Quantifying Contextuality

Samson Abramsky
Joint work with Shane Mansfield and Rui Soares Barbosa

Department of Computer Science, University of Oxford

September 1, 2016
Overview

- Unified, general framework for non-locality and contextuality
- Qualitative hierarchy of contextuality
- **Quantitative measure of contextuality**
Overview

- Unified, general framework for non-locality and contextuality
- Qualitative hierarchy of contextuality
- Quantitative *measure of contextuality*

Why?
Overview

- Unified, general framework for non-locality and contextuality
- Qualitative hierarchy of contextuality
- **Quantitative measure of contextuality**

Why?

- Compare degree of contextuality of empirical models
- …across different measurement scenarios
- Contextuality as a resource
Contextuality
Empirical Data (e.g. CHSH)

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<tr>
<th>((a, b))</th>
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\(o_A \in \{0, 1\}\)

\(o_B \in \{0, 1\}\)

\(m_A \in \{a, a'\}\)

\(m_B \in \{b, b'\}\)
Measurement Scenarios: CHSH

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A measurement scenario is a triple \( \langle X, M, O \rangle \) where:
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A *measurement scenario* is a triple \( \langle X, M, O \rangle \) where:

\( X \) a finite set of measurements — e.g.

\[ X = \{ a, a', b, b' \} \]
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A measurement scenario is a triple $\langle X, \mathcal{M}, O \rangle$ where:

- $X$ a finite set of measurements — e.g.
  $$X = \{a, a', b, b'\}$$

- $\mathcal{M}$ the (maximal) contexts — e.g.
  $$\mathcal{M} = \{\{a, b\}, \{a, b'\}, \{a', b\}, \{a', b'\}\}$$
Measurement Scenarios: CHSH

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$X$ a finite set of measurements — e.g.

$$X = \{a, a', b, b'\}$$

$M$ the (maximal) contexts — e.g.

$$M = \{\{a, b\}, \{a, b'\}, \{a', b\}, \{a', b'\}\}$$

$O$ a finite set — e.g.

$$O = \{0, 1\}$$
Measurement Scenarios: ‘Triangle’

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Measurements:

$$X = \{a, b, c\}$$

Contexts:

$$\mathcal{M} = \{\{a, b\}, \{b, c\}, \{c, a\}\}$$

Outcomes:

$$O = \{0, 1\}$$
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**Measurements:**

$$X = \{a, b, c\}$$

**Contexts:**

$$\mathcal{M} = \{\{a, b\}, \{b, c\}, \{c, a\}\}$$

**Outcomes:**

$$O = \{0, 1\}$$
Measurement Scenarios: 18-vector KS

- A set of 18 variables: \( X = \{A, \ldots, O\} \)

- A set of outcomes: \( O = \{0, 1\} \)

- A measurement cover: \( M = \{C_1, \ldots, C_9\} \)
  whose contexts \( C_i \) correspond to the columns in the following table:

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<thead>
<tr>
<th>( C_1 )</th>
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<th>( C_4 )</th>
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Empirical Models

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• Fix a measurement scenario $\langle X, M, O \rangle$
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Fix a measurement scenario \( \langle X, \mathcal{M}, O \rangle \)

Empirical model: family \( \{ e_C \}_{C \in \mathcal{M}} \) where each \( e_C \in \text{Prob}(O^C) \)
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- Fix a measurement scenario \(\langle X, \mathcal{M}, O \rangle\)

- **Empirical model**: family \(\{e_C\}_{C \in \mathcal{M}}\) where each \(e_C \in \text{Prob}(O^C)\)

- Distribution for each context:

  \[ e_{\{a,b\}} = \text{prob}(o_1, o_2 | a, b), \ldots, \quad e_{\{a',b'\}} = \text{prob}(o_1, o_2 | a', b') \]
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- Fix a measurement scenario ⟨X, M, O⟩

- Empirical model: family \{e_C\}_{C \in M} where each e_C \in \text{Prob}(O^C)

- Distribution for each context:
  \[ e_{\{a,b\}} = \text{prob}(o_1, o_2 | a, b), \ldots, e_{\{a’,b’\}} = \text{prob}(o_1, o_2 | a’, b’) \]

- ‘Local’ consistency:
  \[ \text{prob}(o_1 | a, b) = \text{prob}(o_1 | a, b’) = \text{prob}(o_1 | a), \text{etc.} \]
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- Fix a measurement scenario \( \langle X, \mathcal{M}, O \rangle \)

- **Empirical model**: family \( \{ e_C \}_{C \in \mathcal{M}} \) where each \( e_C \in \text{Prob}(O^C) \)

- Distribution for each context:
  
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Contextuality

Classical data should arise as a convex combination of global assignments:

\((a, a', b, b') \mapsto (0, 0, 0, 0), (a, a', b, b') \mapsto (0, 0, 0, 1), \ldots, (a, a', b, b') \mapsto (1, 1, 1, 1)\)

\[
\begin{array}{c|cccc}
(a, b) & (0, 0) & (0, 1) & (1, 0) & (1, 1) \\
\hline
(a, b) & \frac{1}{2} & 0 & 0 & \frac{1}{2} \\
(a, b') & \frac{3}{8} & \frac{1}{8} & \frac{1}{8} & \frac{3}{8} \\
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Contextuality

Classical data should arise as a convex combination of *global assignments*:

\[(a, a', b, b') \mapsto (0, 0, 0, 0), \quad (a, a', b, b') \mapsto (0, 0, 0, 1), \quad \ldots \quad (a, a', b, b') \mapsto (1, 1, 1, 1)\]

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**Contextuality**

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*Contextuality* is present if such a decomposition is *not* possible
**Contextuality**

Classical data should arise as a convex combination of *global assignments*:

$$(a, a', b, b') \mapsto (0, 0, 0, 0), \ (a, a', b, b') \mapsto (0, 0, 0, 1), \ldots, \ (a, a', b, b') \mapsto (1, 1, 1, 1)$$

<table>
<thead>
<tr>
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<tbody>
<tr>
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<tr>
<td>$(a, b')$</td>
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*Contextuality* is present if such a decomposition is *not* possible

(Contextuality rules out deterministic HVs; non-locality is a special case)
Strong Contextuality

Strong Contextuality:
\textbf{no} event can be extended to a global assignment.
Strong Contextuality

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E.g. K–S models, GHZ, the PR box:

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<tr>
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<tr>
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The Contextual Fraction

**Proposition**

*Every empirical model admits a convex decomposition*

\[ e = \lambda e^{NC} + (1 - \lambda) e^{SC} \]

*into a non-contextual and a strongly contextual model. The maximum value \( \lambda \) for such decompositions, which is attained, is the non-contextual fraction of \( e \), \( NC(e) \).*
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**Contextual fraction:**  \( \text{CF}(e) = 1 - \text{NC}(e) \)

- \( \text{CF}(e) \in [0, 1] \)
- \( e \) is *non-contextual* iff \( \text{CF}(e) = 0 \)
- \( e \) is *strongly contextual* iff \( \text{CF}(e) = 1 \)
Given a measurement scenario $\langle X, M, O \rangle$, the incidence matrix $M$ has rows indexed by $\langle C, s \rangle$, $C \in M$, $s \in O$. The columns of the matrix correspond to the deterministic NCHV models. Every NCHV model is equivalent to a mixture of deterministic models. A probability distribution on (i.e. mixture of) deterministic NCHV models is given by a column vector $c$; while an empirical model over the scenario can be flattened into a row vector $v \in \mathbb{R}^m$, e.g. $v = \{1/2, 0, 0, 1/2, 3/8, 1/8, 1/8, 3/8, 3/8, 1/8, 3/8, 3/8\}$.
Computing the Contextual Fraction

Given a measurement scenario \( \langle X, M, O \rangle \), the incidence matrix \( M \) has

- rows indexed by \( \langle C, s \rangle, \ C \in M, \ s \in O^C \)
- columns indexed by global assignments \( g \in O^X \)

\[
M[\langle C, s \rangle, g] := \begin{cases} 
1 & \text{if } g|_C = s \\
0 & \text{otherwise}
\end{cases}
\]
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$$v_e = \{1/2, 0, 0, 1/2, 3/8, 1/8, 1/8, 3/8, 3/8, 1/8, 3/8, 3/8, 1/8, 1/8, 3/8, 3/8, 1/8\}$$
Checking contextuality of $e$ corresponds to solving

Find $d \in \mathbb{R}^n$

such that $M d = v_e$

and $d \geq 0$
(Non-)Contextual Fraction via Linear Programming

Checking contextuality of $e$ corresponds to solving

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Computing the non-contextual fraction corresponds to solving the following linear program:

Find $c \in \mathbb{R}^n$

maximising $1 \cdot c$

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Bell Inequality Violations
Generalised Bell Inequalities

An **inequality** for a scenario \( \langle X, \mathcal{M}, O \rangle \) is given by:

- A set of coefficients \( \alpha = \{ \alpha(c, s) \}_{c \in \mathcal{M}, s \in O} \)
- A bound \( R \)

For a model \( e \),

\[
B_\alpha(e) = \sum_{c \in \mathcal{M}, s \in O} \alpha(c, s) e(c)
\]

Wlog we can take \( R \) non-negative (in fact, we can take \( R = 0 \)).

A Bell inequality is **tight** if it is saturated by some NC model.
Generalised Bell Inequalities

An **inequality** for a scenario $\langle X, \mathcal{M}, O \rangle$ is given by:

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For a model $e$,

$$B_\alpha(e) \leq R,$$

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$$B_\alpha(e) := \sum_{c \in \mathcal{M}, s \in \mathcal{E}(c)} \alpha_{(c,s)} e_c(s)$$
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- **Bell inequality** if it is satisfied by every NC model
- Bell inequality is **tight** if it is saturated by some NC model
Violation of a Bell inequality

- Bell inequality $\rightarrow$ a bound for $B_\alpha(e)$ amongst NC models
Violation of a Bell inequality

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- For general (no-signalling) models, $B_\alpha(e)$ is limited only by

$$\|\alpha\| := \sum_{C \in \mathcal{M}} \max \{ \alpha(C, s) \mid s \in O^C \}$$
Violation of a Bell inequality

- Bell inequality → a bound for $\mathcal{B}_\alpha(e)$ amongst NC models

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- The **normalised violation** of a Bell inequality $\langle \alpha, R \rangle$ by $e$ is

$$\frac{\max\{0, \mathcal{B}_\alpha(e) - R\}}{\|\alpha\| - R} \in [0,1]$$
Proposition

Let $e$ be an empirical model

- Normalised violation by $e$ of any Bell inequality is at most $\text{CF}(e)$
- There exists a Bell inequality for which this is attained
- This Bell inequality is tight at “the” non-contextual model $e^{NC}$

$$e = \text{NC}(e)e^{NC} + \text{CF}(e)e^{SC}$$
Quantifying Contextuality LP:

Find \( c \in \mathbb{R}^n \)
maximising \( 1 \cdot c \)
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\[ \alpha := 1 - |M| y \]
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Contextual Fraction (Recap)

Fully general: applicable to any measurement scenario

Normalised: allowing comparison across scenarios

0 for non-contextuality
1 for strong contextuality

Computable using linear programming

Precise relationship to violations of Bell inequalities

What else?

Computational tools (Mathematica package) implementing all this

Resource Theory: Monotonicity properties wrt operations that don't introduce contextuality
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Computational tools (*Mathematica* package) to:

1. Calculate quantum empirical models from any (pure or mixed) state and any sets of compatible measurements
2. Calculate the incidence matrix for any measurement scenario
3. Quantify the degree of contextuality of any empirical model using the LP method
4. Find the Bell inequality using the dual LP
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- two-qubit Bell state $|\phi^+\rangle = \frac{|\uparrow\uparrow\rangle + |\downarrow\downarrow\rangle}{\sqrt{2}}$
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- Equatorial measurements at angles $(\phi_1, \phi_2)$
1. Equatorial measurements on $|\phi^+\rangle$

- two-qubit Bell state $|\phi^+\rangle = \frac{|\uparrow\uparrow\rangle + |\downarrow\downarrow\rangle}{\sqrt{2}}$

- Equatorial measurements at angles $(\phi_1, \phi_2)$

- e.g. $(\phi_1, \phi_2) = (0, \pi/3)$ gives Bell–CHSH model

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1. Equatorial measurements on $|\phi^+\rangle$

Plot $CF(e)$ against measurement angles $(\phi_1, \phi_2)$
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Plot $CF(e)$ against measurement angles $(\phi_1, \phi_2)$

Maxima:

$$\{\phi_1, \phi_2\} \in \left\{ \left\{ \frac{\pi}{8}, \frac{5\pi}{8} \right\}, \left\{ \frac{7\pi}{8}, \frac{3\pi}{8} \right\} \right\}$$
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\[
p = \frac{\sqrt{2} + 2}{8}
\]
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Maxima:

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$$p = \frac{\sqrt{2} + 2}{8}$$

Note that these achieve Tsirelson violation of the CHSH inequality.
2. Equatorial measurements on GHZ\(^{(n)}\)

- \(n\)-partite GHZ states, given for \(n > 2\) by:

\[
|\psi_{\text{GHZ}(n)}\rangle = \frac{|\uparrow\rangle \otimes^n + |\downarrow\rangle \otimes^n}{\sqrt{2}}
\]
2. Equatorial measurements on GHZ\((n)\)

- \(n\)-partite GHZ states, given for \(n > 2\) by:
  \[
  \left| \psi_{\text{GHZ}(n)} \right> = \frac{|\uparrow\rangle \otimes^n + |\downarrow\rangle \otimes^n}{\sqrt{2}}
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- For \(n > 2\), Mermin considered Pauli \(X\) or \(Y\) measurements to provide logical proofs of non-locality
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- For \(n > 2\), Mermin considered Pauli \(X\) or \(Y\) measurements to provide logical proofs of non-locality.

- Again, equatorial measurements on the Bloch sphere.
2. Equatorial measurements on GHZ($n$)

Figure: CF(e) for equatorial measurements at $\phi_1$ and $\phi_2$ on each qubit of $\left| \psi_{\text{GHZ}(n)} \right\rangle$ with: (a) $n = 3$; (b) $n = 4$. 
2. Equatorial measurements on GHZ($n$)

- $n = 3$: minima of the plot reach 0 (strong contextuality) at

  \[ \{\phi_1, \phi_2\} \in \left\{ \left\{ \frac{\pi}{2}, 0 \right\}, \left\{ \frac{2\pi}{3}, \frac{\pi}{6} \right\}, \left\{ \frac{5\pi}{6}, \frac{\pi}{3} \right\} \right\} \]
2. Equatorial measurements on GHZ\( (n) \)

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\( (\phi_1, \phi_2) = (\pi/2, 0) \) corresponds to the Pauli Y and X, yielding the usual GHZ model. Other minima: alternative sets of measurements on the GHZ state that still lead to the familiar parity argument.
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- **\(n = 4\):** minima of 0 occur at

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\{\phi_1, \phi_2\} \in \left\{ \left\{ \frac{\pi}{2}, 0 \right\}, \left\{ \frac{5\pi}{8}, \frac{\pi}{8} \right\}, \left\{ \frac{3\pi}{4}, \frac{\pi}{4} \right\}, \left\{ \frac{7\pi}{8}, \frac{3\pi}{8} \right\} \right\}.
\]
2. Equatorial measurements on GHZ($n$)

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($\phi_1, \phi_2 = (\pi/2, 0)$) corresponds to the Pauli $Y$ and $X$, yielding the usual GHZ model. Other minima: alternative sets of measurements on the GHZ state that still lead to the familiar parity argument

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- General $n$: local equatorial measurements at

$$\left(\phi_1, \phi_2\right) \in \left\{ \left\{ \frac{(n + k)\pi}{2n}, \frac{k\pi}{2n} \right\} \mid 0 \leq k < n \right\}$$

on GHZ($n$) state give rise to strong contextuality
Towards a Resource Theory of Contextuality
May be more than one useful measure of contextuality

What properties should a good measure satisfy?
May be more than one useful measure of contextuality

What properties should a good measure satisfy?

Monotone wrt operations that do not introduce contextuality
Contextuality as a Resource

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- Monotone wrt operations that do not introduce contextuality
- Towards a resource theory, as for entanglement (e.g. LOCC), non-locality, ...
Contextuality as a Resource

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- Towards a resource theory, as for entanglement (e.g. LOCC), non-locality, . . .

- Algebra of empirical models, towards a process calculus?
Operations

- relabelling
  \[ e : \langle X, M, O \rangle, \alpha : (X, M) \cong (X', M') \leadsto e[\alpha] : \langle X', M', O \rangle \]

  For \( C \in M, s : \alpha(C) \rightarrow O \),
  \[ e[\alpha]_{\alpha(C)}(s) := e_C(s \circ \alpha^{-1}) \]
Operations

- **relabelling**
  \[ e : \langle X, \mathcal{M}, O \rangle, \ \alpha : (X, \mathcal{M}) \cong (X', \mathcal{M}') \leadsto e[\alpha] : \langle X', \mathcal{M}', O \rangle \]
  
  For \( C \in \mathcal{M}, s : \alpha(C) \rightarrow O, e[\alpha]_{\alpha(C)}(s) := e_C(s \circ \alpha^{-1}) \)

- **restriction**
  \[ e : \langle X, \mathcal{M}, O \rangle, (X', \mathcal{M}') \leq (X, \mathcal{M}) \leadsto e \upharpoonright \mathcal{M}' : \langle X', \mathcal{M}', O \rangle \]
  
  For \( C' \in \mathcal{M}', s : C' \rightarrow O, (e \upharpoonright \mathcal{M}')_{C'}(s) := e_C|_{C'}(s) \)
  
  with any \( C \in \mathcal{M} \) s.t. \( C' \subseteq C \)
Operations

- **relabelling**
  \[ e : \langle X, M, O \rangle, \alpha : (X, M) \cong (X', M') \rightsquigarrow e[\alpha] : \langle X', M', O \rangle \]
  
  For \( C \in M, s : \alpha(C) \rightarrow O \), \( e[\alpha]_{\alpha(C)}(s) := e_C(s \circ \alpha^{-1}) \)

- **restriction**
  \[ e : \langle X, M, O \rangle, (X', M') \leq (X, M) \rightsquigarrow e \upharpoonright M' : \langle X', M', O \rangle \]
  
  For \( C' \in M', s : C' \rightarrow O \), \( (e \upharpoonright M')_{C'}(s) := e_C|_{C'}(s) \)
  
  with any \( C \in M \) s.t. \( C' \subseteq C \)

- **coarse-graining**
  \[ e : \langle X, M, O \rangle, f : O \rightarrow O' \rightsquigarrow e/f : \langle X, M, O' \rangle \]
  
  For \( C \in M, s : C \rightarrow O' \), \( (e/f)_C(s) := \sum_{t : C \rightarrow O, f \circ t = s} e_C(t) \)
Operations

- **mixing**
  
  \[ e : \langle X, \mathcal{M}, O \rangle, \quad e' : \langle X, \mathcal{M}, O \rangle, \quad \lambda \in [0, 1] \leadsto e + \lambda e' : \langle X, \mathcal{M}, O \rangle \]

  For \( C \in \mathcal{M}, s : C \rightarrow O' \),
  
  \[ (e + \lambda e')_C(s) := \lambda e_C(s) + (1 - \lambda)e'_C(s) \]
Operations

- **mixing**
  \[ e : \langle X, M, O \rangle, \quad e' : \langle X, M, O \rangle, \quad \lambda \in [0, 1] \leadsto e + \lambda e' : \langle X, M, O \rangle \]

  \[
  \text{For } C \in M, s : C \rightarrow O', \\
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  \]

- **choice**
  \[ e : \langle X, M, O \rangle, \quad e' : \langle X', M', O \rangle \leadsto e \& e' : \langle X \sqcup X', M \sqcup M', O \rangle \]

  \[
  \text{For } C \in M, (e \& e')_C := e_C \\
  \text{For } D \in M', (e \& e')_D := e'_D
  \]
Operations

- **mixing**
  \[ e : \langle X, M, O \rangle, \ e' : \langle X, M, O \rangle, \ \lambda \in [0, 1] \rightsquigarrow e + \lambda \ e' : \langle X, M, O \rangle \]

  For \( C \in M, s : C \longrightarrow O' \),
  \[(e + \lambda \ e')_C(s) := \lambda e_C(s) + (1 - \lambda)e'_C(s)\]

- **choice**
  \[ e : \langle X, M, O \rangle, \ e' : \langle X', M', O \rangle \rightsquigarrow e \& e' : \langle X \sqcup X', M \sqcup M', O \rangle \]

  For \( C \in M, (e \& e')_C := e_C \)
  For \( D \in M', (e \& e')_D := e'_D \)

- **tensor**
  \[ e : \langle X, M, O \rangle, \ e' : \langle X', M', O \rangle \rightsquigarrow e \otimes e' : \langle X \sqcup X', M \ast M', O \rangle \]

  \[ M \ast M' := \{ C \sqcup D \mid C \in M, D \in M' \} \]
  For \( C \in M, D \in M', s = \langle s_1, s_2 \rangle : C \sqcup D \longrightarrow O \),
  \[(e \otimes e')_{C \sqcup D} \langle s_1, s_2 \rangle := e_C(s_1)e'_D(s_2)\]
Operations and the Contextual Fraction

relabelling

\[ CF(e^{[\alpha]}) = CF(e) \]

restriction

\[ CF(e|\sigma') \leq CF(e) \]

coarse-graining

\[ CF(e/f) \leq CF(e) \]

mixing

\[ CF(e + \lambda e') \leq \lambda CF(e) + (1 - \lambda) CF(e') \]

choice

\[ CF(e \& e') = \max \{ CF(e), CF(e') \} \]

\[ NCF(e \& e') = \min \{ NCF(e), NCF(e') \} \]

tensor

\[ CF(e_1 \otimes e_2) = CF(e_1) + CF(e_2) - CF(e_1)CF(e_2) \]

\[ NCF(e_1 \otimes e_2) = NCF(e_1)NCF(e_2) \]
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  \[ \text{CF}(e/f) \leq \text{CF}(e) \]

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- **NCF**
  \[ \text{NCF}(e \& e') = \min \{ \text{NCF}(e), \text{NCF}(e') \} \]
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- **tensor** (**)
  \[ \text{CF}(e_1 \otimes e_2) = \]
  \[ \text{CF}(e_1) + \text{CF}(e_2) - \text{CF}(e_1)\text{CF}(e_2) \]
  \[ \text{NCF}(e_1 \otimes e_2') = \text{NCF}(e_1)\text{NCF}(e_2) \]
Quantifying Quantum Advantage

We want to use the contextual fraction to quantify advantage in various information-processing tasks. The general form for such results: The greater the violation of the classical bound we want, the more contextuality there has to be. We shall look at one such result in terms of games. The class of games we will consider are a (vast) generalization of XOR games (but can be generalized much further). They subsume what are sometimes called "pseudo-telepathy games".
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Games on Measurement Scenarios

Given a measurement scenario \((X, M, O)\), a game is specified by winning conditions \(W \subseteq O\) for each context \(C \in M\).

An empirical model \(e = \{e_C\}\) can be viewed as a strategy for this game. Given a context \(C\), chosen by Nature uniformly at random, it chooses an outcome according to the distribution \(e_C\).

The success probability of \(e\) is given by

\[
\frac{1}{|M|} \sum_{C \in M} e_C(W_C)
\]

The classical bound for the game is the maximum success probability for any non-contextual strategy.
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Logical Bell inequalities give the classical bound

Say that a game \( \{ W_C \} \) is \textit{\( K \)-consistent} if the maximum cardinality of a consistent sub-family of \( \{ W_C \} \) is \( K \).
Logical Bell inequalities give the classical bound

Say that a game \( \{ W_C \} \) is *K-consistent* if the maximum cardinality of a consistent sub-family of \( \{ W_C \} \) is \( K \).

A sub-family \( \{ W_{C_i} \} \) is consistent if there is an assignment \( \nu : \bigcup_i C_i \rightarrow O \) such that \( \nu|_{C_i} \in W_{C_i} \) for all \( i \).
Logical Bell inequalities give the classical bound

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\subsection*{Theorem}

The classical bound for a \( K \)-consistent game is \( \frac{1}{|M|} K \).
Logical Bell inequalities give the classical bound

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**Theorem**

The classical bound for a \( K\)-consistent game is \( \frac{1}{|M|} K \).

A suitable measure of the non-classicality (or “hardness”) of a \( K\)-consistent game \( G \) is \( \mu_G := |M| - K \).
Relating the contextual fraction to hardness of a task
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**Theorem**

Consider a game $G$, and a strategy (empirical model) $e$, with success probability $p_S(e)$, and failure probability $p_F(e) := 1 - p_S(e)$. Then we have

$$\frac{\mu_G - p_F(e)}{\mu_G} \leq CF(e)$$
Relating the contextual fraction to hardness of a task

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Consider a game \( G \), and a strategy (empirical model) \( e \), with success probability \( p_S(e) \), and failure probability \( p_F(e) := 1 - p_S(e) \). Then we have

\[
\frac{\mu_G - p_F(e)}{\mu_G} \leq CF(e)
\]

This says that for any game with a given level of difficulty \( \mu_G \), the higher we want the success probability for a strategy \( e \) to be, the more contextual \( e \) has to be.
An analogous result for quantum computation

A similar result can be proved for the measurement-based quantum computation paradigm, refining a result by Robert Raussendorf:

**Theorem**

Given a boolean function $f$ with a level of difficulty $\nu_f$ measured by how far it is from being mod 2 linear, then

$$\nu_f - p_{\text{F}}(e) \leq C\nu_f$$

Here $p_{\text{F}}(e)$ refers to the failure probability for $e$, viewed as a generalized MBQC, to compute $f$.

These results are early steps towards developing a quantitative theory of contextuality as a resource for exceeding classical bounds on information processing tasks.
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These results are early steps towards developing a quantitative theory of contextuality as a resource for exceeding classical bounds on information processing tasks.
Contextuality in the presence of signalling

Real experimental data (e.g. recent "loophole-free Bell tests" at Delft, NIST etc.) will typically have signalling effects which need to be filtered out. Also, non-quantum applications may well feature signalling. Given a possibly signalling empirical model \( e \) (i.e. we are not assuming compatibility), we can consider maximal convex decompositions

\[
e = \lambda e_{NS} + (1 - \lambda) e_{SS},
\]

where \( e_{NS} \) is no-signalling, and \( e_{SS} \) is "strongly signalling", i.e. with no no-signalling fraction. We write \( NS(e) \) for the maximum value of \( \lambda \), which is attained. Note that \( NS(e) = 1 \) if and only if \( e \) is no-signalling.
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Computing the No-Signalling Fraction

This can be computed by the following linear program:

Find \( w \in \mathbb{R}^n \) maximising \( \frac{1}{|M|} \cdot w \)

subject to \( Nw = 0 \) and \( w \leq v \) and \( w \geq 0 \).

(1)

Here \( N \) is the No-Signalling matrix.
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and \( w \geq 0 \) \hspace{1cm} (1)
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and \( w \geq 0 \).

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This leads us to a refined version of the contextual fraction, which takes possible signalling in the empirical data into account.

\[ \text{CF}(e) = \text{NS}(e) - \text{NC}(e) \]

Note that this agrees with our previous definition of the contextual fraction in the no-signalling case. This measure expresses how contextual \( e \) is as how no-signalling it is minus how non-contextual it is.
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\textit{how no-signalling} it is minus \textit{how non-contextual} it is.
The Tricolour

This leads us to a refined version of the contextual fraction, which takes possible signalling in the empirical data into account.

$$\text{CF}(e) = \text{NS}(e) - \text{NC}(e).$$

Note that this agrees with our previous definition of the contextual fraction in the no-signalling case.

This measure expresses how contextual $e$ is as how no-signalling it is minus how non-contextual it is.
Real Experimental Data
Real Experimental Data

Recent loophole free Bell tests (Delft, NIST and Vienna)
Real Experimental Data

Recent loophole free Bell tests (Delft, NIST and Vienna)

- **Delft:**

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- Local data: distributions $p(o_1, o_2|a, b)$, $p(o_1, o_2|a', b')$, etc.
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**NO-SIGNALLING**

- Experimental data does not perfectly satisfy no-signalling...
Quantifying Signalling

$e$ is no-signalling iff

$$N v_e = 0$$

where

$$N[i,j] := \begin{cases} 
1 & \text{if } s_j \in O^{C_i} \text{ and } s_j|_{C_i'} = t_i \\
-1 & \text{if } s_j \in O^{C_i'} \text{ and } s_j|_{C_i} = t_i \\
0 & \text{otherwise}
\end{cases}$$

- $(\langle t, C, C' \rangle_i)$ an enumeration of $\{ \langle t, C, C' \rangle | t \in O^{C \cap C'} \text{ and } (C, C') \in \mathcal{M}^2 \}$
- $(s_j)$ an enumeration of $\{ s | t \in O^C \text{ and } C \in \mathcal{M}^2 \}$
e is no-signalling iff

\[ Nv_e = 0 \]

Otherwise we can obtain the no-signalling fraction with the LP

\[
\begin{align*}
\text{maximise} & \quad 1 \cdot z \\
\text{subject to} & \quad Nz = 0 \\
\text{and} & \quad z \leq v_e \\
\text{and} & \quad z \geq 0
\end{align*}
\]
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Setting $\mu = 1\cdot z^*$

$$e = \mu e_{NS} + (1 - \mu)e_{SS}$$
maximise \( 1 \cdot z \)
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and \( z \leq v_e \)
and \( z \geq 0 \)

Setting \( \mu = 1 \cdot z^* \)

\( e = \mu e_{NS} + (1 - \mu) e_{SS} \)

maximise \( 1 \cdot x \)
subject to \( Mx \leq v_{eNS} \)
and \( x \geq 0 \)

Setting \( \lambda = 1 \cdot x^* \)

\( e = \mu \lambda e_{NC} + \mu (1 - \lambda) e_{SC} + (1 - \mu) e_{SS} \)
Analysis of Real Data (Delft)

Decomposition of data:

\[ e_{\text{Delft}} \approx 0.0664 e_{SS} + 0.4073 e_{SC} + 0.5263 e_{NC} \]
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Ratio of signalling to genuine contextuality:

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(Different data?)
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Negative Probabilities

The basis for a measure based on negative probabilities is the following result.

Theorem (Abramsky and Brandenburger 2011)

If $e$ is any compatible empirical model, there is a signed measure $d : X \rightarrow \mathbb{R}$ with

$$\sum_{x \in X} d(x) = 1$$

such that $d|_C = e|_C$ for all $C \in \mathcal{M}$.

Thus if we used signed measures ("negative probabilities") we can find a global section for any compatible empirical model.

We now define a measure of how far it is necessary to deviate from a standard probability distribution to get a global section.

$$\text{NP}(e) := \min \{ (\|d\| - 1)/2 \mid d \text{ is a signed global section for } e \}$$

Here $\|d\| = \sum_{x \in X} |d(x)|$, the $\ell_1$-norm. We take $d^+ - d^-$, where $d = d^+ - d^-$. Clearly if $e$ is non-contextual, $\text{NP}(e) = 0$.

Question: How does NP relate to CF?
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NP(e) := \min_{d} \left\{ \left( \| d \| - 1 \right) / 2 \right\}
\]

Here \( \| d \| = \sum_{x \in X} |d(x)| \), the \( \ell_{1} \)-norm. We take \( d^{+} - d^{-} \), where \( d = d^{+} - d^{-} \).

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