Convergence of MCMC and Loopy BP in the Tree Uniqueness Region for the Hard-Core Model

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joint work with:

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Hard-Core Model

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Gibbs Distribution

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$Z = Z(G, \lambda)$ is the *partition function*.

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- focus on the approximation algorithms

Approximation Algorithms' Approach

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    - in time $\mathrm{poly}(n)$, $\mathrm{poly}(\epsilon^{-1})$ and $\mathrm{poly}(\log(\delta^{-1}))$

$$\Pr[\hat{Z} \in (1 \pm \epsilon) Z(G, \lambda)] > 1 - \delta$$

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Hardness of approximation [Sly 2010]

For triangle-free Δ-regular graphs, where $\Delta \geq 3$, and for all $\lambda > \lambda_c(\Delta)$, it is NP-hard to approximate the partition function within factor $2^{\gamma n}$.

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- Galanis, Ge, Stefankovic, Vigoda, Yang (2011)
- Sly, Sun (2012)
- Galanis, Stefankovic, Vigoda (2012)

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What is $\lambda_c(\varDelta)$? [Kelly 1985]

Critical point for "uniqueness/non-uniqueness" phase transition of the hard-core model on $\varDelta$ regular trees

$$\lambda_c(\varDelta) := \frac{(\varDelta - 1)^{\varDelta - 1}}{(\varDelta - 2)^{\varDelta}} \sim \frac{e}{\varDelta}$$

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Δ-regular tree T of height h

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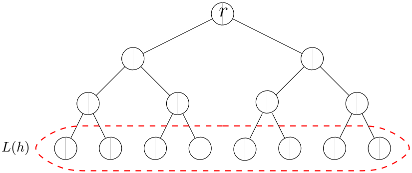

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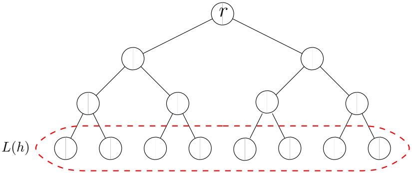

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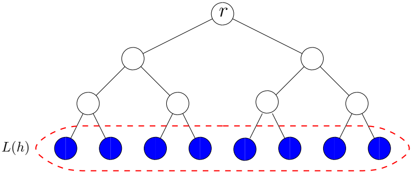

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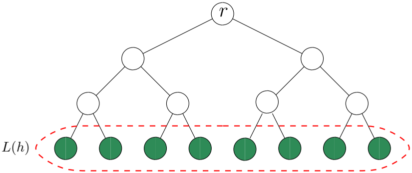

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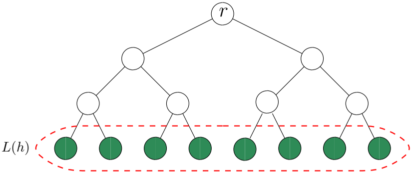

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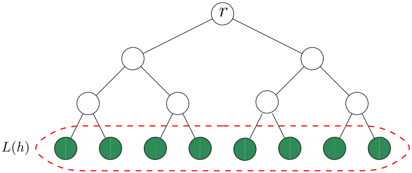

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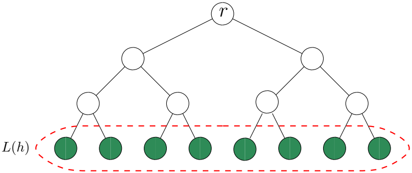

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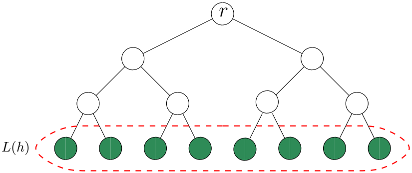

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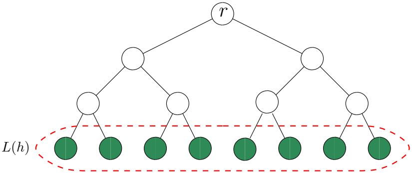

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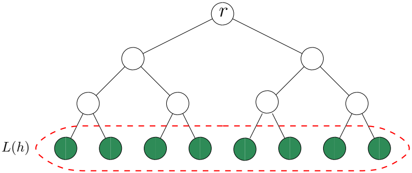

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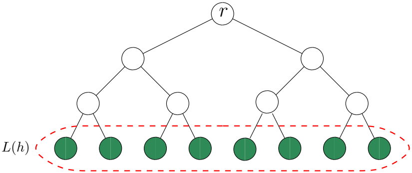

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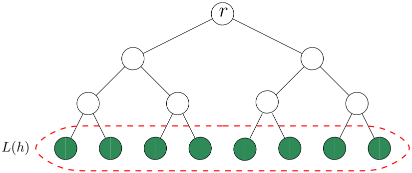

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$$\lambda < \lambda_c(\Delta) \Leftrightarrow \text{Gibbs measure is Unique}$$

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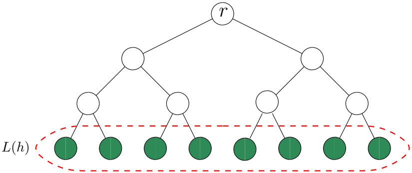

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For every λ we compare

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$$\lambda < \lambda_c(\Delta) \Leftrightarrow \text{we have spatial mixing}$$

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- L. Li, P. Lu, and Y. Yin (2012), (2013)
- Restrepo, Shin, Tetali, Vigoda, and Yang (2013)
- A. Sinclair, P. Srivastava, and Y. Yin (2013)

Performance Weitz's algorithm

Approximation guarantees

For all $\delta > 0$, there exists constant $C = C(\delta) > 0$, for all Δ all G of maximum degree Δ, all $\lambda < (1-\delta)\lambda_c(\Delta)$ all $\epsilon > 0$ Weitz's algorithm returns an estimation $\hat{Z}$ of the partition function $Z(G,\lambda)$ such that

$$(1-\epsilon)Z(G,\lambda) \leq \hat{Z} \leq (1+\epsilon)Z(G,\lambda)$$

in time $O((n/\epsilon)^{C \log \Delta})$.

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it is desirable that the chain mixes "fast"

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The chain converges to the hard-core model with fugacity λ.

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Theorem

For all $\delta > 0$*, there exists* $\Delta_0 = \Delta_0(\delta)$ *for all graphs* $G = (V, E)$ *of maximum degree* $\Delta \geq \Delta_0$ *and* girth ≥ 7*, all* $\lambda < (1-\delta)\lambda_c(\Delta)$*, the mixing time of the Glauber dynamics satisfies*

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Mixing Time . . .

$$T_{mix} = \min\{t : \text{ for all } X_0, d_{tv}(X_t, \mu) \leq 1/4\},$$

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Corollary

The above sampling result yields an FPRAS for estimating the partition function Z. The running time is $O^*(n^2)$.

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Previous work

$T_{mix} = O\left(n \log(n)\right)$ for Glauber dynamics on G of maximum degree Δ and $\lambda < 2/(\Delta - 2)$

- Dyer Greenhill, Luby, Vigoda

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Mossel, Weitz, Wormald (2009)

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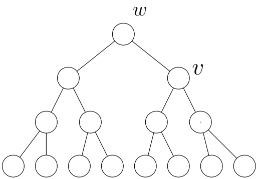

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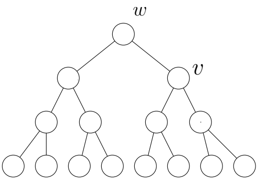

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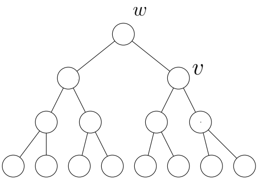

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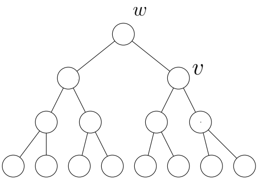

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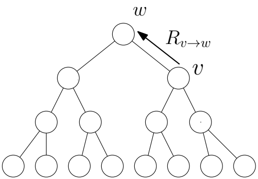

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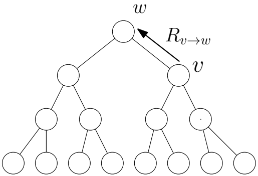

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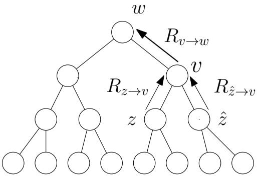

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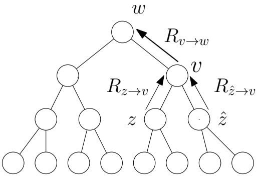
$$R_{v\to w} = \lambda \prod_{z\in N(v)\setminus\{w\}} \frac{1}{1 + R_{z\to v}}$$



BP starts from arbitrary $R^0_{v\to w}$s,
iterates like

$$R^i_{v\to w} = \lambda \prod_{z\in N(v)\setminus\{w\}} \frac{1}{1 + R^{i-1}_{z\to v}}$$

Convergence

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Convergence on trees

There exists i_0 such that for every $i \geq i_0$ and every $(R^0_{v\to w})_{\{v,w\}\in E}$ we have

$$R^i_{v\to w} = R^*_{v\to w}$$

In turn

$$\mu(v \text{ occupied}|w \text{ unoccupied}) = q^* = \frac{R^*_{v\to w}}{1 + R^*_{v\to w}}$$

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BP is an elaborate use of *Dynamic Programing* to compute marginal.

(Loopy) Belief Propagation

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- We do not know whether it converges

(Loopy) Belief Propagation

- We do not know whether it converges
- . . . if does, we do not know where exactly it converges

BP Convergence for girth ≥ 6

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Theorem

For $G=(V,E)$ of maximum degree $\Delta \geq \Delta_0$ and girth ≥ 6, all $\lambda < (1-\delta)\lambda_c(\Delta)$, the following holds: for $i \geq C$, for all $v \in V$, $w \in N(v)$,

$$\left| \frac{q^i_w(v)}{\mu(v \ is\ occupied \mid w \ is\ unoccupied)} - 1 \right| \leq \epsilon$$

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we also have convergence for the BP estimate of $\mu(v \text{ is occupied})$

Path Coupling for Rapid Mixing

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Path Coupling [Bubley and Dyer 1997]

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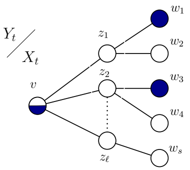
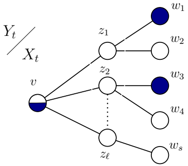


Path Coupling [Bubley and Dyer 1997]

Consider copies $(X_s), (Y_s)$ such that $X_t \oplus Y_t = \{v\}$

$$\mathbb{E}\left[\Phi(X_{t+1}, Y_{t+1}) \middle| X_t, Y_t\right] \leq (1-\gamma)\Phi(X_t, Y_t).$$



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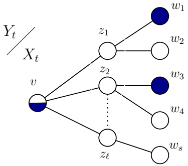

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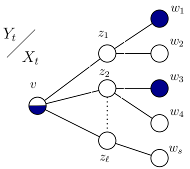

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Path Coupling Example

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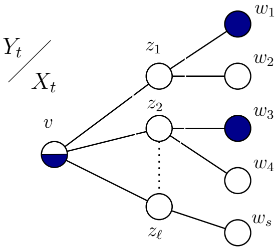
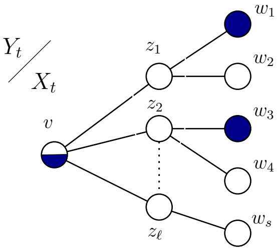


Path Coupling Example

Expected distance

$$\mathbb{E}\left[\Phi(X_{t+1}, Y_{t+1}) \middle| X_t, Y_t\right] = \left(1 - \frac{1}{n}\right)\Phi(v) + \frac{1}{n}\sum_{z_i} \Pr[z_i \in Y_{t+1}] \cdot \Phi(z_i)$$



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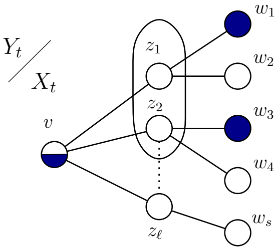

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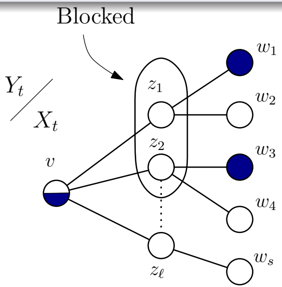

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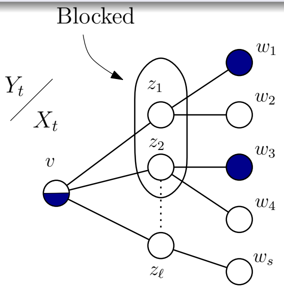
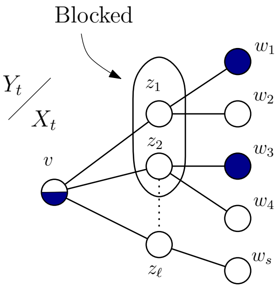


Path Coupling Example

Path coupling condition

$$\Phi(v) > \frac{\lambda}{1+\lambda} \sum_{z_i} \mathbf{1}\{z_i \text{ unblocked in } Y_t\} \cdot \Phi(z_i)$$



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- $\omega^*(z) \approx \mu(z \text{ is unblocked})$

Back to Path Coupling

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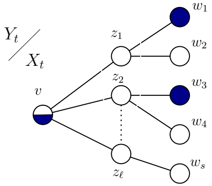

Back to Path Coupling

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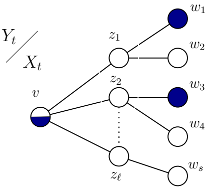

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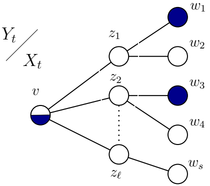

Finding Φ

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For $\rho = 1 - \delta$, there is Φ such that

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$n \times n$ matrix $\mathcal{C}$

$$\mathcal{C}(v, z) = \begin{cases} \frac{\lambda \omega^*(z)}{1+\lambda \omega^*(z)} & \text{if } z \in N(v) \\ 0 & \text{otherwise} \end{cases}$$

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There is a vector $\Phi \in \mathbb{R}^V_{\geq 1}$ such that

$$\mathcal{C}\Phi \leq \rho \cdot \Phi.$$

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Jacobian of Loopy BP

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Relation to Path Coupling

$$\hat{J} = \mathcal{C}$$

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Spectral radius of BP in uniqueness region

We should expect $\rho(\lambda, \Delta) < 1$, because the fixed point ω^* is attractive

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What is Φ

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Local Uniformity with Loopy BP fixed points

Theorem

Let G be of girth ≥ 7 and maximum degree Δ, for $\Delta > \Delta_0$. Let (X_t) be the Glauber dynamics with $\lambda < (1-\delta)\lambda_c(\Delta)$.
For any vertex v, with probability $1 - \exp\left[-\Delta/C\right]$, it holds that

$$\#\ \text{Unblocked Neighbors of } v \text{ in } X_t < \sum_{z \in N(v)} \omega^*(z) + \epsilon\Delta$$

where $t \geq Cn \log \Delta$.

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  - Φ is from the Jacobian of BP operator
- Glauber dynamics (approximately) converges locally to ω^*
  - locally Glauber dynamics behaves approximately like BP fixed points
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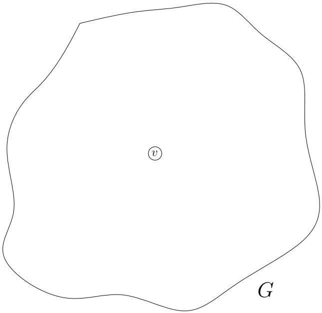
Rapid Mixing with uniformity

Dyer, Frieze, Hayes, Vigoda 2013

Rapid Mixing with uniformity

Dyer, Frieze, Hayes, Vigoda 2013



There is a single disagreement at v

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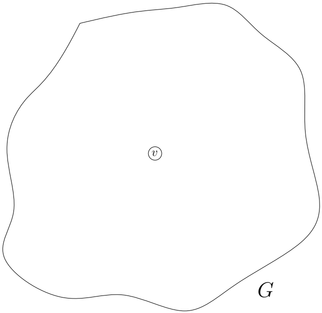

Run the chains for $Cn \log \Delta$ steps, "burn-in"

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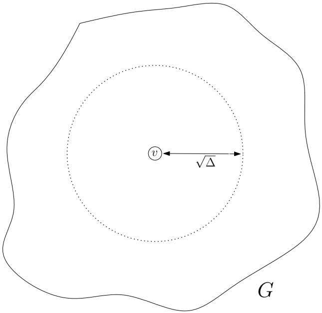

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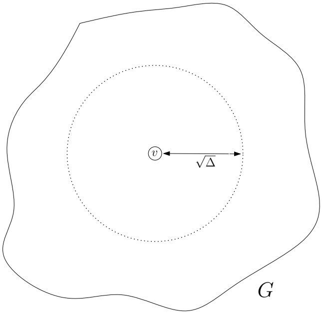

The disagreements spread in the graph during burn-in

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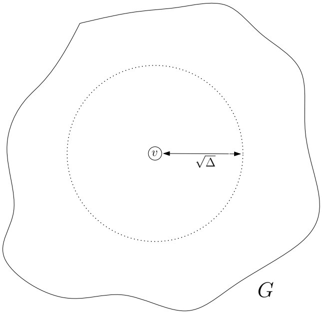

Typically the disagreements do not escape the ball

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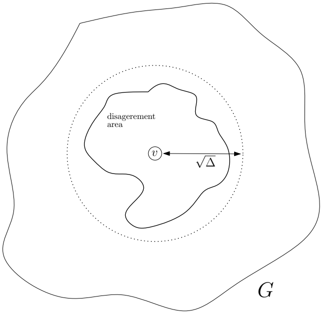

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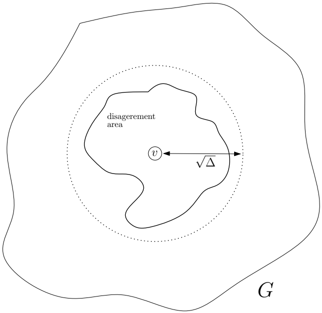

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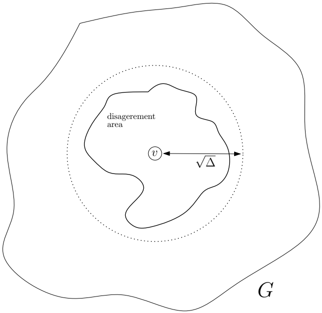

Interpolate and do path coupling for the pairs,

... the pairs now "behave" like ω^*

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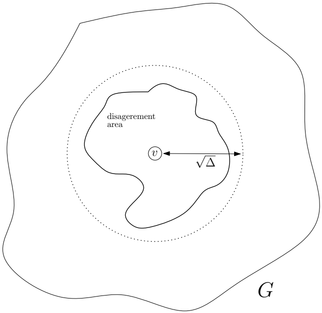

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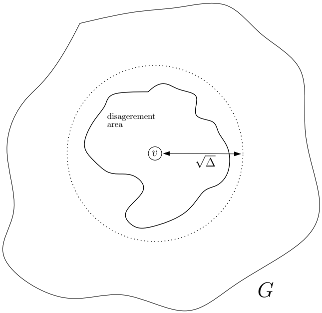

$$\mathbb{E}\left[\Phi(X_{C'n\log\Delta}, Y_{C'n\log\Delta})\,\middle|\, X_0, Y_0\right] \leq (1-\gamma)\Phi(X_0, Y_0)$$

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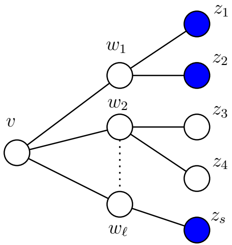

$$\mathbf{R}(\sigma, v) = \Pr_{Y \sim \mu}\left[v \text{ is unblocked in } Y | v \notin Y,\ Y(S_2(v)) = \sigma(S_2(v))\right]$$

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BP for Gibbs measure

Let G be of girth ≥ 6 and maximum degree $\Delta > \Delta_0$. Let X be distributed as in μ with $\lambda < (1-\delta)\lambda_c(\Delta)$.
Then for any vertex v with probability $\geq 1 - \exp\left(-\Delta/C\right)$ it holds that

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BP for Glauber dynamics

Let G be of girth ≥ 7 and maximum degree $\Delta > \Delta_0$. Let (X_t) be the Glauber dynamics with $\lambda < (1-\delta)\lambda_c(\Delta)$.
Then for any vertex v and any $t > Cn \log \Delta$ with probability $\geq 1 - \exp(-\Delta/C)$ it holds that

$$\left| \mathbf{R}(X_t, v) - \prod_{z \sim v} \left(1 - \frac{\lambda}{1+\lambda} \mathbb{E}_{t_z}\left[\mathbf{R}(X_{t_z}, z)\right]\right) \right| < \gamma.$$

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Let G be of girth ≥ 7 and maximum degree $\Delta > \Delta_0$. Let (X_t) be the Glauber dynamics with $\lambda < (1-\delta)\lambda_c(\Delta)$.
For all $\mathcal{I} = [t_0, t_1]$, where $t_0 = Cn \log \Delta$, for every $v \in V$ with probability $1 - (1 + |\mathcal{I}|/n) \exp(-\Delta/C)$, we have that

$$(\forall t \in \mathcal{I}) \quad |\mathbf{R}(X_t, v) - \omega^*(v)| \leq \epsilon.$$

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- Hayes 2012

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Potential function

$$\Psi(x) = (\lambda)^{-1}\mathrm{arcsinh}(\sqrt{\lambda x})$$

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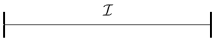

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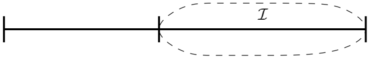

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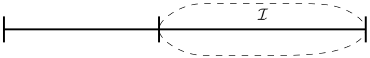
$$|\Psi(\mathbf{R}(X_t, u)) - \Psi(\omega^*(u))| \leq \alpha_{i+1}$$



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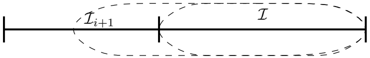

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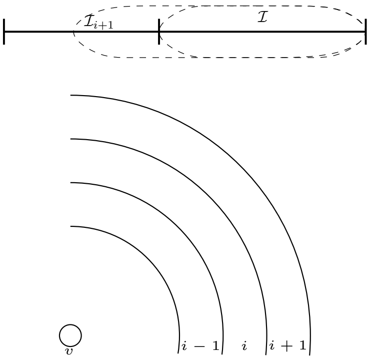

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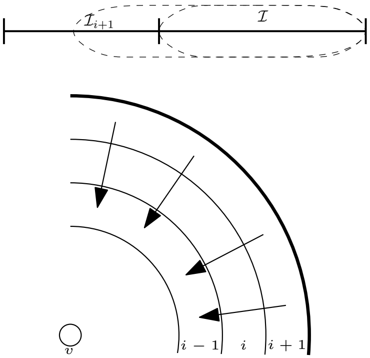

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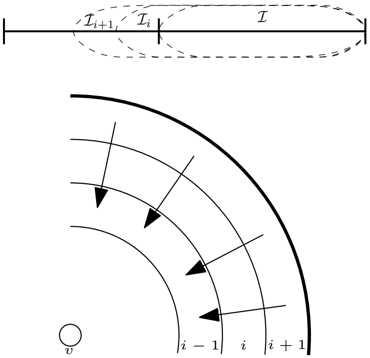

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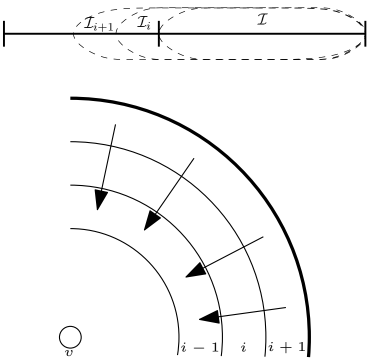

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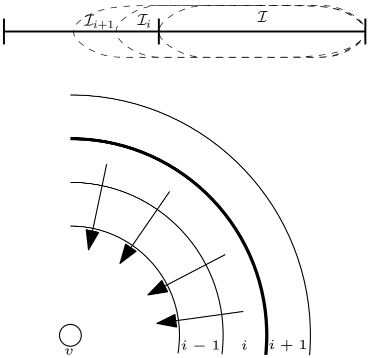

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  - Allows to establish uniformity and weights in a *systematic way*

The End

THANK YOU!