subseteq G$, 
\[
\text{Ex}[ \ # \ F\text{-subgraphs of } X_\beta \ ] \leq n^{\vert V(F) \vert - \beta(F)}
\]
Proof Sketch

• **Theorem** [Li, Razborov, R. 2014]
  
  $\text{AC}^0$ circuits for $\text{SUB}(G)$ require size $n^{\Omega(\text{tw}(G)/\log \text{tw}(G))}$
Proof Sketch

• **Theorem** [Li, Razborov, R. 2014]
  AC⁰ circuits for SUB(G) require size $n^{\Omega(tw(G)/\log tw(G))}$

1. We define a constant $c(\beta) \geq 0$ associated with each threshold weighting $\beta$
Proof Sketch

• **Theorem** [Li, Razborov, R. 2014]  
  $\text{AC}^0$ circuits for $\text{SUB}(G)$ require size $n^{\Omega(\text{tw}(G)/\log \text{tw}(G))}$

1. We define a constant $c(\beta) \geq 0$ associated with each threshold weighting $\beta$

2. The average-case $\text{AC}^0$ circuit complexity of $\text{SUB}(G)$ on $X_{\beta}$ is $n^{\Theta(c(\beta))}$
Proof Sketch

• **Theorem** [Li, Razborov, R. 2014]

AC⁰ circuits for SUB(G) require size $n^{\Omega(tw(G)/\log tw(G))}$

1. We define a constant $c(\beta) \geq 0$ associated with each threshold weighting $\beta$

2. The average-case AC⁰ circuit complexity of SUB(G) on $X_\beta$ is $n^{\Theta(c(\beta))}$

between $n^{c(\beta)}$ and $n^{2c(\beta)}$
Proof Sketch

• **Theorem** [Li, Razborov, R. 2014]  
  $\text{AC}^0$ circuits for SUB(G) require size $n^{\Omega(\text{tw}(G) / \log \text{tw}(G))}$

  1. We define a constant $c(\beta) \geq 0$ associated with each threshold weighting $\beta$

  2. The average-case $\text{AC}^0$ circuit complexity of SUB(G) on $X_\beta$ is $n^{\Theta(c(\beta))}$

  3. For every graph $G$, there exists $\beta$ such that $c(\beta) \geq \Omega(\text{tw}(G) / \log \text{tw}(G))$
Theorem [Li, Razborov, R. 2014] \( \text{AC}^0 \) circuits for \( \text{SUB}(G) \) require size \( n^{\Omega(\text{tw}(G)/\log \text{tw}(G))} \).

1. We define a constant \( c(\beta) \geq 0 \) associated with each threshold weight \( \beta \).
2. The average-case \( \text{AC}^0 \) circuit complexity of \( \text{SUB}(G) \) on \( X_\beta \) is \( n^{\Theta(c(\beta))} \).
3. For every graph \( G \), there exists \( \beta \) such that \( c(\beta) \geq \Omega(\text{tw}(G)/\log \text{tw}(G)) \).

Proof Sketch

This \( \beta \) from the Markov chain of [Arora-Rao-Vazirani 2004], [Marx 2007]
Excluded-Minor Approximation of Tree-Width & Tree-Depth
Recall

- **Def.** The **tree-depth** of a graph $G$ is the minimum height of a tree $T$ such that $G \subseteq \text{closure}(T)$
Recall

• If $F$ is a minor of $G$, then $\text{SUB}(F) \leq \text{SUB}(G)$
  (there is a linear $\text{AC}^0$ reduction from $\text{SUB}(F)$ to $\text{SUB}(G)$)

Credit: Wikipedia (NikelsonH)
Minor-Monotonicity

• $\text{tw}(\cdot)$ and $\text{td}(\cdot)$ are minor-monotone:

$$\text{F is a minor of G } \implies \text{tw}(F) \leq \text{tw}(G) \land \text{td}(F) \leq \text{td}(G)$$
Minor-Monotonicity

• \( \text{tw}(\cdot) \) and \( \text{td}(\cdot) \) are \textit{minor-monotone}:

\[
\text{F is a minor of G} \implies \text{tw}(F) \leq \text{tw}(G) \ \& \ \text{td}(F) \leq \text{td}(G)
\]

• The class \( \{G : \text{td}(G) \leq k\} \) is characterized by a \textit{finite} set of “excluded minors”, but \textit{doubly exponential} in \( k \).
Minor-Monotonicity

• \( \text{tw}(\cdot) \) and \( \text{td}(\cdot) \) are minor-monotone:

\[
F \text{ is a minor of } G \implies \text{tw}(F) \leq \text{tw}(G) \text{ \& } \text{td}(F) \leq \text{td}(G)
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• The class \( \{G : \text{td}(G) \leq k\} \) is characterized by a finite set of “excluded minors”, but doubly exponential in \( k \)

• **Question**: Can \( O(1) \) many minors approximate \( \text{td}(\cdot) \)?
Minor-Monotonicity

- $\text{tw}(\cdot)$ and $\text{td}(\cdot)$ are minor-monotone:
  
  $$F \text{ is a minor of } G \implies \text{tw}(F) \leq \text{tw}(G) \ & \ \text{td}(F) \leq \text{td}(G)$$

- The class $\{G : \text{td}(G) \leq k\}$ is characterized by a finite set of “excluded minors”, but doubly exponential in $k$

- Question: Can $O(1)$ many minors approximate $\text{td}(\cdot)$?

  longest-path($G$) (i.e. $1 +$ largest excluded path minor) gives an exponential approximation of $\text{td}(G)$
Minor-Monotonicity

• \( \text{tw}(\cdot) \) and \( \text{td}(\cdot) \) are minor-monotone:

\[ F \text{ is a minor of } G \implies \text{tw}(F) \leq \text{tw}(G) \land \text{td}(F) \leq \text{td}(G) \]

• The class \( \{G : \text{td}(G) \leq k\} \) is characterized by a finite set of “excluded minors”, but doubly exponential in \( k \)

• Question: Can \( O(1) \) many minors approximate \( \text{td}(\cdot) \)?

We seek a polynomial approximation of \( \text{td}(G) \)
• **Grid Minor Theorem** [Chekuri, Chuzhoy 2014]
  Every graph of \textit{tree-width} \( \geq k^c \) has a \( k \times k \) grid minor.
• **Grid Minor Theorem** [Chekuri, Chuzhoy 2014]
  Every graph of **tree-width** $\geq k^c$ has a $k \times k$ grid minor.

That is, grid minors give a *polynomial* approximation of $\text{tw}(G)$
• **Grid Minor Theorem**  [Chekuri, Chuzhoy 2014]
  Every graph of *tree-width* $\geq k^c$ has a $k \times k$ grid minor.

• **COROLLARY**
  If $\text{SUB}(\text{Grid}_{k \times k})$ has circuit size $n^{\Omega(k)}$ for all $k$, then $\text{SUB}(G)$ has circuit size $n^{\Omega(\text{tw}(G)^\epsilon)}$ for all graphs $G$. 
• “Grid/Tree/Path Minor Thm” [Kawarabayashi, R. 2016] Every graph of \textbf{tree-depth} $\geq k^c$ has one of the following minors:
  - $k \times k$ grid
  - complete binary tree of height $k$
  - path of length $2^k$
• “Grid/Tree/Path Minor Thm” [Kawarabayashi, R. 2016]
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  o complete binary tree of height $k$
  o path of length $2^k$

  These three obstructions give a polynomial approximation of $\text{td}(G)$
• “Grid/Tree/Path Minor Thm” [Kawarabayashi, R. 2016]
  Every graph of \textbf{tree-depth} \geq k^c has one of the following minors:
  \begin{itemize}
    \item k \times k grid
    \item complete binary tree of height k
    \item path of length \(2^k\)
  \end{itemize}

• \textbf{COROLLARY}
  If \text{SUB(Grid}_{k \times k}) and \text{SUB(Tree}_k) and \text{SUB(Path}_{2^k}) have \text{AC}^0 formula size \(n^{\Omega(k)}\) for all \(k\), then \text{SUB(G)} has \text{AC}^0 formula size \(n^{\Omega(td(G)^{\epsilon})}\) for all graphs \(G\).
• **[LRR 2014]**  \( \text{SUB}(\text{Grid}_{k \times k}) \) has \( AC^0 \) formula size \( n^{\Omega(k)} \)

• **[R 2014]**  \( \text{SUB}(\text{Path}_{2^k}) \) has \( AC^0 \) formula size \( n^{\Omega(k)} \)

• **[R 2016]**  \( \text{SUB}(\text{Tree}_k) \) has \( AC^0 \) formula size \( n^{\Omega(k)} \)

**COROLLARY**

If \( \text{SUB}(\text{Grid}_{k \times k}) \) and \( \text{SUB}(\text{Tree}_k) \) and \( \text{SUB}(\text{Path}_{2^k}) \) have \( AC^0 \) formula size \( n^{\Omega(k)} \) for all \( k \), then \( \text{SUB}(G) \) has \( AC^0 \) formula size \( n^{\Omega(td(G) \wedge \varepsilon)} \) for all graphs \( G \).
• [LRR 2014] SUB(Grid\(_{k \times k}\)) has AC\(^0\) formula size \(n^{\Omega(k)}\)
• [R 2014] SUB(Path\(_{2^k}\)) has AC\(^0\) formula size \(n^{\Omega(k)}\)
• [R 2016] SUB(Tree\(_{k}\)) has AC\(^0\) formula size \(n^{\Omega(k)}\)

The AC\(^0\) formula size of SUB(G) is \(n^{\Omega(td(G)^{\varepsilon})}\)

• COROLLARY
If SUB(Grid\(_{k \times k}\)) and SUB(Tree\(_{k}\)) and SUB(Path\(_{2^k}\)) have AC\(^0\) formula size \(n^{\Omega(k)}\) for all \(k\), then SUB(G) has AC\(^0\) formula size \(n^{\Omega(td(G)^{\varepsilon})}\) for all graphs G.
“Poly-rank” homomorphism preservation theorem
Classical Preservation Theorems

- Los-Tarski / Lyndon / Hom. Preservation Theorem
  A first-order formula $\varphi$ is preserved under injective / surjective / all homomorphisms if, and only if, it is equivalent to a first-order formula $\psi$ that is existential / positive / existential-positive.
Failure on Finite Structures

- Los-Tarski / Lyndon False on Finite Structures
  
  [Tait 1959], [Ajtai-Gurevich 1997]

There exists a first-order formula that is preserved under injective (resp. surjective) homomorphisms on finite structures, yet is not equivalent on finite structures to any existential (resp. positive) formula.
Failure on Finite Structures

• Los-Tarski / Lyndon False on Finite Structures
  [Tait 1959], [Ajtai-Gurevich 1997]
  There exists a first-order formula that is preserved under injective (resp. surjective) homomorphisms on finite structures, yet is not equivalent on finite structures to any existential (resp. positive) formula.

• Non-uniform circuit version:

  \[
  \text{Monotone-AC}^0 \neq \text{Monotone} \cap \text{AC}^0
  \]
Survival on Finite Structures

• Hom. Preservation Theorem on Finite Structures

[ R. 2005 ]

If a first-order formula $\varphi$ of quantifier-rank $k$ is preserved under homomorphisms on finite structures, then it is equivalent on finite structures to an existential-positive formula $\psi$ of quantifier-rank $f(k)$, where $f : \mathbb{N} \rightarrow \mathbb{N}$ is a computable function.
Survival on Finite Structures

• Hom. Preservation Theorem on Finite Structures

[ R. 2005 ]

If a first-order formula \( \varphi \) of quantifier-rank \( k \) is preserved under homomorphisms on finite structures, then it is equivalent on finite structures to an existential-positive formula \( \psi \) of quantifier-rank \( f(k) \), where \( f : \mathbb{N} \to \mathbb{N} \) is a computable function.

• Proof gives a non-elementary upper bound on \( f(k) \).
f(k) = k on *Infinite* Structures

- “Equi-rank” Hom. Preservation Theorem

[R. 2005]

If a first-order formula $\varphi$ of quantifier-rank $k$ is preserved under homomorphisms *on infinite structures*, then it is equivalent *on infinite structures* to an existential-positive formula $\psi$ of quantifier-rank $k$. 
f(k) \leq \text{poly}(k)

• “Poly-rank” Hom. Pres. Theorem on Finite Structures [R. 2016]

If a first-order formula $\varphi$ of quantifier-rank $k$ is preserved under homomorphisms on finite structures, then it is equivalent on finite structures to an existential-positive formula $\psi$ of quantifier-rank $f(k)$, where $f(k) \leq \text{poly}(k)$. 
f(k) \leq \text{poly}(k)

- "Poly-rank" Hom. Pres. Theorem on Finite Structures

If a first-order formula $\varphi$ of quantifier-rank $k$ is preserved under homomorphisms on finite structures, then it is equivalent on finite structures to an existential-positive formula $\psi$ of quantifier-rank $f(k)$, where $f(k) \leq \text{poly}(k)$.

Proof gives reduction to $n^{\Omega(\text{td}(G)^{\epsilon})}$ AC$^0$ formula size lower bound for $\text{SUB}(G)$
If a first-order formula $\varphi$ of quantifier-rank $k$ is preserved under homomorphisms on finite structures, then it is equivalent on finite structures to an existential-positive formula $\psi$ of quantifier-rank $f(k)$, where $f(k) \leq \text{poly}(k)$.

$f(k) \leq 2^{O(k)}$ follows from lower bound for $k$-STCONN of [R. 2014]
f(k) ≤ poly(k)

• “Poly-rank” Hom. Pres. Theorem on Finite Structures

If a first-order formula \( \varphi \) of quantifier-rank \( k \) is preserved under homomorphisms on finite structures, then it is equivalent on finite structures to an existential-positive formula \( \psi \) of quantifier-rank \( f(k) \), where \( f(k) \leq \text{poly}(k) \).

f(k) ≤ non-elementary(k)

follows from lower bound for k-STCONN of [Ajtai 1989]
f(k) ≤ poly(k)

• “Poly-rank” Hom. Pres. Theorem on Finite Structures [R. 2016]

If a first-order formula $\varphi$ of quantifier-rank $k$ is preserved under homomorphisms on finite structures, then it is equivalent on finite structures to an existential-positive formula $\psi$ of quantifier-rank $f(k)$, where $f(k) \leq \text{poly}(k)$.

• Non-uniform circuit version:

\[
\text{HomPres} \cap \text{AC}^0 = \exists^+\text{FO} \subseteq \{\text{poly-size monotone DNFs}\}
\]
Summary (Last Slide!)

• Complexity of SUB(G) is tied to natural structural parameters of G and to fundamental questions in complexity (P vs. NP, L vs. NL, NC$^1$ vs. P)

• Connection between AC$^0$ & FO & tw(G)/td(G):
  
  \[ \text{AC}^0 \text{ circuit size} \approx \text{FO variable width} \approx \text{tree-width}(G) \]
  \[ \text{AC}^0 \text{ formula size} \approx \text{FO quantifier rank} \approx \text{tree-depth}(G) \]

• Natural family of input distributions $X_\beta$: hard-on-average for optimal choice of $\beta$
Thank you!