Bootstrap Percolation on Random Graphs

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Part I.

Bootstrap Percolation on Graph G

with Infection Threshold $r \in \mathbb{N}$ (fixed)

- it starts from a set of initially infected vertices
- in each step

  every vertex with $\geq r$ infected neighbours becomes infected
- once a vertex has become infected, it remains infected forever

Bootstrap Percolation on

- Bethe lattice [Chalupa–Leath–Reich 79]
- Infinite trees [Balogh–Peres–Pete 06]
- Grid $[n]^d$ [Balogh–Bollobás–Duminil-Copin–Morris 12]
- Random regular graphs [Balogh –Pittel 07; Janson 09]
- Binomial random graph $\boldsymbol{G}(n, p)$ [Janson–Łuczak–Turova–Vallier 12]
- Inhomogeneous random graphs [Fountoulakis–K.–Koch–Makai 16+]
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Part II.

Bootstrap Percolation on $\boldsymbol{G}(n, p)$ with infection threshold $r \geq 2$ (fixed)

$A \;=\;$ random set of initially infected vertices of size a

by symmetry we can assume $A \;=\; \{1, 2, \ldots, a\}$

$A_f \;=\;$ final set of infected vertices

Bootstrap Percolation on $G(n, p)$

[JANSON–ŁUCZAK–TUROVA–VALLIER 12]

Behaviour of $|A_f|$

- $n^{-1} \ll p \ll n^{-1/r}$

  whp

  either only a few additional vertices are infected, $|A_f| = o(n)$

  or almost every vertex becomes infected, $|A_f| = n - o(n)$

- $p = O(n^{-1})$
- $p = \Theta(n^{-1/r})$
- $p = \omega(n^{-1/r})$

Bootstrap Percolation on $G(n,p)$ Revisited

[Janson–Łuczak–Turova–Vallier 12; K.–Makai 16]

$r \geq 2, \;\; n^{-1} \ll p \ll n^{-1/r}$

$$t_0 = \left((1+\delta)\frac{(r-1)!}{np^r}\right)^{\frac{1}{r-1}}, \text{ where } \delta = \max\left\{(np^r)^{\frac{1}{2(r-1)}}, (np)^{-\frac{1}{4(r-1)}}\right\} = o(1)$$

$$\pi(t) = \mathbb{P}[\,\mathrm{Bin}(t,p) \geq r\,], \qquad t \geq 0$$

$$a_c = -\min_{0\leq t\leq t_0} \frac{n\pi(t) - t}{1-\pi(t)} \sim \left(1-\frac{1}{r}\right)\left(\frac{(r-1)!}{np^r}\right)^{\frac{1}{r-1}}$$

$$t_c = \arg\min_{0\leq t\leq t_0} \frac{n\pi(t) - t}{1-\pi(t)} \sim \left(\frac{(r-1)!}{np^r}\right)^{\frac{1}{r-1}} \ll p^{-1}$$

ω_0 = any function satisfying $\omega_0 \gg \sqrt{a_c}$

- If $|A| = a_c - \omega_0$, w/p $\geq 1 - \exp(-\Theta(\omega_0^2/t_0))$, $|A_f| < t_c$
- If $|A| = a_c + \omega_0$, w/p $\geq 1 - \exp(-\Theta(\omega_0^2/t_0))$, $|A_f| = n - o(n)$

Part III.

Proof Ideas

(1) Reformulation

(2) Martingale concentration

(3) Subcritical case

(4) Supercritica case

Proof Ideas – Reformulation

[Scalia-Tomba 85]

- Set $A(0) = A = \{1, \dots, a\}$ and $Z(0) = \emptyset$
- For each $t \in \mathbb{N}$

  if $A(t-1) \setminus Z(t-1) \neq \emptyset$, select $u_t \in A(t-1) \setminus Z(t-1)$ and

  set $Z(t) = Z(t-1) \cup \{u_t\}$; otherwise set $Z(t) = Z(t-1)$

  For each $i \in \{1, \dots, n-a\}$ let

  $$X(t,i) = \begin{cases} 1 & \text{if vertex } a+i \text{ has } \geq r \text{ neighbours in } Z(t) \\ 0 & \text{otherwise} \end{cases}$$

  Set $A(t) = A(0) \cup \{\, a+i \;:\; X(t,i) = 1, j = 1, \dots, n-a \,\}$

- Let $T = \min\{\, t \;:\; A(t) = Z(t) \,\} = \min\{\, t \;:\; |A(t)| = t \,\}$

  $A_f =$ final set of infected vertices $= A(T)$

Proof Ideas – Martingale

- Let
$$\hat{\pi}(t) = \begin{cases} \mathbb{P}\left[\mathrm{Bin(t,p)} \geq r\right] & \text{if} \quad t \leq T \\ \mathbb{P}\left[\mathrm{Bin(T,p)} \geq r\right] & \text{if} \quad t > T \end{cases}$$

- For each $(t, i) \geq (0, n-a)$ (in lexicographical order) set

$$M(t,i) = \sum_{j=1}^{i} \frac{X(t,j) - \hat{\pi}(t)}{1 - \hat{\pi}(t)} + \sum_{j=i+1}^{n-a} \frac{X(t-1,j) - \hat{\pi}(t-1)}{1 - \hat{\pi}(t-1)}$$

- For each $t \geq 1$
$$|A(t)| = a + \sum_{i=1}^{n-a} X(t,i)$$
$$= a + M(t, n-a)\,(1 - \hat{\pi}(t)) + (n-a)\,\hat{\pi}(t)$$

- $M(0, n-a), \ldots, M(n, n-a)$ forms a martingale

  i.e. $\mathbb{E}\left[\, M(t,i) \mid M(0,n-a), \ldots, M((t,i)-1)\,\right] = M((t,i)-1)$

Proof Ideas – Subcritical Case

- Recall
$$a_c = \frac{t_c - n\,\pi(t_c)}{1-\pi(t_c)}$$
$$|A(t_c)| = a + M(t_c, n-a)\,(1-\hat{\pi}(t_c)) + (n-a)\,\hat{\pi}(t_c)$$

- Since $\hat{\pi}(t_c) \le \pi(t_c)$ and $a = a_c - \omega_0 = \dfrac{t_c - n\,\pi(t_c)}{1-\pi(t_c)} - \omega_0,$

  we have
$$\begin{aligned} |A(t_c)| &\le a + M(t_c, n-a) + (n-a)\,\pi(t_c) \\ &= t_c - \omega_0(1-\pi(t_c)) + M(t_c, n-a) \\ &< t_c \end{aligned}$$

  where the last inequality holds with probability at least
$$1 - \exp\left(-(1+o(1))\frac{r\omega_0^2}{2(t_0 + r\omega_0/3)}\right)$$

  using martingale concentration

Proof Ideas – Supercritical Case

- Martingale argument $\implies$ $\#$ infected vertices $\geq t_0 + \omega_0/2$

- $A_1 =$ set of infected vertices of size $t_0 + \omega_0/4$

  $A_2 =$ set of infected vertices of size $\sim \omega_0/4$

- $\hat{B} =$ set of vertices with exactly $r-1$ neighbours in A_1

  $G[\hat{B}] \sim \boldsymbol{G}((1+\epsilon)\,p^{-1},\ p)$ contains a giant comp. B of size $\frac{2\epsilon}{1+2\epsilon}\,p^{-1}$

  $\exists$ a vertex in B that is connected to a vertex in A_2

- $C =$ set of vertices with $\geq r$ neighbours in B $\implies$ $|C| \geq \delta^{-1}\,p^{-1}$

- $D =$ set of vertices with $\geq r$ neighbours in C $\implies$ $|D| = n - o(n)$