Enumerator polynomials: Completeness and Intermediate Complexity

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Joint work with Nitin Saurabh

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Hamiltonian cycles $\iff$ Monomials of HamC.

eg K_4:
1-2-3-4-1: $X_{12}X_{23}X_{34}X_{41}$
1-2-4-3-1: $X_{12}X_{24}X_{43}X_{31}$
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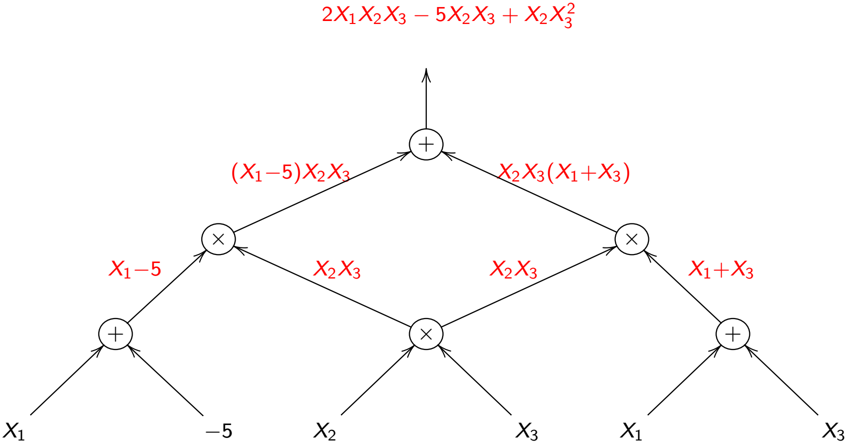
- HamC must be "hard". In what computation model?

Algebraic computation models: Circuits



Arithmetic Circuit Families

Circuit family (C_n) computes polynomial family (p_n).

Family $\{f_n\}_{n>0}$ is a p-family if degree and number of variables in f_n grows polynomially in n.

Now onwards, only p-families.

Algebraic Complexity Classes

- VP: *p-computability*; polynomial size circuits.
- VNP: *p-definability*; exponential sums of partial Boolean instantiations of polynomials in VP.
  $(f_n) \in$ VNP if there exist $(g_m) \in VP$ and polynomial $r(n)$:
  $$f_n(\tilde{x}) = \sum_{\tilde{y} \in \{0,1\}^{t(n)}} g_{r(n)}(\tilde{x}, \tilde{y})$$

(Defined by Valiant in 1979; algebraic analogues of P, NP.)

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projections of g		not projections of g
$y_1 + y_2$	$= g(y_1, 1, y_2, 1)$	$y_1^2 y_2$
$y_1y_2 + 5$	$= g(y_1, y_2, 1, 5)$	(too high degree)
$y_1y_2 + y_2y_3$	$= g(y_1, y_2, y_2, y_3)$	$y_1 + y_2 + y_3$
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  $f \leq_{proj} g$ if circuit for g can be used to compute f, with no extra gates.

- p-projection: $f_n \leq_{proj} g_{m(n)}$ for some poly $m(.)$.

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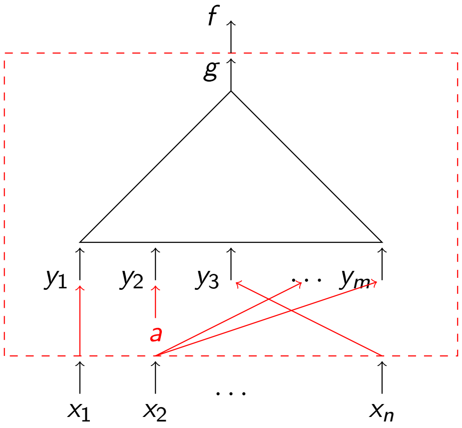

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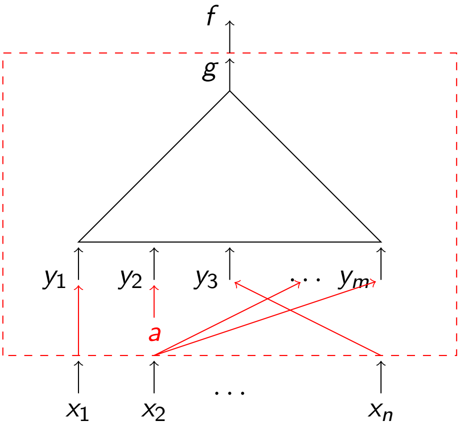

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f is a p-projection of g if $m(n) \in n^{O(1)}$.

Other Hard "Enumerator" Polynomials

- Enumerating Cliques:

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- Enumerating Bipartite Perfect Matchings:

$$\text{Perm}_n \triangleq \sum_{\substack{M \text{ a perfect} \\ \text{matching in } K_{n,n}}} \left(\prod_{(u_i,v_j)\in M} X_{i,j} \right) = \sum_{\sigma\in S_n} \left(\prod_{i\in[n]} X_{i,\sigma(i)} \right)$$

VNP-complete with respect to p-projections
(over fields of characteristic $\neq 2$).

A remarkable enumerator polynomial

$$\mathsf{Cut}_n(X) \triangleq \sum_{(A,B) \text{ partition of } [n]} \left(\prod_{i \in A, j \in B} X_{i,j} \right).$$

eg: $\mathsf{Cut}_3(X) = 1 + X_{1,2}X_{1,3} + X_{1,2}X_{2,3} + X_{1,3}X_{2,3}$.

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Theorem (Bürgisser (1999))

Over the field GF[2],
(Cut_n) *is neither in* VP, *nor* VNP-*hard (with respect to p-projections),*
unless all languages in $\oplus$P *(Mod*$_2$*P) have polynomial-size circuits*
and hence PH *collapses to second level.*

Intermediate Complexity

- (Boolean world) Ladner's theorem (1975): If $P \neq NP$, then there is a language in NP that is neither in P nor NP-hard.
- (Algebraic world) Bürgisser (1999): Over every field, if $VP \neq VNP$, then there is a polynomial family in VNP that is neither in VP nor VNP-hard.

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  Over other fields?
- Over $\mathbb{R}$, Cut_n is in fact VNP-complete. [deRugy-Altherre 2012]

Intermediate Complexity over finite fields

Fix field $\mathbb{F}_q$ of size q, characteristic p.

$$\mathrm{Cut^q}_n(X) \triangleq \sum_{(A,B) \text{ partition of } [n]} \left(\prod_{i \in A, j \in B} (X_{i,j})^{q-1} \right)$$

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Since 1999, these were the only known intermediate-complexity polynomials.

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  - Clique encoded differently.
  - Vertex Cover
  - Closed Walks
  - 3-dimensional matchings
  - 3-SAT

Clique polynomial, redefined

Old definition:

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For other fields $\mathbb{F}_q$:

$$\mathsf{CIS^q}_n \triangleq \sum_{T \subseteq E_n} \left(\prod_{e \in T} (X_e)^{q-1} \right) \left(\prod_{v \text{ incident on } T} (Y_v)^{q-1} \right)$$

3Sat polynomial (over GF[2])

Cl_n: Set of all possible 3-literal clauses on n variables.

$$\mathrm{Sat}_n \triangleq \sum_{a \in \{0,1\}^n} \left(\prod_{i \in [n] : a_i = 1} X_i \right) \left(\prod_{\substack{c \in \mathrm{Cl}_n : \\ a \text{ satisfies } c}} Y_c \right)$$

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Vertex Cover polynomial (over GF[2])

$$\mathsf{VC}_n \triangleq \sum_{S \subseteq V_n} \left(\prod_{e \in E_n :\ e \text{ is incident on } S} X_e \right) \left(\prod_{v \in S} Y_v \right)$$

3-Dimensional Matching polynomial (over GF[2])

$$\mathrm{3DM^q}_n := \sum_{M \subseteq A_n \times B_n \times C_n} \left(\prod_{e \in M} X_e \right) \left(\prod_{\substack{v \in M \\ \text{(counted only once)}}} Y_v \right)$$

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H: Hardness. The monomials of h encode solutions to a problem that is #P-hard via parsimonious reductions.
(Hence, if h is in VP, then $\oplus$P has small circuits.)

Why Sat is intermediate

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Ease: Given a 0-1 assignment to $\tilde{X}$ and $\tilde{Y}$, $\mathsf{Sat}_n(\tilde{x}, \tilde{y})$ equals $\# \{a\colon x_i = 0 \implies a_i = 0 \text{ and } y_c = 0 \implies a \text{ does not satisfy } c\}$.

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Hard: Given any 3-CNF formula F on n variables with m clauses,
For clauses $c \in F$, set all $Y_c = t$; set other Y_c to 1. Set all X_i to 1.

$$\mathrm{Sat}_n(t) = \sum_{a\in\{0,1\}^n} \left(\prod_{\substack{c\ \in F: \\ a \text{ satisfies } c}} t \right) = \sum_{a\in\{0,1\}^n} t^{(\text{number of clauses sat by } a)}$$

Coefficient of t^m equals $\#F \pmod 2$.

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Discard vertices v with $Y_v = 0$; discard edges e touching discarded vertices or with $X_e = 0$.

ℓ edges remain. Each subset of these edges contributes 1.
Value: $2^\ell \pmod 2$; 1 iff $\ell = 0$.

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Hard: Given any graph $G = (V, E)$,

Set all $Y_v = t$; Set $X_e = z$ if $e \in E$, $X_e = 1$ otherwise.

$$\mathsf{CIS}(z, t) = \sum_{T \subseteq E_n} z^{|T \cap E(G)|} t^{(\text{number of vertices incident on } T)}$$

Coefficient of $z^{\binom{k}{2}} t^k$ = Number of cliques of size k, $\pmod 2$.

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$$\mathrm{Clow}_n \triangleq \sum_{\substack{w=\langle v_0,v_1,\ldots,v_{n-1}\rangle: \\ \forall j>0, \quad v_0<v_j}} \left(\prod_{i\in[n]} X_{(v_{i-1},v_{i \bmod n})} \right) \left(\prod_{v\in\{v_0,v_1,\ldots,v_{n-1}\}} Y_v \right)$$

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$$\text{Clow}(z, t) = \sum_{w\text{: clow of length } n} z^{|w\cap E|} t^{(\text{number of vertices in } w)}$$

Coefficient of $z^n t^n$ = Number of Hamilton cycles $\pmod 2$.

Enumerating Graph Homomorphisms

Graphs G, H.

Homomorphism from G to H:
a map $\phi : V(G) \to V(H)$ preserving adjacencies.

- Object of interest: Homomorphism from G to H
- Q1: Is there a homomorphism $G \to H$?
- Q2: How many homomorphisms?
- Q3: Describe all homomorphisms; Enumerate them symbolically.

Enumerator Polynomial for Homomorphisms

Graphs G, H.
Variables on edges of H. (Think of G as fixed.)

$$f_{G,H} \triangleq \sum_{\phi:\text{homomorphism } G \to H} \left(\prod_{(u,v)\in E(G)} Y_{(\phi(u),\phi(v))} \right)$$

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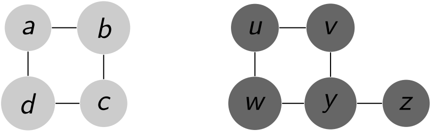
(G_n), (H_n): p-families of graphs. (size grows polynomially with n)

$f_n = f_{G_n,H_n}$.

Homomorphism Polynomials (continued)



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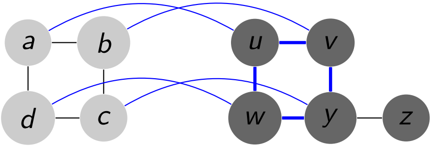
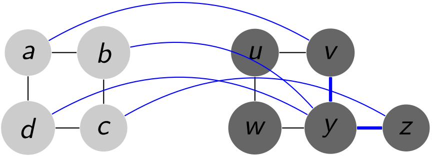


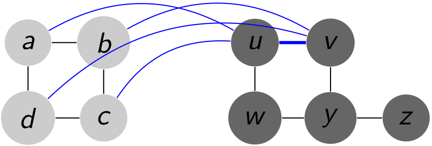
Homomorphism	Monomial
$a \to u$ $b \to v$ $c \to y$ $d \to w$	$Y_{u,v} Y_{v,y} Y_{y,w} Y_{u,w}$

Homomorphism Polynomials (continued)



Homomorphism	Monomial
$a \to v$	
$b \to y$	$Y_{v,y}^2 Y_{y,z}^2$
$c \to z$	
$d \to y$	

Homomorphism Polynomials (continued)



Homomorphism	Monomial
$a \to u$	
$b \to v$	$Y_{u,v}^4$
$c \to u$	
$d \to v$	

Rigid, incomparable graphs

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- A rigid: the only homomorphism from A to A is the identity. Asymptotically, almost all graphs are rigid.
- $A \rightarrow B$: there exists a homomorphism from A to B.

  $A \not\rightarrow B$: there exists no homomorphism from A to B.

  A, B, incomparable: $A \not\rightarrow B$ and $B \not\rightarrow A$.

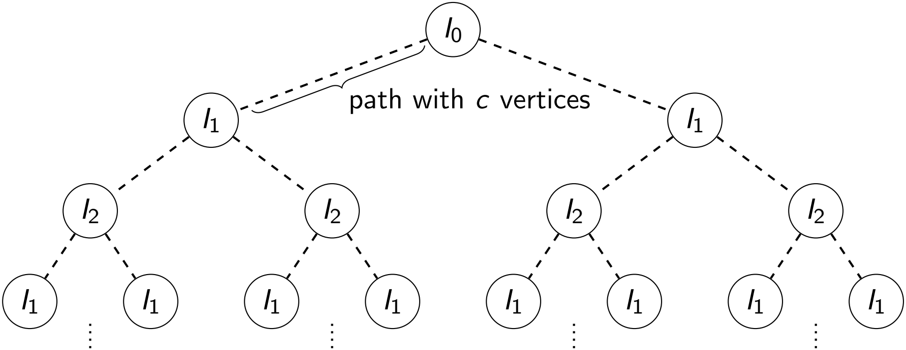
  Asymptotically, almost all pairs of graphs are incomparable.

Describing our graph families

The family (G_n):

I_0, I_1, I_2: any three rigid pairwise incomparable graphs.

Mark three nodes in each as attachment points. $c = |I_0| + |I_1| + |I_2|$.



What we show:

- The family (G_n): complete binary tree with $2^{\lceil \log n \rceil}$ leaves, "inflated" by three rigid pairwise-incomparable graphs, and "stretched" with long paths.
- The family (H_n): complete graph on n^6 vertices.

$$f_{G,H} = \sum_{\psi : V(G) \to n^6} \left(\prod_{(u,v) \in E(G)} Y_{(\psi(u),\psi(v))} \right)$$

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- The family (G_n): simple path, “stretched”, endpoints “inflated” to rigid pairwise-incomparable graphs.
- The family (H_n): complete graph on n^2 vertices.
- The family $(f_{G,H})$ is complete for VBP w.r.t. p-projections.

What's the big deal?

- For VP, first natural complete family whose definition is independent of circuits and where completeness is w.r.t. p-projections.

  (Earlier work by Durand,Malod,M,Rugy-Altherre,Saurabh (2014) gave completeness w.r.t. oracle reductions, or for more artificial homomorphisms with labels and weights.)

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- For VBP, complete polynomials were known – determinant, iterated matrix multiplication. This is one more.
- Our upper bounds hold whenever G_n is bounded tree-width / path-width and H_n is complete.

  (Dynamic programming approach using nice normal-form tree-width/path-width decompositions of G_n.)

Monotone p-projections

- Even more restrictive than p-projections.
- Recall projection: $f \leq_{proj} g$ if circuit for g can be used to compute f, with no extra gates.

  Now monotone projections: $f \leq_{m-proj} g$ if circuit for g can be used to compute f, with no extra gates, without using "negative" constants.

  (Makes sense over totally ordered semi-ring.
  eg $\mathbb{R}$, $\mathbb{Q}$, Boolean semi-ring.)

Why bother?

Goal: to get lower bounds for restricted circuits.

- Jukna: If HamC_n is a monotone p-projection of Perm_n, then monotone Boolean circuits for the Permanent must be of $2^{n^{\Omega(1)}}$ size.
  Current best lower bound: $n^{\log n}$ size. (Razborov 1985)
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- Grochow 2015: Any monotone projection from Perm to HamC needs exponential blowup.
  If $\mathrm{HamC}_n \leq_{m-proj} \mathrm{Perm}_{t(n)}$, then $t(n) = 2^{\Omega(n)}$.

What we show:

- Over the reals (or any totally ordered semi-ring), the families Sat and Clow are not monotone p-projections of Perm.
- Any monotone affine projection from Perm to Sat must have a blow-up of at least $2^{\Omega(\sqrt{n})}$.
- Any monotone affine projection from Perm to Clow must have a blow-up of at least $2^{\Omega(n)}$.

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- More recently, Nitin Saurabh showed: Any monotone affine projection from Perm to Clique must have a blow-up of at least $2^{\Omega(\sqrt{n})}$.

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- Suppose Sat_n is a monotone projection of Perm_t. Then Sat_n polytope can be described with $O(t(n))$ inequalities. (using Grochow 2015)
- For Clow_n, polytope in $\mathbb{R}^{n^2}$, convex hull of clows.
  The Travelling SalesPerson (TSP) polytope is embedded in it.
  Any extension of TSP needs $2^{\Omega(n)}$ inequalities. (Rothvoss 2014)

Summary

1. Over finite fields, five families of enumerator polynomials shown to have complexity intermediate between VP and VNP, assuming the PH does not collapse to second level.
2. Over $\mathbb{R}$ and $\mathbb{Q}$, two of these families proved to require exponential blowup when expressed as monotone p-projections of the permanent.
3. Enumerator polynomials for graph homomorphisms: Rich canvas.
   - First natural family of polynomials defined independent of circuits and shown VP-complete w.r.t. p-projections.
   - Smooth transition to VBP-complete family.
   - VNP-complete variants also exist.

Future Directions

- Can we find polynomials with intermediate complexity over all fields? all fields with non-0 characteristic? all finite fields? all finite fields with characteristic p? finite fields with infinitely many different characteristics?
- Are there polynomials with intermediate complexity over some finite fields but obtainable as monotone p-projections of the permanent?
- Can we find polynomials enumerating homomorphisms, with intermediate complexity?

Thank You!