Logic and Quantum Information
Lecture V: Mere Possibilities

Samson Abramsky

Department of Computer Science
The University of Oxford
We have already seen an interplay between probabilistic and possibilistic notions. We now put this in a more general setting. Recall firstly that a probability distribution of finite support on a set $X$ can be specified as a function $d: X \to \mathbb{R} \geq 0$ where $\mathbb{R} \geq 0$ is the set of non-negative reals, satisfying the normalization condition $\sum_{x \in X} d(x) = 1$. This guarantees that the range of the function lies within the unit interval $[0, 1]$. The finite support condition means that $d$ is zero on all but a finite subset of $X$. The probability assigned to an event $E \subseteq X$ is then given by $d(E) = \sum_{x \in E} d(x)$. 

Samson Abramsky (Department of Computer Science) The University of Oxford Logic and Quantum Information Lecture V: Mere Possibility
Generalizing Distributions

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where $\mathbb{R}_{\geq 0}$ is the set of non-negative reals, satisfying the normalization condition

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This guarantees that the range of the function lies within the unit interval \([0, 1]\).

The finite support condition means that \( d \) is zero on all but a finite subset of \( X \). The probability assigned to an event \( E \subseteq X \) is then given by

\[
d(E) = \sum_{x \in E} d(x).
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Generalization: Semirings

This is easily generalized by replacing $\mathbb{R}_{\geq 0}$ by an arbitrary commutative semiring, which is an algebraic structure $(\mathbb{R}, +, 0, \cdot, 1)$, where $(\mathbb{R}, +, 0)$ and $(\mathbb{R}, \cdot, 1)$ are commutative monoids satisfying the distributive law:

$$a \cdot (b + c) = a \cdot b + a \cdot c.$$ 

Examples include the non-negative reals $\mathbb{R}_{\geq 0}$ with the usual addition and multiplication, and the booleans $\mathbb{B} = \{0, 1\}$ with disjunction and conjunction playing the roles of addition and multiplication respectively.

There is also the example of signed measures, giving by taking the reals $\mathbb{R}$.

We can now define a functor $D_R$ of $\mathbb{R}$-distributions, parameterized by a commutative semiring $R$. Given a set $X$, $D_R(X)$ is the set of $\mathbb{R}$-distributions of finite support. The functorial action is defined exactly as for the probabilistic case, as the push-forward of a measure. If $f: X \rightarrow Y$, $D_R(f): D_R(X) \rightarrow D_R(Y)$:

$$D_R(f)(d)(U) = d(f^{-1}(U)).$$
Generalization: Semirings

This is easily generalized by replacing $\mathbb{R}_{\geq 0}$ by an arbitrary \textit{commutative semiring}, which is an algebraic structure $(R, +, 0, \cdot, 1)$, where $(R, +, 0)$ and $(R, \cdot, 1)$ are commutative monoids satisfying the distributive law:

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The Possibilistic Collapse

In the boolean case, B-distributions on $X$ correspond to non-empty finite subsets of $X$. In this boolean case, we have a notion of possibilistic contextuality, where we have replaced probabilities by boolean values, corresponding to possible or impossible.

Note that there is a homomorphism of semirings from $\mathbb{R}_{\geq 0}$ to B, which sends positive probabilities to 1 (possible), and 0 to 0 (impossible). This lifts to a map on distributions, which sends a probability distribution to its support. This in turn sends probabilistic empirical models $\{d_C \}_{C \in M}$ to possibilistic empirical models.

We refer to this induced map as the possibilistic collapse.
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We refer to this induced map as the **possibilistic collapse**.
Given a measurement scenario \((X, M, O)\), and a semiring \(R\), we have the notion of a compatible family of \(R\)-distributions \(\{e_C\}_{C \in M}\), where \(e_C \in \mathcal{D}_R(O_C)\).

We will write \(\text{EM}(\Sigma, R)\) for the set of empirical models over the scenario \(\Sigma\) and the semiring \(R\).

We refer to probabilistic empirical models for \(R = \mathbb{R}_{\geq 0}\), and possibilistic empirical models for \(R = \mathbb{B}\).

All the notions relating to contextuality, global sections etc. work in the same way as before, across this broader variety of situations.
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Generalizing Empirical Models

Given a measurement scenario \((X, \mathcal{M}, O)\), and a semiring \(R\), we have the notion of a compatible family of \(R\)-distributions \(\{e_C\}_{C \in \mathcal{M}}\), where \(e_C \in \mathcal{D}_R(O^C)\).

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All the notions relating to contextuality, global sections etc. work in the same way as before, across this broader variety of situations.
The contextuality hierarchy revisited

A homomorphism of semirings $h: R \rightarrow S$ induces a natural transformation $\bar{h}$ from the presheaf of $R$-valued distributions to the presheaf of $S$-valued distributions. In particular, if $h: R_{\geq 0} \rightarrow B$ is the unique semiring homomorphism from the positive reals to the booleans, then $\bar{h}$ is the possibilistic collapse.

Given a global section $d_{\text{g}}$ for an empirical model $e \in \text{EM}(\Sigma, R)$, it is easy to see that $\bar{h}(d_{\text{g}})$ is a global section for $\bar{h}(e)$. Thus we have the following result.

Proposition If $\bar{h}(e)$ is contextual, then so is $e$. In particular, if the possibilistic collapse of a probabilistic empirical model $e$ is contextual, then $e$ is contextual. The converse is not true.
The contextuality hierarchy revisited

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The contextuality hierarchy revisited

A homomorphism of semirings \( h : R \rightarrow S \) induces a natural transformation \( \tilde{h} \) from the presheaf of \( R \)-valued distributions to the presheaf of \( S \)-valued distributions.

In particular, if \( h : \mathbb{R}_{\geq 0} \rightarrow B \) is the unique semiring homomorphism from the positive reals to the booleans, then \( \tilde{h} \) is the possibilistic collapse.

Given a global section \( d_g \) for an empirical model \( e \in EM(\Sigma, R) \), it is easy to see that \( \tilde{h}(d_g) \) is a global section for \( \tilde{h}(e) \). Thus we have the following result.

**Proposition**

If \( \tilde{h}(e) \) is contextual, then so is \( e \). In particular, if the possibilistic collapse of a probabilistic empirical model \( e \) is contextual, then \( e \) is contextual. The converse is not true.
In this generality, these notions can be seen to arise in a wide range of situations in classical computer science. In particular, as we shall now see, there is an isomorphism between the formal description we have given for the quantum notions of non-locality and contextuality, and basic definitions and concepts in relational database theory.

Samson Abramsky, 'Relational databases and Bell’s theorem', In Search of Elegance in the Theory and Practice of Computation: Essays Dedicated to Peter Buneman, Springer 2013.
Relational databases

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<table>
<thead>
<tr>
<th>branch-name</th>
<th>account-no</th>
<th>customer-name</th>
<th>balance</th>
</tr>
</thead>
<tbody>
<tr>
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<td>Newton</td>
<td>£2,567.53</td>
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<tr>
<td>Hanover</td>
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<td>Leibniz</td>
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</table>
From possibility models to databases

Consider again the Hardy model:

\[(0, 0) (0, 1) (1, 0) (1, 1)\]

\[(a_1, b_1)\]

\[(a_2, b_2)\]

Change of perspective:

\[a_1, a_2, b_1, b_2\] attributes

0, 1 data values

joint outcomes of measurements tuples
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<table>
<thead>
<tr>
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</tr>
</thead>
<tbody>
<tr>
<td>((a_1, b_1))</td>
<td>1</td>
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<tr>
<td>((a_1, b_2))</td>
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<tbody>
<tr>
<td>(a₁, b₁)</td>
<td>1</td>
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</tr>
<tr>
<td>(a₁, b₂)</td>
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<tr>
<td>(a₂, b₁)</td>
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<td>1</td>
<td>1</td>
<td>1</td>
<td>0</td>
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</tbody>
</table>

Change of perspective:

- \(a_1, a_2, b_1, b_2\) attributes
- \(0, 1\) data values
- joint outcomes of measurements tuples
The Hardy model as a relational database

The four rows of the model turn into four relation tables:

<table>
<thead>
<tr>
<th>$a_1$</th>
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<tbody>
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<td>0</td>
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<th>$a_1$</th>
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<td>0</td>
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<td>0</td>
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</tbody>
</table>

What is the DB property corresponding to the presence of non-locality/contextuality in the Hardy table?
There is no universal relation: no table

whose projections onto

\{$a_i, b_i$\}, $i = 1, 2$, yield the above four tables.
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A dictionary

- Relational databases
- Measurement scenarios
- Attribute measurement
- Set of attributes defining a relation table
- Compatible set of measurements
- Database schema
- Measurement cover
- Tuple
- Local section (joint outcome)
- Relation/set of tuples
- Boolean distribution on joint outcomes
- Universal relation instance
- Global section/hidden variable model
- Acyclicity
- Vorob’ev condition

We can also consider probabilistic databases and other generalisations; cf. prove-
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The no-signalling polytope

- Fix a measurement scenario \( \langle X, O, M \rangle \).
- \( \mathcal{N} \): set of probabilistic empirical models.
The no-signalling polytope

- Fix a measurement scenario \( \langle X, O, M \rangle \).
- \( \mathcal{N} \): set of probabilistic empirical models.
- convex set: convex combination (done componentwise)

\[
(re + (1 - r)e')_C := re_C + (1 - r)e'_C
\]

gives another empirical model.
The no-signalling polytope

- Fix a measurement scenario $\langle X, O, M \rangle$.
- $\mathcal{N}$: set of probabilistic empirical models.
- Convex set: convex combination (done componentwise)
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  (re + (1 - r)e')_c := re_c + (1 - r)e'_c
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  gives another empirical model.
- Explicitly represent models as points in $\mathbb{R}^N$, with $N = \sum_{c \in M} |C|$. 
The no-signalling polytope

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- explicitly represent models as points in $\mathbb{R}^N$, with $N = \sum_{C \in \mathcal{M}} |C|$.
- $\mathcal{N}$ is a polytope: defined by a finite number of linear constraints.
The structure of the no-signalling polytope

- **NS**: set of probabilistic empirical models
- **F**: the face lattice of this polytope (vertices, edges, . . . )
- **S**: possibilistic models of the form poss(e) for some $e \in \text{NS}$
  - ordered contextwise by support

Then

$$\mathcal{F} \cong S_{\perp}$$
In fact, the result applies to a much wider class of polytopes.

**NS** is defined by constraints:

- Non-negativity;
- Linear equations: viz. normalisation and no-signalling.

In geometric terms: \( \text{NS} = \mathcal{H}_{\geq 0} \cap \text{Aff} (\text{NS}) \)

where \( \text{Aff} (\text{NS}) \) is the affine subspace generated by \( \text{NS} \),

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For any \( P \) is **standard form**, there is an order-isomorphism between:
- \( \mathcal{F}(P) \), the face lattice of \( P \).
- \( S(P) \), set of “supports” of points in \( P \), ordered by inclusion.
A $\mathcal{V}$-polytope is the convex hull $\text{Conv}(S)$ of a finite set of points $S \subseteq \mathbb{R}^n$. 
Polytopes

- A \textit{V-polytope} is the convex hull $\text{Conv}(S)$ of a finite set of points $S \subseteq \mathbb{R}^n$.
- An \textit{H-polytope} is a bounded intersection of a finite set of closed half-spaces in $\mathbb{R}^n$.

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Fundamental Theorem of Polytopes: the two notions coincide.
Face lattice

- \(a \cdot x \geq b\) is valid for \(P\) if it is satisfied by every \(x \in P\).
- A valid inequality defines a face \(F\) of \(P\):
  \[
  F := \{x \in P : a \cdot x = b\}.
  \]
- \(\mathcal{F}(P)\): the set of faces of \(P\); \(\mathcal{F}^+(P)\): the set of non-empty faces.
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- $\mathcal{F}(P)$ is partially ordered by set inclusion.
- It is a finite lattice.
- It is atomistic, coatomistic, and graded.
- Meets in $\mathcal{F}(P)$ are given by intersection of faces, joins defined indirectly.

Called the face lattice of $P$, aka the combinatorial type of $P$. 
Relative interior

Relative interior of a set $S$:

$$\text{relint}(S) = \{x \in S : \exists \epsilon > 0. \text{Aff}(S) \cap B_{\epsilon}(x) \subseteq S\}$$

For a convex set:

$$\text{relint}(S) = \{x \in S : \forall y \in S. \exists \epsilon > 0. (1 + \epsilon)x - \epsilon y \in S\}$$

Intuitively: a point $x$ is in the relative interior if the line segment $[y, x]$ from any point $y$ of $S$ in to $x$ can be extended beyond $x$ while remaining in $S$. 
Carrier face

Every polytope $P$ can be written as the disjoint union of the relative interiors of its non-empty faces:

$$P = \bigsqcup_{F \in \mathcal{F}^+(P)} \text{relint } F.$$

This means that for any polytope $P$ we can define a map

$$\text{carr} : P \to \mathcal{F}^+(P)$$

which assigns to each point $x$ of $P$ its carrier face — the unique face $F$ such that $x \in \text{relint } F$. 
Supports

Polytope $P$ in **standard form**: $P = \mathcal{H}_{\geq 0} \cap \text{Aff}(P)$. 
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  $$(\text{supp}x)_i = \begin{cases} 
  0, & x_i = 0 \\
  1, & x_i > 0 
  \end{cases}$$

$S(P) := \{\text{supp}x : x \in P\}$, ordered componentwise.

Join of $u, v$ is componentwise boolean disjunction:

$$(u \lor v)_i := u_i \lor v_i.$$
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- So \( S(P)_\bot \) is a finite lattice.
\[ \mathcal{F}^+(P) \quad \text{Carr} \quad S(P) \]

\[ \text{carr } x \subseteq \text{carr } y \iff \text{supp } x \leq \text{supp } y \]
\[ \mathcal{F}^+(P) \cong S(P) \]

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Clearly, $x^\sigma \cdot z \geq 0$ is valid for all $z \in P$, and defines a face

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- Choose $\epsilon$ such that $\epsilon z \leq x$.
- $v := (1 + \epsilon)x - \epsilon z \geq 0$.
- Hence, $v \in F_x$. 

Samson Abramsky (Department of Computer Science\Logic and Quantum InformationLecture V: Mere Possil
\[ \mathcal{F}^+(P) \quad P \quad S(P) \]

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$$P$$

\[\mathcal{F}^+(P) \cong S(P)\]

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Lattice iso: \(\mathcal{F}(P) \cong S(P)_{\perp}\)
Some consequences

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- The vertices of the no-signalling polytope are exactly the probability models with minimal support. Moreover, there is only one probability model for each such minimal support.
- Therefore, the extremal empirical models are exactly those models which are completely and uniquely determined by their supports.
- These vertices of the polytope can be written as the disjoint union of the non-contextual, deterministic models – the vertices of the polytope of classical models – and the strongly contextual models with minimal support.
But ... 

- Note the mention of support!
- We still start from probabilistic models and take their supports.

Can we characterise the combinatorial type of **NS** using **only** possibilistic notions?
Recall that empirical models are families of **consistent distributions**. These can be defined over any commutative semiring $R$. $\mathbb{R}_{\geq 0}$ gives probabilistic models. $B$ gives **possibilistic models**.
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Using the (unique) semiring homomorphism $\mathbb{R}_{\geq 0} \rightarrow B$, we have a map

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The support lattice \( S(\text{NS}_{\mathbb{R}_{\geq 0}}) \) is the image of this map.

Can we give an **intrinsic characterisation** of the image of the possibilistic collapse map, using only possibilistic notions?
\[ S(\text{NS}_{R \geq 0}) \neq \text{NS}_B \]

i.e. there exist possibilistic empirical models that are not the support of any (probabilistic) empirical model (Abramsky, 2012).

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That is, are minimal possibilistic models always realisable as supports?

Also, NO!
\[X = \{A, B, C, D\}\]
\[\mathcal{M} = \{\{A, B\}, \{A, C\}, \{A, D\}, \{B, C\}, \{B, D\}, \{C, D\}\}\]
\[O = \{0, 1, 2\}\]

Possible assignments:

- \(AB \mapsto 00, 10, 21\)
  
  \[
  \begin{array}{ccc}
  a & b & c \\
  \end{array}
  \]

- \(AC \mapsto 00, 11, 21\)
  
  \[
  \begin{array}{ccc}
  d & e & f \\
  \end{array}
  \]

- \(AD \mapsto 01, 10, 21\)
  
  \[
  \begin{array}{ccc}
  k & l & m \\
  \end{array}
  \]

- \(BC \mapsto 00, 11\)
  
  \[
  \begin{array}{ccc}
  g & h \\
  \end{array}
  \]

- \(BD \mapsto 00, 11\)
  
  \[
  \begin{array}{ccc}
  i & j \\
  \end{array}
  \]

- \(CD \mapsto 01, 10\)
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$$a = k, b = l, g = i, h = j, c = n, d = k, e = l, f = m$$
$$c = h, h = o, g = n, i = o, j = n, c = j, l = o, d = n.$$
All variables must be equated.

- Minimality: set any variable to zero, then all must be zero.
- Only remaining non-trivial equation is $a = a + a$.
- No non-zero, real solution!
A Bell-type example

\[ X_{\text{Bell}} = \{ A_1, B_1, C_1, D_1, A_2, B_2, C_2, D_2 \} \]
\[ M_{\text{Bell}} = \{ A_1, B_1, C_1, D_1 \} \times \{ A_2, B_2, C_2, D_2 \} \]
\[ O = \{ 0, 1, 2 \} \]

Possible sections:

- \( A_1 A_2 \mapsto 00, 11, 22 \)
- \( B_1 B_2, C_1 C_2, D_1 D_2 \mapsto 00, 11 \)
- \( A_1 B_2, A_2 B_1 \mapsto 00, 10, 21 \)
- \( A_1 C_2, A_2 C_1 \mapsto 00, 11, 21 \)
- \( A_1 D_2, A_2 D_1 \mapsto 01, 10, 21 \)
- \( B_1 C_2, B_2 C_1 \mapsto 00, 11 \)
- \( B_1 D_2, B_2 D_1 \mapsto 00, 11 \)
- \( C_1 D_2, C_2 D_1 \mapsto 01, 10 \)
A Bell-type example
Still an open question

Can we give an **intrinsic characterization** of the image of the possibilistic collapse map, using only possibilistic notions?
The Kochen-Specker Theorem

The Kochen-Specker theorem (1967) offers a state-independent proof of strong contextuality in QM.

Our previous arguments for quantum realizability of contextual models have hinged on using particular quantum states.

The Kochen-Specker argument rests on properties of certain families of measurements which hold for any quantum state.

A trade-off: Bell’s theorem has weaker conclusions, but also weaker assumptions.
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The Kochen-Specker Property

We fix the set of outcomes to be $\mathcal{O} = \{0, 1\}$. Thus measurement scenarios will be determined simply by hypergraphs $(X, U)$. Given $C \in U$, we say that $s \in \mathcal{O}_C$ satisfies the KS property if $s(x) = 1$ for exactly one $x \in C$.

The Kochen-Specker model over $(X, U)$ is defined by setting $d_C$, for each $C \in U$, to be the set of all $s \in \mathcal{O}_C$ which satisfy the KS property. Note that the model is uniquely determined once we have given $(X, U)$.

Note that, if we regard the elements of $X$ as propositional variables, we can think of $s \in \mathcal{O}_C$ as a truth-value assignment. Then the KS property for an assignment $s$ is equivalent to $s$ satisfying the following formula:

$$\text{ONE}(C) := \bigvee_{x \in C} (x \land \bigwedge_{x' \in C \setminus \{x\}} \neg x')$$
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Then the KS property for an assignment $s$ is equivalent to $s$ satisfying the following formula:

$$\text{ONE}(C) := \bigvee_{x \in C} (x \land \bigwedge_{x' \in C \setminus \{x\}} \neg x')$$
KS Constructions

A KS construction is a KS model \((X, U)\) which is strongly contextual. Explicitly, this is equivalent to saying that the formula \(\bigwedge C \in U \text{ONE}(C)\) is unsatisfiable.

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A Kochen-Specker construction

This uses a set $X$ of 18 variables, $\{A, \ldots, O\}$.

A measurement cover $U = \{U_1, \ldots, U_9\}$, where the columns $U_i$ are the sets $A A H H B I P P Q B E I K E Q R R C F C G M N D F M G J L N O J L O$. 

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For each \( x \in X \), we define \( U(x) := \{ C \in U : x \in C \} \).

Proposition (SA, A. Brandenburger)
If the Kochen-Specker model on \((X, U)\) is non-contextual, then every common divisor of \( \{|U(x)| : x \in X\} \) must divide \(|U|\).

Applying this to the above example, we note that the cover \( M \) has 9 elements, while each element of \( X \) appears in two members of \( M \). Thus the Kochen-Specker model on \((X, M)\) is contextual.
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Subsumed by our cohomology results.
Quantum Representations

What do these combinatorial questions have to do with quantum mechanics? A contextual Kochen-Specker model ($X, U$) gives rise to a quantum mechanical witness of contextuality whenever:

We can label $X$ with unit vectors in $\mathbb{R}^n$, for some fixed $n$, such that $U$ consists of those subsets $C$ of $X$ which form orthonormal bases of $\mathbb{R}^n$.

The point of our previous example is that we can label the 18 elements of $X$ with vectors in $\mathbb{R}^4$ such that the four-element subsets in $M$ are orthogonal. This yields one of the most economical known examples of a KS construction.

By contrast, the Specker triangle is not quantum realizable.
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Given such a family of vectors, we can construct observables corresponding to each compatible family where the outcomes encode the eigenvectors. This means that for any state, the result of measuring that state with this observable must always yield an outcome satisfying the KS property. Hence we get a state-independent proof of strong contextuality in QM.
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People

Adam Brandenburger, Lucien Hardy, Shane Mansfield, Rui Soares Barbosa, Ray Lal, Mehrnoosh Sadrzadeh, Phokion Kolaitis, Georg Gottlob, Carmen Constantin, Kohei Kishida, Linde Wester, Giovanni Caru
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The Penrose Tribar