Optimal Online Algorithms via Linear Scaling

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Multiple-Choice Secretary Problem
[Hajiaghayi, Kleinberg, Parkes, EC 2004], [Kleinberg, SODA 2005], ...

- $n$ items arrive online over time, each with a weight
- Weights are adversarial
  Arrival order is uniformly random
- Upon arrival of an item:
  Immediate decision whether to accept or reject
- Constraint: Accept at most $k$ items
- Objective: Maximize sum of weights of accepts items

**Theorem (Kleinberg, SODA 2005)**

There is a $1 - O(1/\sqrt{k})$-competitive algorithm, but no $1 - o(1/\sqrt{k})$-competitive algorithm.
Algorithm

In round $\ell$, upon arrival of item $j$

- Let $S^{(\ell)}$ be $\left\lfloor \frac{\ell}{n}k \right\rfloor$ items of highest value that arrived so far
- If $j \in S^{(\ell)}$ and $|\text{Accepted}| < k$
  Set $\text{Accepted} := \text{Accepted} \cup \{j\}$

Value of tentative selection

Let $p := 9 \sqrt{\frac{1}{k}}$. For $pn \leq \ell \leq (1 - p)n$, we have

- $E[v(S^{(\ell)})] \geq \frac{\ell}{n} \text{OPT} \left( 1 - 9 \sqrt{\frac{1}{\ell n k}} \right)$
- $E[v_j C^{\ell}] = \frac{1}{\ell} E[v(S^{(\ell)})] \geq \frac{1}{n} \text{OPT} \left( 1 - 9 \sqrt{\frac{1}{\ell n k}} \right)$

$C^{\ell} = \begin{cases} 1 & \text{if } j \in S^{(\ell)} \\ 0 & \text{otherwise} \end{cases}$
Algorithm

In round $\ell$, upon arrival of item $j$

- Let $S^{(\ell)}$ be $\lfloor \frac{\ell}{n} k \rfloor$ items of highest value that arrived so far
- If $j \in S^{(\ell)}$ and $|\text{Accepted}| < k$
  
  Set $\text{Accepted} := \text{Accepted} \cup \{j\}$

Conflict probability

Let $p := 9 \sqrt{\frac{1}{k}}$. For $pn \leq \ell \leq (1 - p)n$, we have

- $E[|\text{Accepted}|] \leq \sum_{\ell' = 1}^{\ell - 1} E[C_{\ell'}] \leq \frac{\ell}{n} k$
- $\Pr[|\text{Accepted}| \geq k] \leq \exp\left(-\frac{n - \ell}{n} \sqrt{k}\right)$
Putting the pieces together

For \( pn \leq \ell \leq (1 - p)n \) we have:

- \( E[v_j C_\ell] \geq \frac{1}{n} \text{OPT} \left( 1 - 9 \sqrt{\frac{1}{\ell n k}} \right) \)
- \( \text{Pr} [|\text{Accepted}| \geq k] \leq \exp \left( - \frac{n-\ell}{n} \sqrt{k} \right) \)

Despite dependencies:

\[
E[\text{value from round } \ell] \geq \frac{1}{n} \text{OPT} \left( 1 - 9 \sqrt{\frac{1}{\ell n k}} \right) (1 - \exp \left( - \frac{n-\ell}{n} \sqrt{k} \right))
\]

Adding up:

\[
E[\text{ALG}] \geq \sum_{\ell=\rho n}^{(1-p)n} \frac{1}{n} \text{OPT} \left( 1 - 9 \sqrt{\frac{1}{\ell n k}} \right) (1 - \exp \left( - \frac{n-\ell}{n} \sqrt{k} \right))
= \left( 1 - O \left( \sqrt{\frac{1}{k}} \right) \right) \text{OPT}
\]
Outline

1. Multiple-Choice Secretary Problem

2. Online Packing LPs [K., Radke, Tönnis, Vöcking, STOC 2014]

3. Temp Secretary Problem [K., Tönnis, ESA 2016]

4. Summary and Outlook
Online Packing Linear Programs

\[
\begin{align*}
\max & \quad c^T x \\
\text{s.t.} & \quad \begin{bmatrix}
A
\end{bmatrix} \begin{bmatrix}
x \\
b
\end{bmatrix} \leq \begin{bmatrix}
b
\end{bmatrix} \\
\text{for all } j: & \quad 0 \leq x_j \leq 1
\end{align*}
\]

\[B = \min_{i \in [m]} \frac{b_i}{\max_{j \in [n]} a_{i,j}}\]
$(1 - \epsilon)$-competitive algorithms if $B$ is large enough

Feldman, Henzinger, Korula, Mirrokni, Stein (ESA 2010): $B = \Omega\left(\frac{m \log(nK)}{\epsilon^3}\right)$

Agrawal, Wang, Ye (2009): $B = \Omega\left(\frac{m \log(nK/\epsilon)}{\epsilon^2}\right)$

Molinaro and Ravi (ICALP 2012): $B = \Omega\left(\frac{m^2 \log(m/\epsilon)}{\epsilon^2}\right)$

K., Radke, Tönnis, Vöcking (STOC 2014): $B = \Omega\left(\frac{\log \frac{d}{\epsilon^2}}{\epsilon^2}\right)$

matches lower bound by Agrawal et al., 2009
Algorithm

In round $\ell$, upon arrival of variable $x_j$

- $\tilde{x}^{(\ell)} := \text{optimal solution to “known” LP with scaled capacities } \frac{\ell}{n} b$
- $x_j^{(\ell)} := 1$ with probability $\tilde{x}_j^{(\ell)}$, all other entries 0
- If $A(y + x^{(\ell)}) \leq b$, set $y := y + x^{(\ell)}$

Theorem

$E[ALG] = \left(1 - O\left(\sqrt{\frac{\log d}{B}}\right)\right) OPT$

$E[ALG] = \Omega\left(d^{-\frac{2}{B-1}}\right) OPT$
For $pn \leq \ell \leq (1 - p)n$ we have:

- $E[c^T x(\ell)] \geq \frac{1}{n} \text{OPT} \left(1 - 9 \sqrt{\frac{1 + \ln d}{\frac{\ell}{n} B}}\right)$
- $\Pr \left[ \left(\sum_{\ell' \leq \ell} A x(\ell')\right)_i > b_i \right] \leq \frac{1}{d} \exp \left(- \frac{n - \ell}{n} \sqrt{B}\right)$

Despite dependencies:

$E[c^T y(\ell)] \geq \frac{1}{n} \text{OPT} \left(1 - 9 \sqrt{\frac{1 + \ln d}{\frac{\ell}{n} B}}\right) \left(1 - \exp \left(- \frac{n - \ell}{n} \sqrt{B}\right)\right)$

Adding up:

$E[\text{ALG}] \geq \sum_{\ell = pn}^{(1 - p)n} \frac{1}{n} \text{OPT} \left(1 - 9 \sqrt{\frac{1 + \ln d}{\frac{\ell}{n} B}}\right) \left(1 - \exp \left(- \frac{n - \ell}{n} \sqrt{B}\right)\right)$

$= \left(1 - O \left(\sqrt{\frac{\log d}{B}}\right)\right) \text{OPT}$
Algorithm

In round $\ell$, upon arrival of variable $x_j$

- $\tilde{x}^{(\ell)} :=$ optimal solution to “known” LP with scaled capacities $\frac{\ell}{n} b$
- $x_j^{(\ell)} := 1$ with probability $\tilde{x}_j^{(\ell)}$, all other entries 0
- If $A(y + x^{(\ell)}) \leq b$, set $y := y + x^{(\ell)}$

Theorem

- $E[ALG] = \left(1 - O\left(\sqrt{\frac{\log d}{B}}\right)\right) OPT$
- $E[ALG] = \Omega\left(d^{-\frac{2}{B-1}}\right) OPT$
Algorithm

In round $\ell$, upon arrival of variable $x_j$

- $\tilde{x}^{(\ell)} := \text{optimal solution to “known” LP with scaled capacities } \frac{\ell}{n} b$
- $x_j^{(\ell)} := 1$ with probability $\tilde{x}_j^{(\ell)}$, all other entries 0
- If $A(y + x^{(\ell)}) \leq b$, set $y := y + x^{(\ell)}$

Value of tentative selection

Let $p := 9 \sqrt{\frac{1+\ln d}{B}}$. For $pn \leq \ell \leq (1 - p)n$, we have

- $E[c^T \tilde{x}^{(\ell)}] \geq \frac{\ell}{n} \text{OPT} \left( 1 - 9 \sqrt{\frac{1+\ln d}{\ell n B}} \right)$
- $E[c^T x^{(\ell)}] = \frac{1}{\ell} E[c^T \tilde{x}^{(\ell)}] \geq \frac{1}{n} \text{OPT} \left( 1 - 9 \sqrt{\frac{1+\ln d}{\ell n B}} \right)$
Algorithm

In round $\ell$, upon arrival of variable $x_j$

- $\tilde{x}^{(\ell)} :=$ optimal solution to “known” LP with scaled capacities $\frac{\ell}{n}b$
- $x_j^{(\ell)} := 1$ with probability $\tilde{x}_j^{(\ell)}$, all other entries 0
- If $A(y + x^{(\ell)}) \leq b$, set $y := y + x^{(\ell)}$

Conflict probability

Let $p := 9\sqrt{\frac{1+\ln d}{B}}$. For $pn \leq \ell \leq (1 - p)n$, we have

$$Pr \left[ \left( \sum_{\ell' \leq \ell} Ax^{(\ell')} \right)_i > b_i \right] \leq \frac{1}{d} \exp \left( -\frac{n - \ell}{n} \sqrt{B} \right)$$
n items arrive online over time, each with a weight and a length

Weights are adversarial
Arrival times are drawn i.i.d. from a distribution (e.g. $U[0, 1]$)

Upon arrival of an item:
Immediate decision whether to accept or reject

Constraint: Accept at most $k$ items simultaneously

Objective: Maximize sum of weights of accepts items
Algorithm for Identical Lengths

[K., Tönnis, ESA 2016]

Important Observation:
If all items have length $\gamma$, we can accept at most $k \lceil 1/\gamma \rceil$ arriving in $[0, 1]$

\begin{align*}
\textbf{for} \text{ every arriving item } j \textbf{ do} & \\
& \text{Set } t := \tau_j; \\
& \text{Let } S(t) \text{ be the } \lfloor tk/\gamma \rfloor \text{ highest-valued items } i \text{ with } \tau_i \leq t; \\
& \textbf{if } j \in S(t) \textbf{ then} \\
& \quad \textbf{if } \text{Accepted} \cup \{j\} \text{ is a feasible schedule then} \\
& \quad \quad \text{Set } \text{Accepted} := \text{Accepted} \cup \{j\};
\end{align*}

Theorem

If all lengths are $\gamma$, the algorithm is

- $1/2 - O(\sqrt{\gamma})$-competitive and
- $1 - O(1/\sqrt{k}) - O(\sqrt{\gamma})$-competitive
Summary

- General simple template for online algorithms:
  Pretend you could start from scratch now, but leave some space!

- Optimal guarantees for multiple-choice secretary problem and online packing LPs

- Simple and better algorithm for temp secretary problem

- Approach also works for submodular objective functions

  [K., Tönnis, unpublished]
What about cost-minimization problems, e.g., scheduling?
[Göbel, K., Tönnis, ESA 2015]

What about non-uniformly drawn permutations?
[K., Kleinberg, Niazadeh, STOC 2015]

Are there algorithms that are good in multiple models?

Thank you!
Questions?