Analytical Approach to Parallel Repetition

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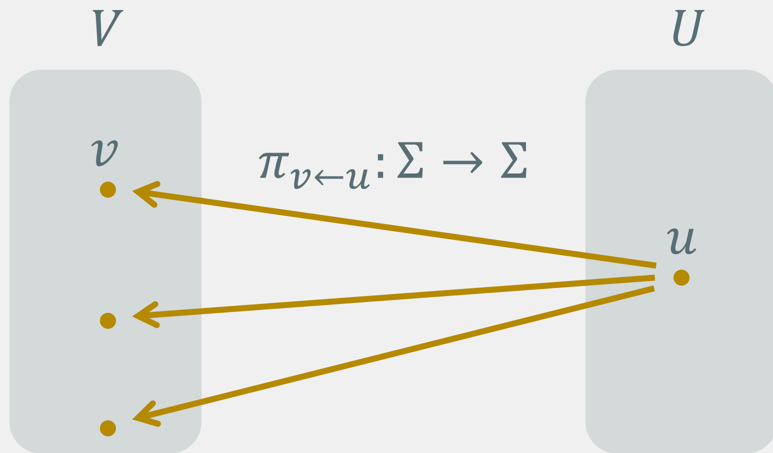
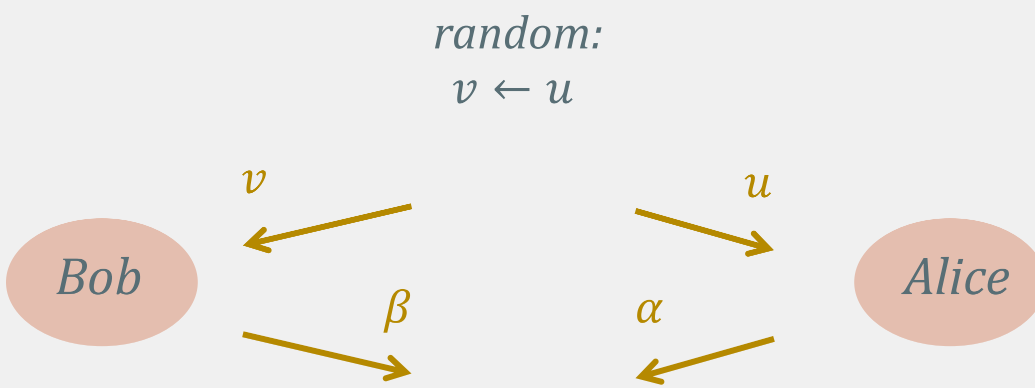
constraint graph



bipartite
d-regular
(for simplicity)

game G



no communication between A & B

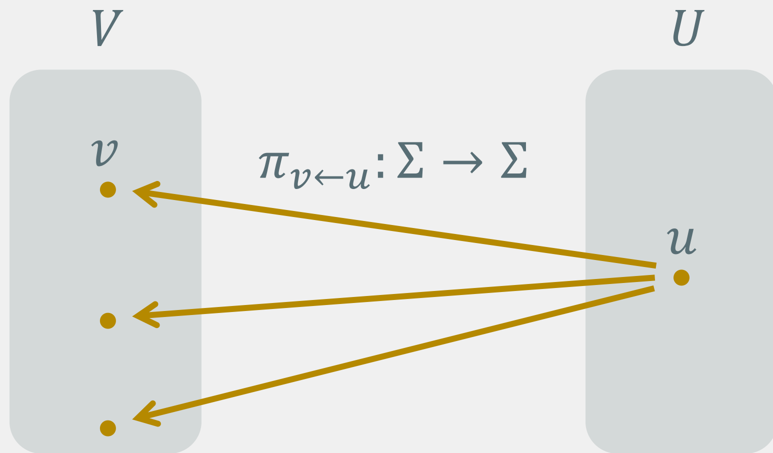
strategy $g\colon V \to \Sigma$

strategy $f\colon U \to \Sigma$

WIN: $\beta = \pi_{v\leftarrow u}(\alpha)$ projection constraint

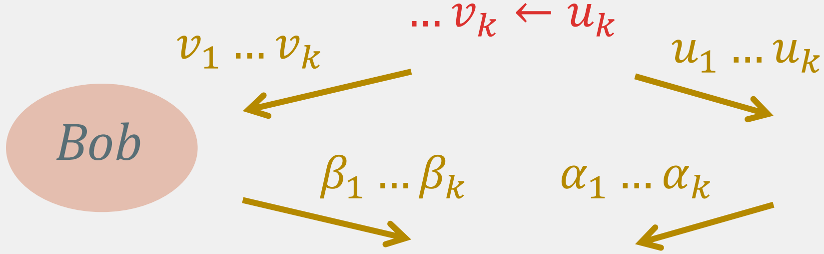
LABEL COVER
given: game G
find: value(G)

$$\text{value}(G) \coloneqq \max_{f,g} \mathbb{P}_{v\leftarrow u}\{g(v) = \pi_{v\leftarrow u} \circ f(u)\}$$



parallel repeated game $G^{\otimes k}$

random:
$v_1 \leftarrow u_1$
$\ldots v_k \leftarrow u_k$



strategy
$g \colon V^k \to \Sigma^k$

strategy
$f \colon U^k \to \Sigma^k$

WIN: $\beta_1 = \pi_{v_1 \leftarrow u_1}(\alpha_1)$
$\ldots \beta_k = \pi_{v_k \leftarrow u_k}(\alpha_k)$

goal: bound value$(G^{\otimes k})$ in terms of value(G) and k

previous bounds

(long history, notoriety)

parallel repetition theorem [Raz'95, improved: Holenstein'07, Rao'08]
(for projection games)

$$\text{value}(G) \leq 1 - \varepsilon \implies \text{value}\left(G^{\otimes k}\right) \leq \left(1 - \varepsilon^{\cancel{3}\,2}/2\right)^k$$

(tight even for games with XOR constraints) [Raz'08]

main application: hardness amplification for LABEL COVER
1 vs δ approximation is NP-hard (basis of inapproximability results)

What happens if* $\mathbf{value}(G) = o(1)$ *… or* $k \leq 1/\varepsilon$ *?

parallel repetition theorem [Raz'95, improved: Holenstein'07, Rao'08]

(for projection games)

$$\mathrm{value}(G) \leq 1 - \varepsilon \implies \mathrm{value}\left(G^{\otimes k}\right) \leq (1 - \varepsilon^2/2)^k$$

our results

analytical framework to analyze parallel repetition

(contrast to previous information-theoretic approach)

new bounds

low value: $\mathrm{value}(G) \leq \rho \implies \mathrm{value}\left(G^{\otimes k}\right) \leq (2\rho)^{k/2}$

(for projection constraints)

few repetitions: $\mathrm{value}(G) \leq 1 - \varepsilon \implies \mathrm{value}\left(G^{\otimes k}\right) \leq (1 - \varepsilon)^{\sqrt{k}}$

(for projection constraints, $k \ll 1/\varepsilon^2$)

new bounds

low value: $\text{value}(G) \le \rho \implies \text{value}\left(G^{\otimes k}\right) \le (2\rho)^{k/2}$

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few repetitions: $\text{value}(G) \le 1-\varepsilon \implies \text{value}\left(G^{\otimes k}\right) \le (1-\varepsilon)^{\sqrt{k}}$

(for projection constraints, $k \ll 1/\varepsilon^2$)

implications

$2^{n^{\Theta(\varepsilon)}}$-time algorithm is *optimal** for approximation ratio $(1-\varepsilon)\ln n$

[Cygan-Kowalik-Wykurz'09]

optimal NP-hardness for SET COVER (and better NP-hardness for LABEL COVER)

$(1-\varepsilon)\ln n$-approximation, via [Moshkovitz–Raz, Feige, Moshkovitz]

Raz's parallel-repetition counterexample tight even for small k

some G have value $\le 1-\varepsilon$ but $\text{value}\left(G^{\otimes k}\right) \ge 1-\varepsilon\sqrt{k}$

(answers question of O'Donnell)

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* under standard complexity assumptions, $\text{NP} \not\subseteq \text{TIME}\left(2^{n^{o(1)}}\right)$

proof overview

show game parameter $\mathrm{relax}(\cdot)$ with

1. $\mathrm{relax}(G) \geq \mathrm{value}(G)$ for all G *(relaxation)*

2. $\mathrm{relax}(G^{\otimes k}) = \mathrm{relax}(G)^k$ for all G, k *(multiplicativity)*

3. $\mathrm{relax}(G) \lesssim \mathrm{value}(G)$ for all G *(approximation)*

proof of parallel-repetition bound

$$\mathrm{value}(G^{\otimes k}) \overset{1.}{\leq} \mathrm{relax}(G^{\otimes k}) \overset{2.}{=} \mathrm{relax}(G)^k \overset{3.}{\lesssim} \mathrm{value}(G)^k$$

proof overview

show game parameter relax$(\cdot)$ with

1. relax$(G) \geq$ value(G) for all G *(relaxation)*

2. relax$\left(G^{\otimes k}\right) =$ relax$(G)^k$ for all G, k *(multiplicativity)*

3. relax$(G) \lesssim$ value(G) for all G *(approximation)*

BASIC SDP satisfies 1 & 2 but not 3 [Feige–Lovász'92] (no efficient param. can satisfy 3)

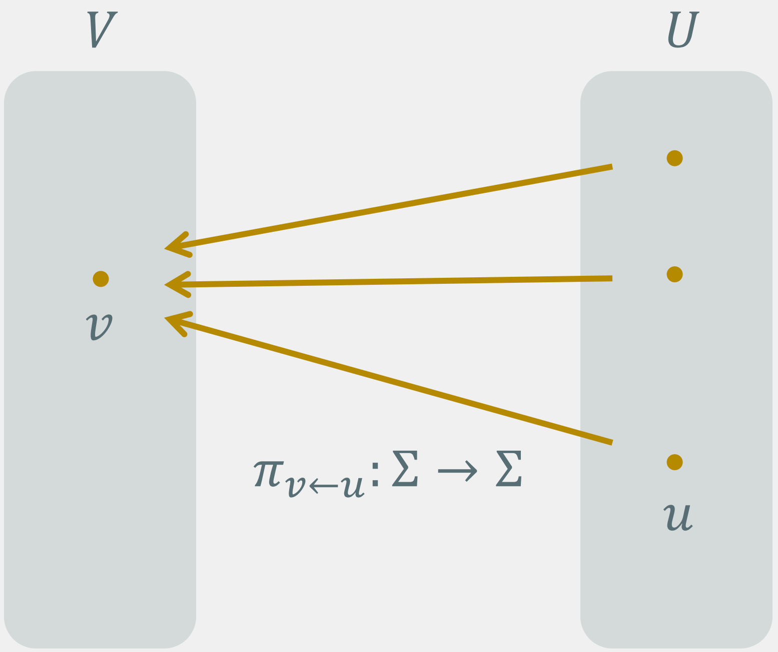
our parameter is the analog over cone of *completely positive matrices*

(instead of cone of p.s.d. matrices)

very similar to "Hellinger value" [Barak-Hardt-Haviv-Rao-Regev-S.'08]

analytical setup

constraint graph



analytical setup

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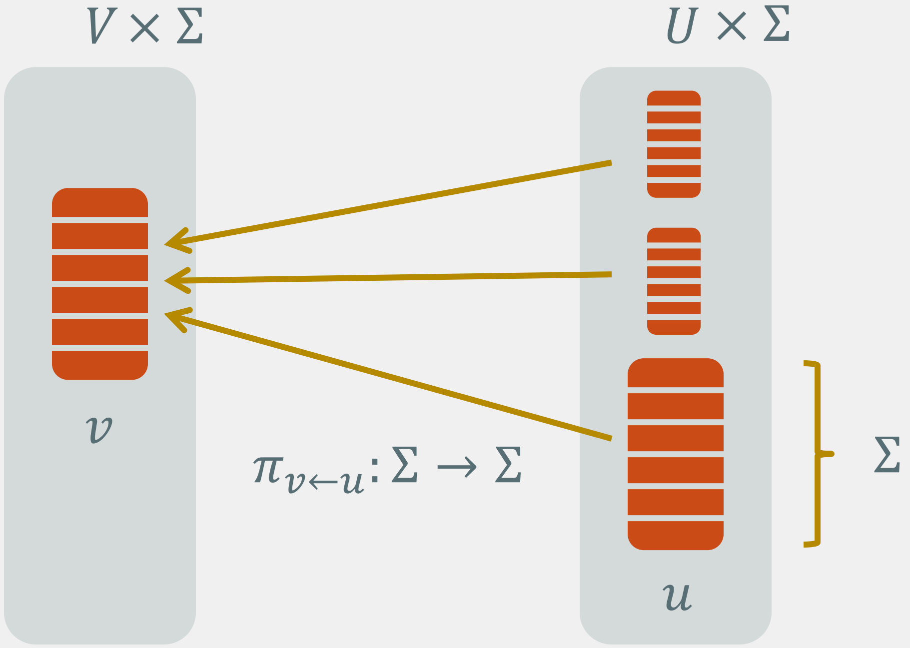
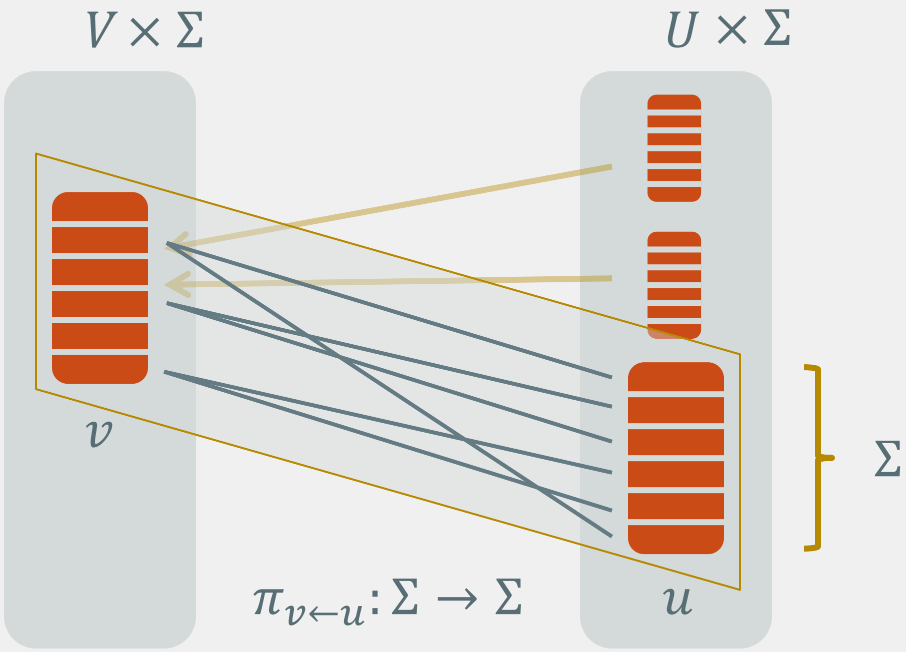


analytical setup

label-extended graph G

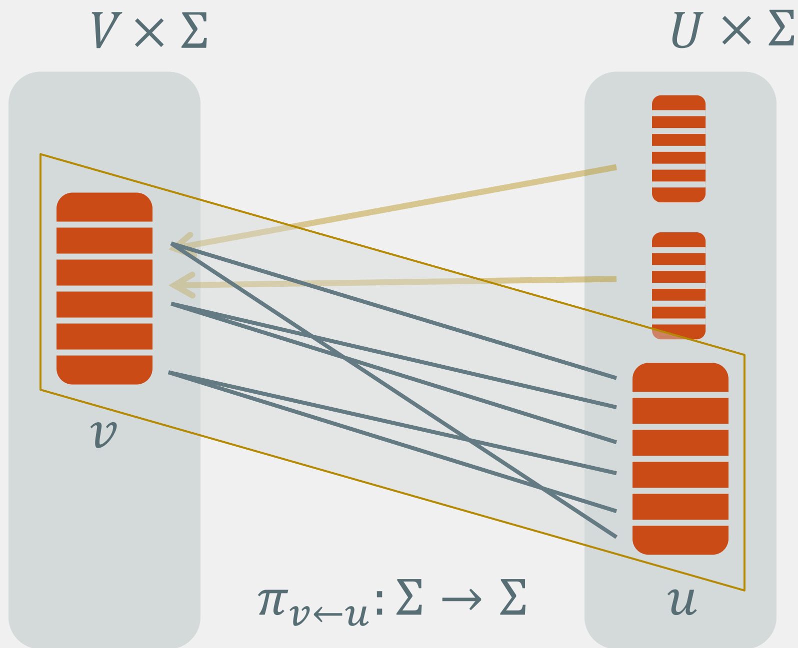
~~constraint graph~~



analytical setup

label-extended graph G

$g: V \times \Sigma \to \mathbb{R}$



$f: U \times \Sigma \to \mathbb{R}$

$f: U \times \Sigma \to \mathbb{R}$ is *assignment* if $f \geq 0$ and $\sum_\alpha f(u, \alpha) = 1$ for all $u \in U$

linear operator

= adjacency matrix of label-extended graph

$G: \mathbb{R}^{U \times \Sigma} \to \mathbb{R}^{V \times \Sigma}$

For assignment f, $Gf(v, \beta)$ = prob. that random v-neighbor "demands" β

$$Gf(v, \beta) \coloneqq \mathbb{E}_{u:\, v \leftarrow u} \sum_{\alpha:\, \beta = \pi_{v \leftarrow u}(\alpha)} f(u, \alpha)$$

success probability for assignments f, g

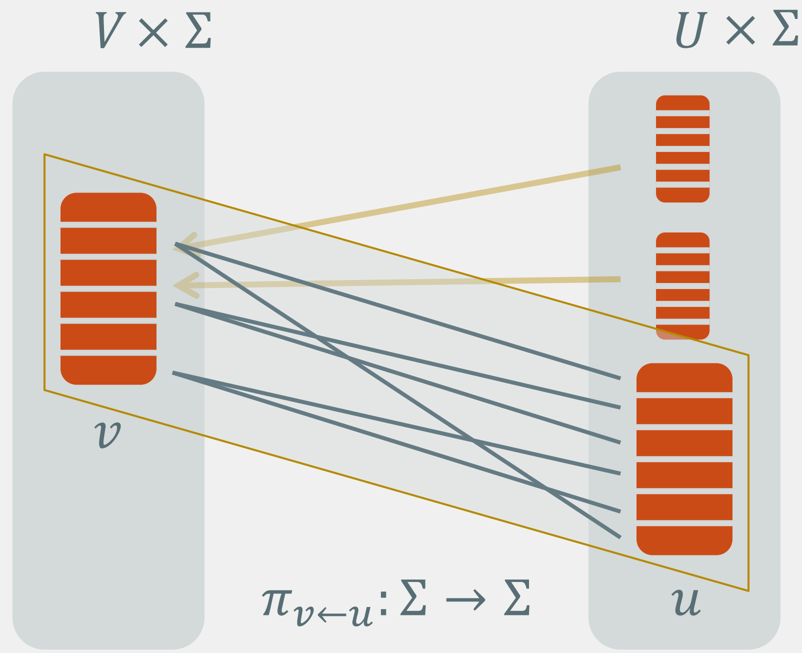
bilinear form

$$\langle Gf, g \rangle \coloneqq \mathbb{E}_v \sum_\beta Gf(v, \beta) \cdot g(v, \beta)$$

$\text{value}(G) = \max \langle Gf, g \rangle$ over assignments f, g

analytical setup

label-extended graph G



$f \colon U \times \Sigma \to \mathbb{R}$

$g \colon V \times \Sigma \to \mathbb{R}$

$G \colon \mathbb{R}^{U \times \Sigma} \to \mathbb{R}^{V \times \Sigma}$ $\quad$ $H \colon \mathbb{R}^{U' \times \Sigma'} \to \mathbb{R}^{V' \times \Sigma'}$

tensor product $\quad$ $G \otimes H \colon \mathbb{R}^{U \times U' \times \Sigma \times \Sigma'} \to \mathbb{R}^{V \times V' \times \Sigma \times \Sigma'}$

= parallel repetition

$$(G \otimes H) f(v, v', \beta, \beta') \coloneqq \mathbb{E}_{\substack{v \leftarrow u \\ v' \leftarrow u'}} \sum_{\substack{\beta = \pi_{v \leftarrow u}(\alpha) \\ \beta' = \pi_{v' \leftarrow u'}(\alpha')}} f(u, u', \alpha, \alpha')$$

analytical setup

$\|Gf\|_2^2 = \mathbb{E}_v \sum_\beta Gf(v,\beta)^2$

good proxy for value(G)

claim: $\text{value}(G) \le \max_{\text{assignment } f} \|Gf\|_2 \le \text{value}(G)^{1/2}$

proof: $\text{value}(G) = \langle Gf, g\rangle \le \|Gf\|_2 \cdot \underbrace{\|g\|_2}_{\le 1}$

$\|Gf\|_2 = \langle Gf, \underbrace{Gf}_{\text{assignment for V}}\rangle^{1/2} \le \text{value}(G)^{1/2}$

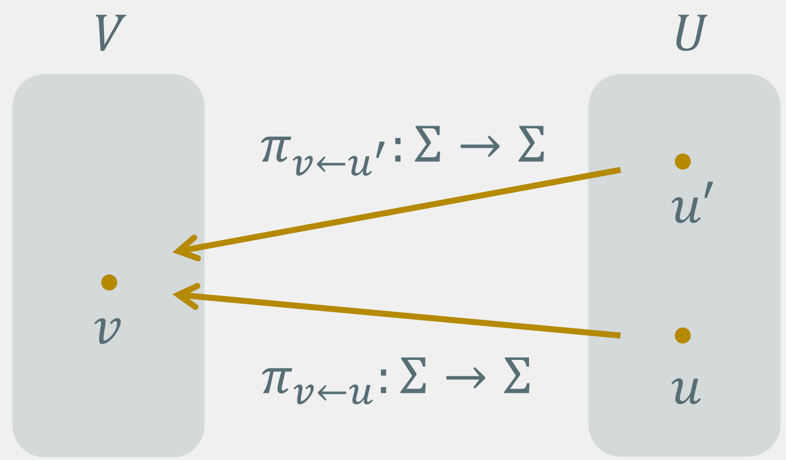
assignment for V (because G is projecting)

squared game $G^T G$:

sample v

sample two neighbors u, u'

WIN if *collision*
$\pi_{v\leftarrow u}(f(u)) = \pi_{v\leftarrow u'}(f(u'))$



warm-up theorem

Suppose constraint graph is expanding (often wlog)

Then, $\text{value}(G^{\otimes k}) > (1-\eta)^k$ implies $\text{value}(G) > 1 - O(\eta)$

two steps

$\exists$ assignment f
$\|G^{\otimes k} f\| \geq (1-\eta)^k$

$\exists$ nonnegative f
$\|Gf\| \geq (1-\eta)\|Tf\|$

$\exists$ assignment f
$\|Gf\| \geq 1-\eta$

relaxation & multiplicativity

approximation (rounding)

trivial game T

∃ assignment f
$\|G^{\otimes k} f\| \geq (1-\eta)^k$

∃ nonnegative f
$\|Gf\| \geq (1-\eta)\|Tf\|$

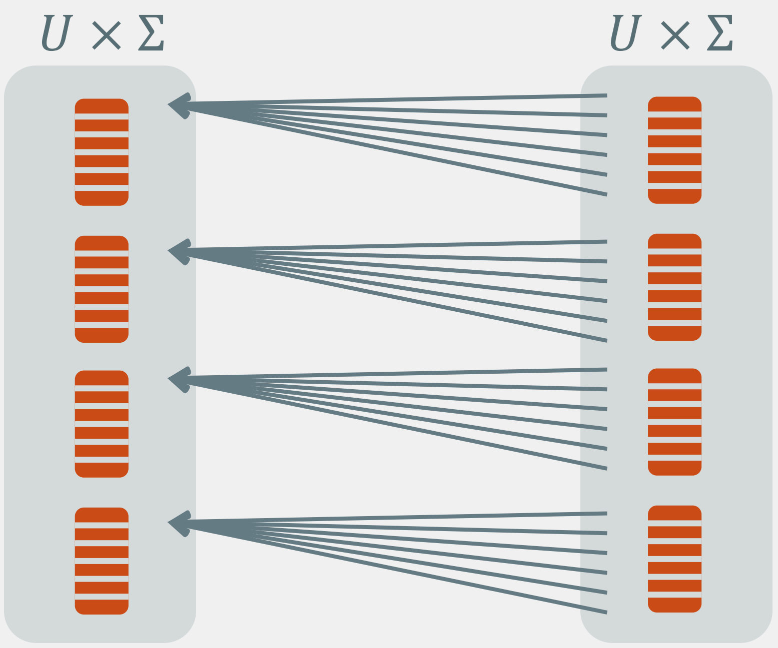
∃ assignment f
$\|Gf\| \geq 1-\eta$

trivial game T

$\|Tf\| = 1$
for every assignment f

$$\max_{\text{assign. } f} \|(T \otimes H)f\| = \max_{\text{assign. } f} \|Hf\|$$
for all games H

T does not help to win in parallel repetition



$\exists$ assignment f
$\|G^{\otimes k} f\| \geq (1-\eta)^k$

$\exists$ nonnegative f
$\|Gf\| \geq (1-\eta)\|Tf\|$

$\exists$ assignment f
$\|Gf\| \geq 1-\eta$

Wlog: $\|(T \otimes G^{\otimes k-1})f\| \leq (1-\eta)^{k-1}$

Otherwise, can take k smaller
(using $\max_f \|(T \otimes G^{\otimes k-1})f\| = \max_f \|G^{\otimes k-1} f\|$)

$\exists$ assignment f
$\|G^{\otimes k} f\| \geq (1-\eta)^k$

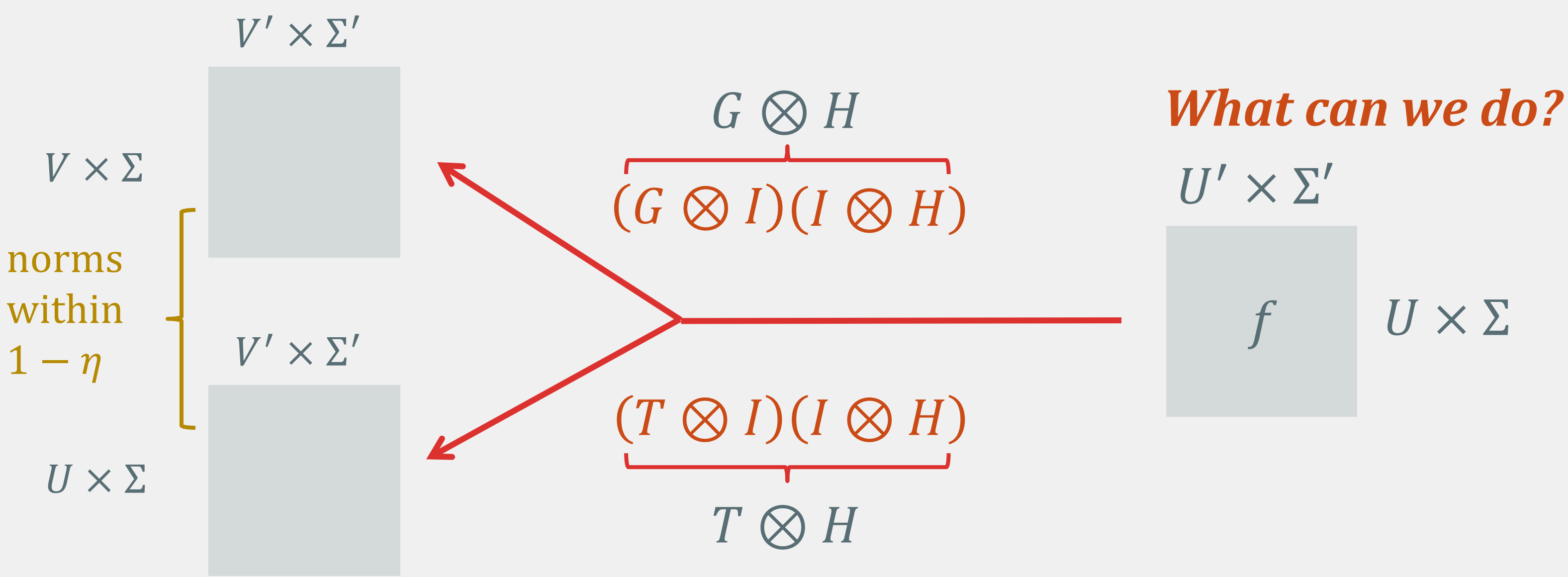
$\exists$ nonnegative f
$\|Gf\| \geq (1-\eta)\|Tf\|$

$\exists$ assignment f
$\|Gf\| \geq 1-\eta$

$$\text{\textit{Wlog:}} \quad \|(T \otimes \underbrace{G^{\otimes k-1}}_{H \,:=})f\| \leq (1-\eta)^{k-1}$$

$H: \mathbb{R}^{U' \times \Sigma'} \to \mathbb{R}^{V' \times \Sigma'}$

$$\text{\textit{Have:}} \quad \|(G \otimes H)f\| \geq (1-\eta)\|(T \otimes H)f\|$$



∃ assignment f
$\|G^{\otimes k} f\| \geq (1-\eta)^k$

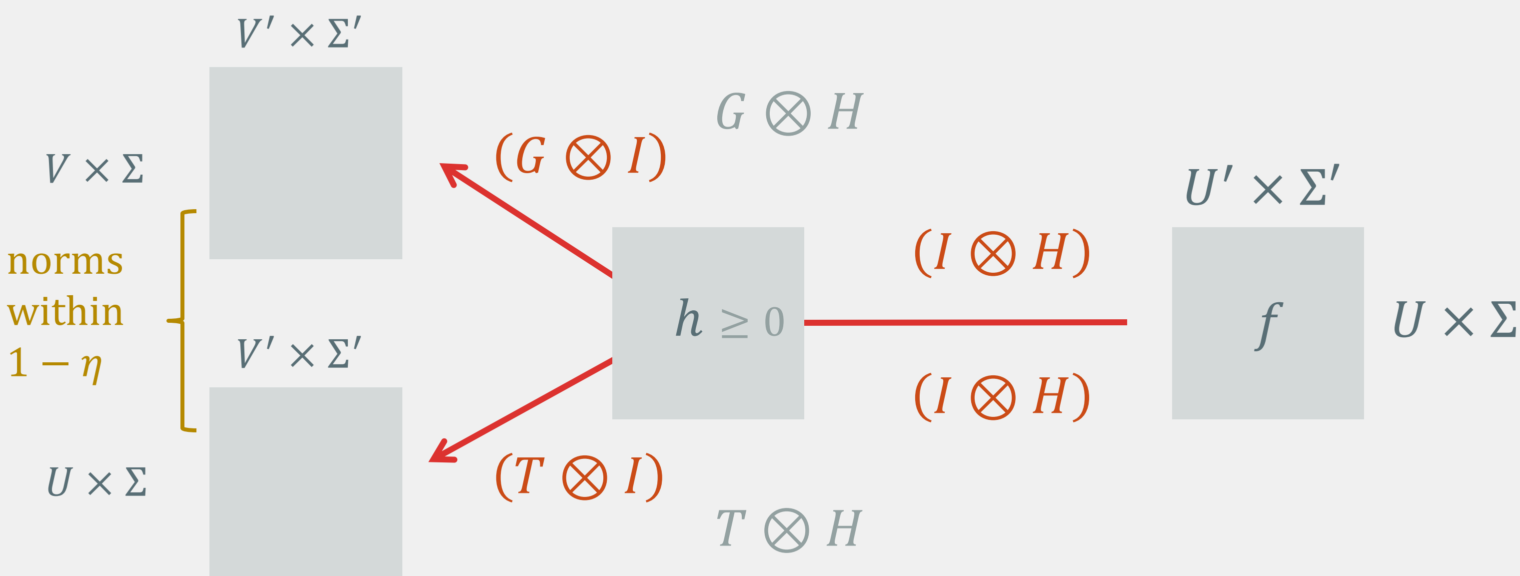
∃ nonnegative f
$\|Gf\| \geq (1-\eta)\|Tf\|$

∃ assignment f
$\|Gf\| \geq 1-\eta$

$$H := \overbrace{G^{\otimes k-1}}$$

Wlog: $\left\|\left(T \otimes G^{\otimes k-1}\right) f\right\| \leq (1-\eta)^{k-1}$ $\qquad H: \mathbb{R}^{U' \times \Sigma'} \to \mathbb{R}^{V' \times \Sigma'}$

Have: $\|(G \otimes H) f\| \geq (1-\eta)\|(T \otimes H) f\|$



$\exists$ assignment f
$\|G^{\otimes k} f\| \geq (1-\eta)^k$

$\Rightarrow$

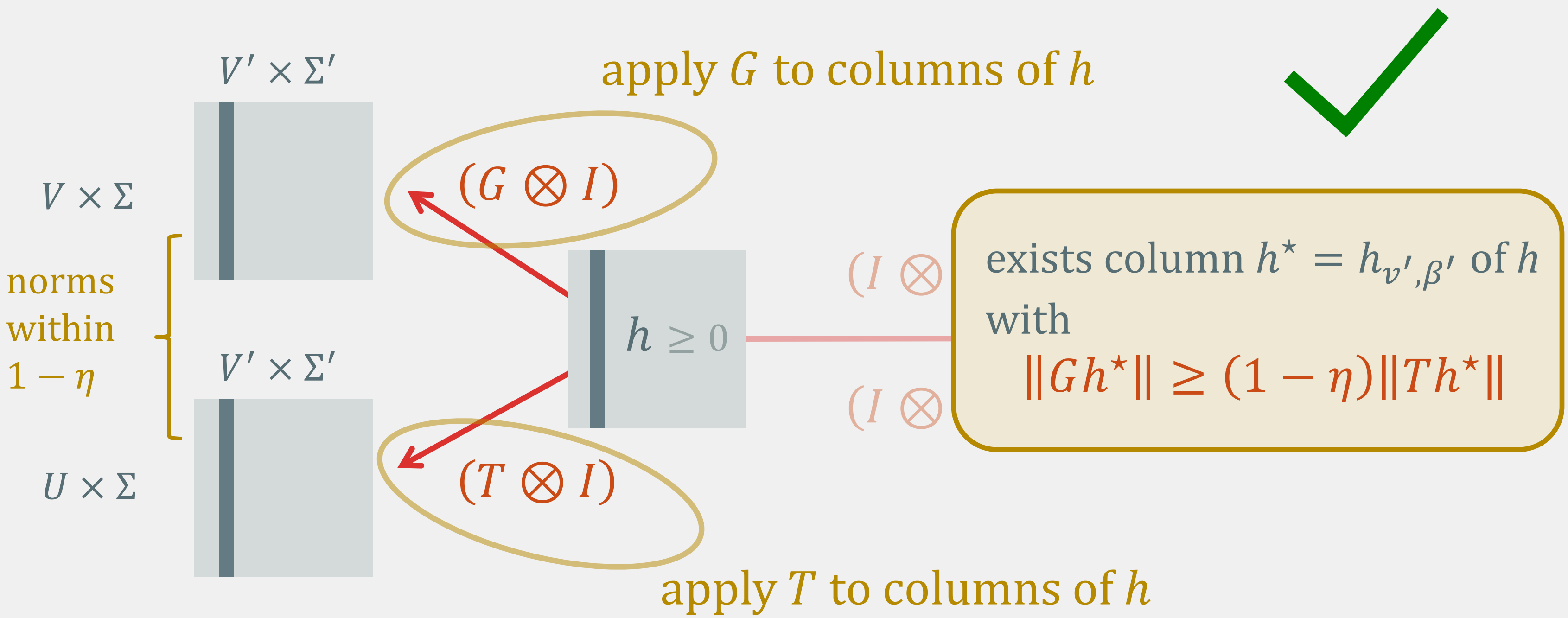
$\exists$ nonnegative f
$\|Gf\| \geq (1-\eta)\|Tf\|$

$\Rightarrow$

$\exists$ assignment f
$\|Gf\| \geq 1-\eta$

Wlog: $\|(T \otimes \overbrace{G^{\otimes k-1}}^{H \,:=})f\| \leq (1-\eta)^{k-1}$ $\qquad H\colon \mathbb{R}^{U' \times \Sigma'} \to \mathbb{R}^{V' \times \Sigma'}$

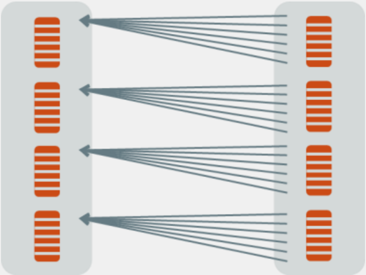
Have: $\|(G \otimes H)f\| \geq (1-\eta)\|(T \otimes H)f\|$



$\exists$ assignment f
$\|G^{\otimes k} f\| \geq (1-\eta)^k$

$\exists$ nonnegative f
$\|Gf\| \geq (1-\eta)\|Tf\|$

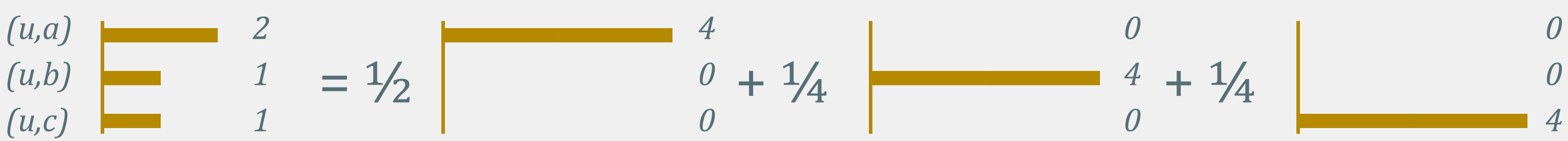
$\exists$ assignment f
$\|Gf\| \geq 1-\eta$



Explicitly: $\|Tf\|^2 = \mathbb{E}_u \left(\sum_\alpha f(u,\alpha)\right)^2$

Wlog: f is "deterministic" ($f(u,\alpha) \neq 0$ for at most one α per u)

Otherwise, write f as distribution over such functions with fixed $\|T \cdot\|$. Use convexity of $f \mapsto \|G\, f\|$.



$\exists$ assignment f
$\|G^{\otimes k} f\| \geq (1-\eta)^k$

$\exists$ nonnegative f
$\|Gf\| \geq (1-\eta)\|Tf\|$

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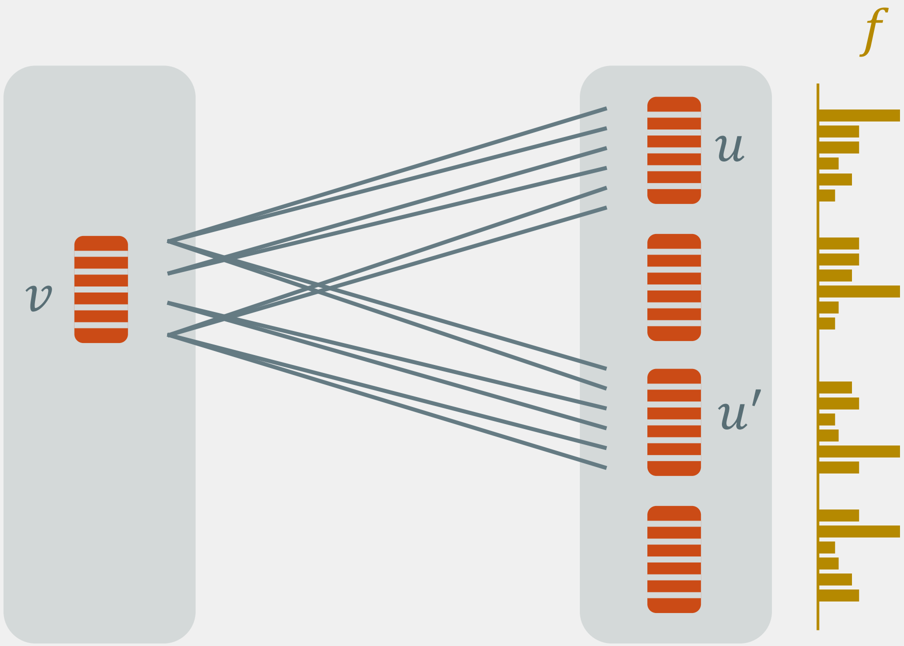
Wlog: f is "deterministic" ($f(u,\alpha) \neq 0$ for at most one α per u)



∃ assignment f
$\|G^{\otimes k} f\| \geq (1-\eta)^k$

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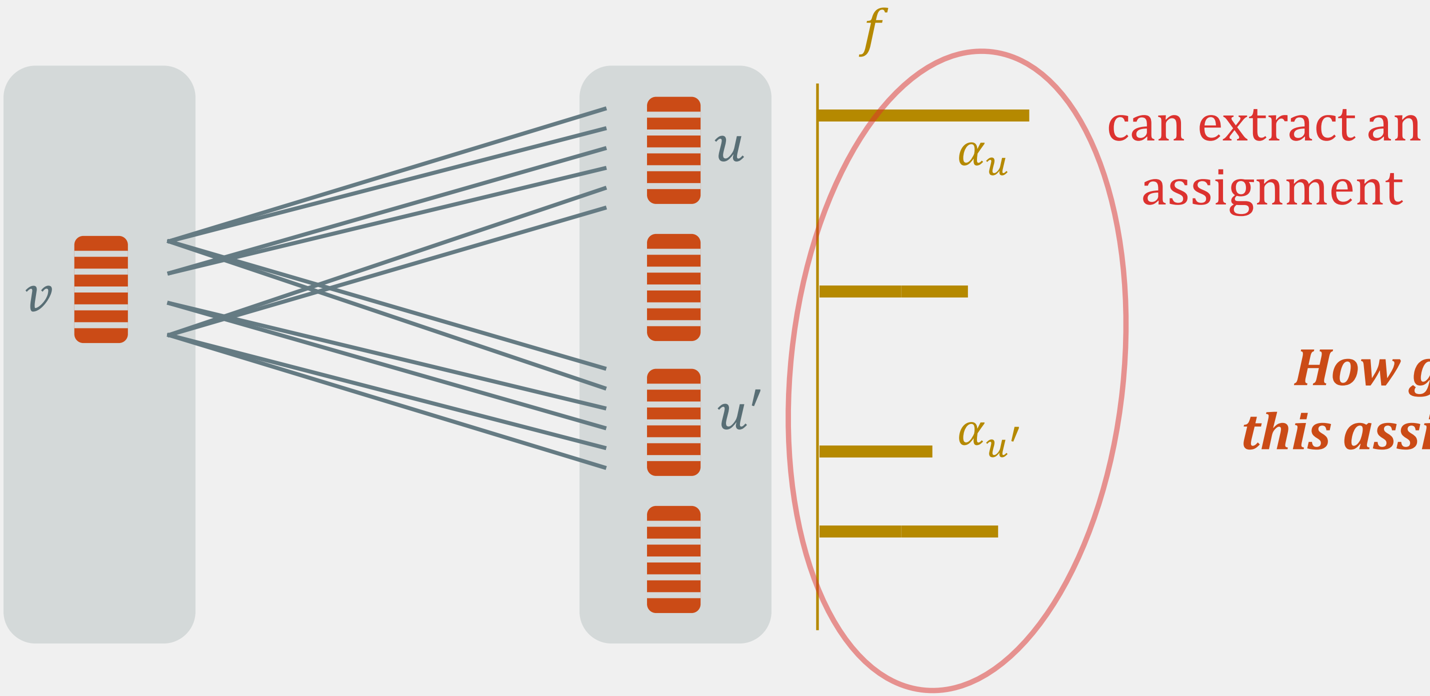
Wlog: f is "deterministic" ($f(u,\alpha) \neq 0$ for at most one α per u)

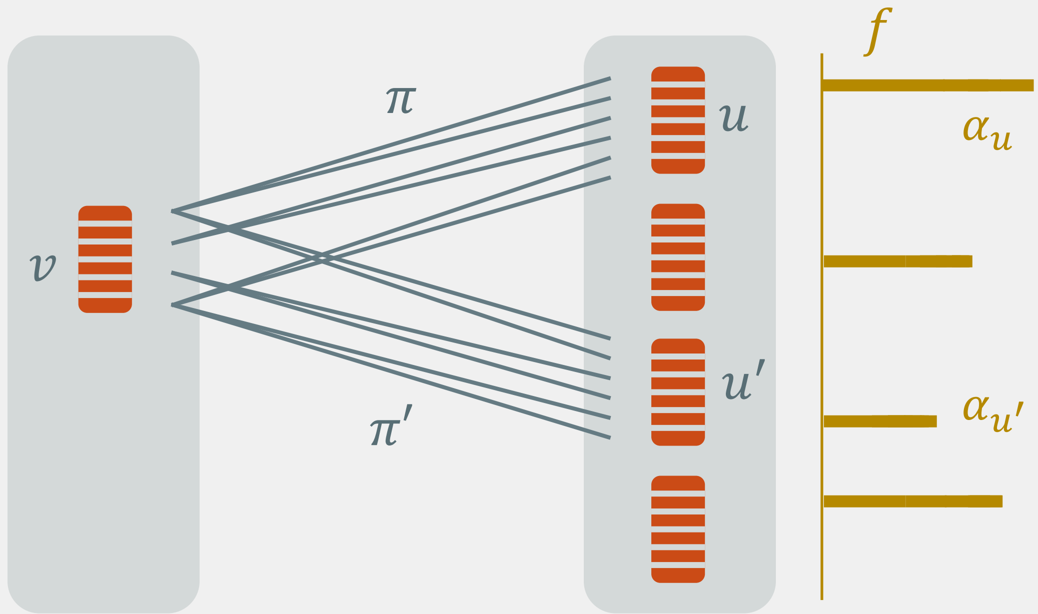


How good is this assignment?

$\exists$ assignment f
$\|G^{\otimes k} f\| \geq (1-\eta)^k$

$\exists$ nonnegative f
$\|Gf\| \geq (1-\eta)\|Tf\|$

$\exists$ assignment f
$\|Gf\| \geq 1-\eta$



can extract an assignment

expander! (squared constraint graph)

$$(1-\eta)\|f\|^2 \leq \|Gf\|^2 = \mathbb{E}_{u\sim u'} f(u,\alpha_u) f(u',\alpha_{u'}) \cdot Q_{u,u'}$$

$Q_{u,u'} = \mathbb{P}_{\substack{v\leftarrow u\\ v\leftarrow u'}}\{\pi(\alpha_u) = \pi'(\alpha_{u'})\}$

$(1-\eta)$-correlated with expander
➔ $O(\eta)$-close to constant function!

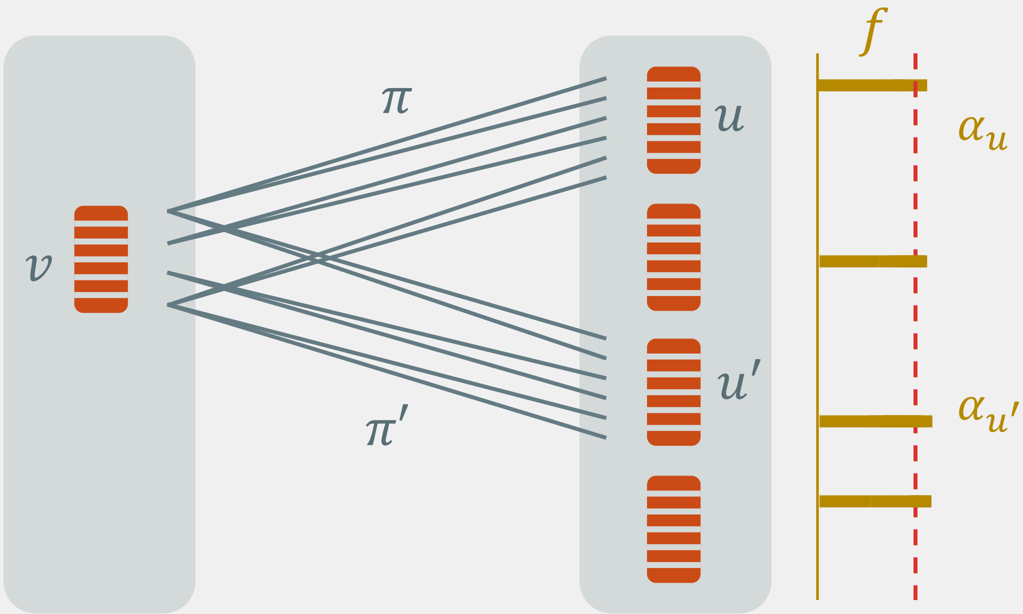
∃ assignment f
$\|G^{\otimes k} f\| \geq (1-\eta)^k$

➔

∃ nonnegative f
$\|Gf\| \geq (1-\eta)\|Tf\|$

➔

∃ assignment f
$\|Gf\| \geq 1-\eta$



can extract an assignment

assignment has value $\geq 1 - O(\eta)$

✓

expander! (squared constraint graph)

$\mathbb{P}_{v \leftarrow u, v \leftarrow u'}\{\pi(\alpha_u) = \pi'(\alpha_{u'})\}$

$$(1-\eta)\|f\|^2 \leq \|Gf\|^2 = \mathbb{E}_{u \sim u'} f(u,\alpha_u) f(u',\alpha_{u'}) \cdot Q_{u,u'}$$

$(1-\eta)$-correlated with expander
➔ $O(\eta)$-close to constant function! ⇝ $\mathbb{E}_{u\sim u'} Q_{u,u'} \geq 1 - O(\eta)$

value of assignment!

extensions

$$\frac{\|Gf\|}{\|Tf\|} \rightarrow \underbrace{\frac{\|(G \otimes I)f\|}{\max_u \|(T_u \otimes I)f\|}}$$

corresponds to
operator norm for $G \otimes I$

non-expanding constraint graphs

compare against family of trivial games T_u

use *Cheeger-style rounding* to extract "partial assignments"

use *correlated sampling* to combine them

low-value regime (value$(G) = o(1)$)

use *low-correlation* version of Cheeger (like for d-to-1 games [S.'10])

few repetitions (value$\left(G^{\otimes k}\right) \geq 0.9$)

value$(G) = 1 - \varepsilon$
value$(H) = 1 - t \cdot \varepsilon$
$\rightarrow$ value$(G \otimes H)$
$\leq 1 - \left(t + \frac{1}{t}\right) \cdot \varepsilon$

show: intermediate non-negative function is close to 0/1

careful rounding to exploit near-integrality

open questions

operator-theoretic viewpoint

applications for other PCP constructions?

combination with information-theoretic approach?

Thank you!
Question?