Logic and Quantum Information
Lecture II: The Topology of Paradox

Samson Abramsky

Department of Computer Science
The University of Oxford
What Do ‘Observables’ Observe?

Surely objective properties of a physical system, which are independent of our choice of which measurements to perform — of our measurement context.

More precisely, this would say that for each possible state of the system, there is a function $\lambda$ which for each measurement $m$ specifies an outcome $\lambda(m)$, independently of which other measurements may be performed.

This point of view is called non-contextuality. It is equivalent to the assumption of a classical source. However, this view is impossible to sustain in the light of our actual observations of (micro)-physical reality.
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However, this view is **impossible to sustain** in the light of our **actual observations of (micro)-physical reality**.
Hidden Variables: The Mermin instruction set picture

\[ a \mapsto 0, \quad b \mapsto 1 \]

Alice

\[ a, a', \ldots \]

Bob

\[ b, b', \ldots \]

Source

\[ 0110 \]

Target

\[ 0110 \]

\[ aa' bb' \]

...
The ‘Hardy Paradox’: Bell’s theorem without inequalities
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Hardy models: those whose support satisfies

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Which ‘instruction set’ λ could the outcomes (0, 0) for measurements (a₁, b₁) could come? Clearly, we must have

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However, this would require the outcome \((0, 0)\) for measurements \((a₂, b₁)\) to be possible, and this is **precluded**.
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Thus Hardy models are **contextual**. They cannot be explained by a classical source.
Bundle Pictures

Logical Contextuality

- Ignore precise probabilities
- Events are possible or not
- E.g. the Hardy model:

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Bundle Pictures

Logical Contextuality

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- E.g. the Hardy model:

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The PR Box

The PR Box achieves the algebraic maximum of 4 for our logical Bell inequality. In terms of the XOR game, it is a winning strategy.
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Visualizing Contextuality

The Hardy table and the PR box as bundles
Visualizing Contextuality

The Hardy table and the PR box as bundles

A hierarchy of degrees of contextuality:

Bell  <  Hardy  <  GHZ
Contextuality, Logic and Paradoxes

A Liar cycle of length $N$ is a sequence of statements $S_1$: $S_2$ is true, $S_2$: $S_3$ is true, ..., $S_{N-1}$: $S_N$ is true, $S_N$: $S_1$ is false.

For $N = 1$, this is the classic Liar sentence $S$: $S$ is false.

Following Cook, Walicki et al. we can model the situation by boolean equations:

$x_1 = x_2, \ldots, x_{n-1} = x_n, x_n = \neg x_1$

The "paradoxical" nature of the original statements is now captured by the inconsistency of these equations.
Contextuality, Logic and Paradoxes

Liar cycles. A Liar cycle of length $N$ is a sequence of statements

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The “paradoxical” nature of the original statements is now captured by the inconsistency of these equations.
We can regard each of these equations as fibered over the set of variables which occur in it:
\[
\begin{align*}
\{x_1, x_2\} &: x_1 = x_2 \\
\{x_2, x_3\} &: x_2 = x_3 \\
&\quad \vdots \\
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Any subset of up to \(n - 1\) of these equations is consistent; while the whole set is inconsistent.

Up to rearrangement, the Liar cycle of length 4 corresponds exactly to the PR box.

The usual reasoning to derive a contradiction from the Liar cycle corresponds precisely to the attempt to find a univocal path in the bundle diagram.
Contextuality in the Liar; Liar cycles in the PR Box

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Paths to contradiction

Suppose that we try to set $a_2$ to 1. Following the path on the right leads to the following local propagation of values:

- $a_2 = 1$
- $b_1 = 1$
- $a_1 = 1$
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- $a_2 = 0$
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We have discussed a specific case here, but the analysis can be generalised to a large class of examples.
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The Robinson Consistency Theorem

A classic result:

Theorem (Robinson Joint Consistency Theorem)

Let $T_i$ be a theory over the language $L_i$, $i = 1, 2$. If there is no sentence $\varphi$ in $L_1 \cap L_2$ with $T_1 \vdash \varphi$ and $T_2 \vdash \neg \varphi$, then $T_1 \cup T_2$ is consistent.

Thus this theorem says that two compatible theories can be glued together. In this binary case, local consistency implies global consistency.

Note, however, that an extension of the theorem beyond the binary case fails. That is, if we have three theories which are pairwise compatible, it need not be the case that they can be glued together consistently.

A minimal counter-example is provided at the propositional level by the following "triangle":

$T_1 = \{ x_1 \leftrightarrow \neg x_2 \}$,

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This example is well-known in the quantum contextuality literature as the Specker triangle.
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A measurement scenario is a triple \((X, \mathcal{M}, O)\) where:

- \(X\) is a set of variables which can be measured, observed or evaluated.
- \(\mathcal{M}\) is a family of sets of variables, those which can be measured together. These form the contexts.
- \(O\) is a set of possible outcomes or values for the variables.

Example: In our tables, the set of variables is \(X = \{a, a', b, b'\}\). The measurement contexts are:

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The outcomes are \(O = \{0, 1\}\). A joint outcome or event in a context \(C\) is \(s \in O^C\), e.g. \(s = \{a \mapsto 0, b \mapsto 1\}\).
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Formalizing Contextuality: Measurement Scenarios

A measurement scenario is a triple \((X, \mathcal{M}, O)\) where:

- \(X\) is a set of variables which can be measured, observed or evaluated

- \(\mathcal{M}\) is a family of sets of variables, those which can be measured together. These form the contexts.

- \(O\) is a set of possible outcomes or values for the variables.

Example:
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A Kochen-Specker construction

This uses a set $X$ of 18 variables, $\{A, \ldots, O\}$, and a measurement cover $U = \{U_1, \ldots, U_9\}$, where the columns $U_i$ are the sets $U_1 \cup U_2 \cup U_3 \cup U_4 \cup U_5 \cup U_6 \cup U_7 \cup U_8 \cup U_9$. The original K-S construction used 117 variables!
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<table>
<thead>
<tr>
<th>$U_1$</th>
<th>$U_2$</th>
<th>$U_3$</th>
<th>$U_4$</th>
<th>$U_5$</th>
<th>$U_6$</th>
<th>$U_7$</th>
<th>$U_8$</th>
<th>$U_9$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$A$</td>
<td>$A$</td>
<td>$H$</td>
<td>$H$</td>
<td>$B$</td>
<td>$I$</td>
<td>$P$</td>
<td>$P$</td>
<td>$Q$</td>
</tr>
<tr>
<td>$B$</td>
<td>$E$</td>
<td>$I$</td>
<td>$K$</td>
<td>$E$</td>
<td>$K$</td>
<td>$Q$</td>
<td>$R$</td>
<td>$R$</td>
</tr>
<tr>
<td>$C$</td>
<td>$F$</td>
<td>$C$</td>
<td>$G$</td>
<td>$M$</td>
<td>$N$</td>
<td>$D$</td>
<td>$F$</td>
<td>$M$</td>
</tr>
<tr>
<td>$D$</td>
<td>$G$</td>
<td>$J$</td>
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<td>$N$</td>
<td>$O$</td>
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<td>$L$</td>
<td>$O$</td>
<td></td>
</tr>
</tbody>
</table>

The original K-S construction used 117 variables!
Empirical Models

Let \((X, M, O)\) be a measurement scenario. An empirical model for this scenario is a family \(\{d_C \mid C \in M\}\) where \(d_C \in \text{Prob}(O^C)\) for \(C \in M\). In other words, the empirical model specifies a probability distribution over the events in each context.
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Let \((X, \mathcal{M}, O)\) be a measurement scenario. An **empirical model** for this scenario is a family

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These distributions are the rows of our probability tables.
The measurement contexts are \( \{a, b\} \), \( \{a', b\} \), \( \{a, b'\} \), \( \{a', b'\} \). Each measurement has possible outcomes 0 or 1. The matrix entry at row \((a', b)\) and column \((0, 1)\) indicates the event \( a' \mapsto 0, b \mapsto 1 \). Each row of the table specifies a probability distribution on events \( O \) for a given choice of measurements \( C \).
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Mathematical Structure of Probability Tables

<table>
<thead>
<tr>
<th>A</th>
<th>B</th>
<th>(0,0)</th>
<th>(1,0)</th>
<th>(0,1)</th>
<th>(1,1)</th>
</tr>
</thead>
<tbody>
<tr>
<td>(a)</td>
<td>(b)</td>
<td>0</td>
<td>1/2</td>
<td>1/2</td>
<td>0</td>
</tr>
<tr>
<td>(a')</td>
<td>(b)</td>
<td>3/8</td>
<td>1/8</td>
<td>1/8</td>
<td>3/8</td>
</tr>
<tr>
<td>(a)</td>
<td>(b')</td>
<td>3/8</td>
<td>1/8</td>
<td>1/8</td>
<td>3/8</td>
</tr>
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<td>(a')</td>
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<td>3/8</td>
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<table>
<thead>
<tr>
<th>A</th>
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<th>(0, 0)</th>
<th>(1, 0)</th>
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</tr>
</thead>
<tbody>
<tr>
<td>a</td>
<td>b</td>
<td>0</td>
<td>1/2</td>
<td>1/2</td>
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</tr>
<tr>
<td>a'</td>
<td>b</td>
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<td>1/8</td>
<td>1/8</td>
<td>3/8</td>
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<td>a'</td>
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</tr>
</tbody>
</table>

The **measurement contexts** are

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Each measurement has possible outcomes 0 or 1. The matrix entry at row \((a', b)\) and column \((0, 1)\) indicates the **event**

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<table>
<thead>
<tr>
<th></th>
<th>B</th>
<th>(0, 0)</th>
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<th>(0, 1)</th>
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</tr>
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<tbody>
<tr>
<td>a</td>
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<td>0</td>
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The measurement contexts are

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Gluing functional sections

If $s(U | U \cap V) = s(V | U \cap V)$, they can be glued to form $U \cup V \rightarrow O$ such that $s(U | U) = s(U)$ and $s(V | V) = s(V)$.
Gluing functional sections

If $s_U|_{U \cap V} = s_V|_{U \cap V}$, they can be glued to form

$$s : U \cup V \rightarrow O$$

such that $s|_U = s_U$ and $s|_V = s_V$. 
The need for restriction

We would like to express the condition that an empirical model is compatible, i.e. “locally consistent.” We want to do this by saying that the distributions “agree on overlaps.” For all $C, C' \in \mathcal{M}$:

$$d_{C}_{|C \cap C'} = d_{C'}_{|C \cap C'}.$$

Cf. the usual notion of compatibility of a family of functions defined on subsets.

A formula for restriction of distributions: if $C' \subseteq C$, $d \in \text{Prob}(O_C)$, $d_{|C'}(s) := \sum_{t \in O_{C'}, t|C = s} d(t)$.

This is just marginalization: if $C = C' \sqcup C''$, then $O_C = O_{C'} \times O_{C''}$.

So compatibility says that the distributions on different contexts have consistent marginals.
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Compatibility and No-Signalling

There is an important physical principle of **No-Signalling**:

Suppose that $C = \{a, b\}$, and $C' = \{a, b'\}$, where $a$ is a variable measured by an agent Alice, while $b$ and $b'$ are variables measured by Bob, who may be spacelike separated from Alice. Then under relativistic constraints, Bob's choice of measurement — $b$ or $b'$ — should not be able to affect the distribution Alice observes on the outcomes from her measurement of $a$.

This is captured by saying that the distribution on $\{a\} = \{a, b\} \cap \{a, b'\}$ is the same whether we marginalize from the distribution $e_{C}$, or the distribution $e_{C'}$.

This condition is generalized by compatibility — and this general form is satisfied by quantum systems.
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Samson Abramsky (Department of Computer Science, The University of Oxford)
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No-Signalling for Alice-Bob Tables

Consider the following schematic representation of an Alice-Bob table:

<table>
<thead>
<tr>
<th></th>
<th>a</th>
<th>b</th>
<th>c</th>
<th>d</th>
<th>e</th>
<th>f</th>
</tr>
</thead>
<tbody>
<tr>
<td>a</td>
<td></td>
<td></td>
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<td>i</td>
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</tr>
<tr>
<td>a</td>
<td></td>
<td></td>
<td>o</td>
<td>p</td>
<td>q</td>
<td>r</td>
</tr>
</tbody>
</table>

where we have labelled the entries with the letters c, ..., r.

The no-signalling conditions for the non-empty intersections of contexts are given by the following equations:

\[
\begin{align*}
  c + e &= k + m, \\
  d + f &= l + n, \\
  g + i &= o + q, \\
  h + j &= p + r.
\end{align*}
\]

\[
\begin{align*}
  c + d &= g + h, \\
  e + f &= i + j, \\
  k + l &= o + p, \\
  m + n &= q + r.
\end{align*}
\]

You can check that these conditions are satisfied by the Bell table.

Moreover, the PR box has a unique family of distributions which satisfy these conditions.
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<table>
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<tr>
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</tr>
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<td>i</td>
<td>j</td>
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    g + i &= o + q, \\
    h + j &= p + r \\
    c + d &= g + h, \\
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\end{align*}
\]
No-Signalling for Alice-Bob Tables

Consider the following schematic representation of an Alice-Bob table:

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where we have labelled the entries with the letters c, ..., r.

The no-signalling conditions for the non-empty intersections of contexts are given by the following equations:

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\begin{align*}
  c + e &= k + m, \\
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You can check that these conditions are satisfied by the Bell table.

Moreover, the PR box has a **unique family of distributions** which satisfy these conditions.
Contextuality defined

An empirical model \( d \in \mathcal{C} \) on a measurement scenario \((X,M,O)\) is non-contextual if there is a distribution \( d \in \text{Prob}(O|X) \) such that, for all \( C \in M \):

\[
d|C = d_C.
\]

That is, we can glue all the local information together into a global consistent description from which the local information can be recovered. We call such a \( d \) a global section. If no such global section exists, the empirical model is contextual.

The import of Bell's theorem and similar results is that there are empirical models arising from quantum mechanics which are contextual.
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The import of Bell's theorem and similar results is that there are empirical models arising from quantum mechanics which are contextual.
There is a class of empirical models, for each measurement scenario \((X, M, O)\), which are quantum realizable. That is, we can find quantum states and local observables which generate the family of distributions \(\{d_C \}_{C \in M}\).

It turns out that all quantum realizable models are compatible. Compatibility is in fact the general form of an important physical principle known as No-Signalling, which ensures the consistency of quantum mechanics with Special Relativity.

However, there are compatible (i.e., No-Signalling) empirical models which are not quantum realizable. We thus get a strict hierarchy of empirical models: 
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Empirical Models as Vectors

We can regard an empirical model $\{d_C\} \in M$ as a vector $v_C = (v_C, s_C) \in M$, where $s_C \in O_C$, $v_C, s_C : d_C(s)$ in a high-dimensional real vector space.

Note that, in a Bell-type scenario with $n$ parties, $k$ measurement choices at each site, and $\ell$ possible outcomes for each measurement, the dimension is $k^n \ell^n$.

Note that empirical models over a given measurement scenario are closed under convex combinations:

$\left(\mu d_C(s) + (1 - \mu) d'_C(s)\right) = \mu d_C(s) + (1 - \mu) d'_C(s)$.

Moreover, convex combinations of compatible models are compatible.
Empirical Models as Vectors

We can regard an empirical model \( \{d_C\}_{C \in \mathcal{M}} \) as a vector

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\mathbf{v} = (\mathbf{v}_{C,s})_{C \in \mathcal{M}, s \in O^C}, \quad \mathbf{v}_{C,s} := d_C(s)
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The Quantum Set
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A subtle convex set sandwiched between two polytopes.
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\[ SC \]
\[ Q \]
\[ NC \]
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A subtle convex set sandwiched between two polytopes.

Key question: find compelling principles to explain why Nature picks out the quantum set.
The Support of a Model

The support of an empirical model \( \{ C \} \) \( \in \mathcal{M} \) is defined as follows. For each \( C \in \mathcal{M} \), we define
\[
S(C) := \{ s \in \mathcal{O}_C | d_C(s) \neq 0 \}
\]
If the empirical model is compatible, so is the support in the following sense: for all \( C, C' \in \mathcal{M} \),
\[
\{ s \mid C \cap C' : s \in S(C) \} = \{ s' \mid C \cap C' : s' \in S(C') \}
\]
Thus the support satisfies No-Signalling at the level of possibilities. This is equivalent to saying that, for all \( C \subseteq C' \), the restriction map \( \rho_{C' C} : S(C') - S(C) :: s \mapsto s \mid C \) is surjective.
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Degrees of contextuality

Firstly, we say that a global assignment \( t \in O \) is consistent with the support of a model if for all \( C' \in M \), \( t \mid C' \) is in the support at \( C' \).

An empirical model is logically contextual if some possible joint outcome \( s \in O_C \) in the support is not accounted for by any global assignment \( t \in O_X \) which is consistent with the support of the model. That is, for no such \( t \) do we have \( t \mid C = s \).

Geometrically, this is saying that some local section cannot be extended to a global one. Equivalently, that the support of the model cannot be covered by the consistent global assignments.

It is strongly contextual if its support has no global section; that is, there is no consistent global assignment. This says that no possible joint outcome is accounted for by any global section!

Obviously, strong contextuality implies logical contextuality.
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A Hierarchy

We can distinguish three degrees of contextuality among models:

- Strong contextuality implies logical contextuality, which implies (probabilistic) contextuality.
- The Bell model is contextual, but not logically contextual.
- The Hardy model is logically contextual, but not strongly contextual.
- The PR box is strongly contextual.

Thus we have a strict hierarchy

\[ \text{probabilistic contextuality} < \text{logical contextuality} < \text{strong contextuality} \]

The model arising from the GHZ quantum state (with 3 or more parties) with \( X \), \( Y \) measurements at each site is strongly contextual.

Thus in terms of well-known quantum examples, we have

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A Hierarchy

We can distinguish three degrees of contextuality among models:

- Strong contextuality implies logical contextuality, which implies (probabilistic) contextuality.
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Thus we have a strict hierarchy

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Degrees of contextuality for quantum states

We can lift these concepts to define a novel way of classifying quantum states in terms of their degree of contextuality. In particular, we shall focus on \( n \)-qubit pure states. If we fix local observables for each party, such a state gives rise to a probability model as above. We can lift the properties of models to states. We say that a state is strongly contextual if for some choice of local observables for each party, the resulting empirical model is strongly contextual.

We can similarly define logical contextuality for states; we say that a state is logically contextual if for some choice of local observables, the resulting empirical model is logically contextual; while the state is not strongly contextual.

Finally, a state is weakly contextual if it is contextual, but neither of the previous two cases apply.
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The Characterization Problem

This gives rise to a natural and challenging problem:

**Problem**

Characterize the multipartite states in terms of their maximum degree of contextuality.

We believe that an answer to this problem will shed considerable light on the structure of multipartite states, not least because it will necessitate solving the following task:

Given a multipartite state, find local observables which witness its highest degree of contextuality.
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SA, Carmen Constantin and Shenggang Ying. *Hardy is (almost) everywhere.* Information and Computation 2016. arXiv:1506.01365

This paper provides an algorithm which given an $n$-qubit entangled state, constructs $n + 2$ local observables leading to a logically contextual model.

Proof of correctness is non-trivial. This leads us on to the main question which is the natural next challenge: For which quantum states can we find local observables which give rise to a strongly contextual empirical model? This question remains open, and appears difficult!
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