Counting with Bounded Treewidth

Yitong Yin
Nanjing University

Joint work with Chihao Zhang
(SJTU ⟷ ?)
**Spin Systems**

**vertices are variables** (domain \([q]\))

**edge are constraints:**

\[
\forall e=(u,v) \in E \\
A_e : [q] \times [q] \rightarrow \mathbb{C}
\]

**configuration** \(\sigma \in [q]^V\)

**partition function:**

\[
Z = \sum_{\sigma \in [q]^V} \prod_{e=(u,v) \in E} A_e(\sigma_u, \sigma_v)
\]
Holant Problem

graph $G=(V,E)$

edges are variables (domain $[q]$)
vertices are constraints ($signatures$)

$\forall v \in V, \quad f_v : [q]^{\deg(v)} \rightarrow \mathbb{C}$

configuration $\sigma \in [q]^E$

$$\text{holant} = \sum_{\sigma \in [q]^E} \prod_{v \in V} f_v \left( \sigma \mid E(v) \right)$$

$E(v) = (e_1, \ldots, e_d)$
incident edges of $v$

$q^m$ configurations where $m=|E|$
\begin{align*}
\text{graph } G &= (V,E) \\
\forall \sigma \in [q]^V \\
w(\sigma) &= \prod_{e=(u,v)\in E} A_e(\sigma_u, \sigma_v) \\
Z &= \sum_{\sigma \in [q]^V} w(\sigma)
\end{align*}

\text{incidence graph } B(V,E,F) \\
\text{where} \\
\text{EQ}(x_1, \ldots, x_d) &= \begin{cases} 
1 & x_1 = \cdots = x_d \\
0 & \text{otherwise}
\end{cases}
\text{holant} = \sum_{\sigma \in [q]^F \atop v \in V \cup E} \prod_{v \in V \cup E} f_v \left( \sigma \mid_{F(v)} \right)
Graph $G = (V, E)$  

Boolean symmetric $f : \{0, 1\}^d \to \mathbb{C}$

$f = [f_0, f_1, \ldots, f_d]$

where $f_i = f(x)$ for $\sum_j x_j = i$

Counting matchings:

At every vertex $v \in V$,

$f_v = [1, 1, 0, 0\ldots, 0]$  

the at-most-one function

$$\text{holant} = \sum_{\sigma \in \{0, 1\}^E} \prod_{v \in V} [1, 1, 0, \ldots, 0] (\sigma |_{E(v)})$$
graph $G=(V,E)$

Boolean symmetric $f: \{0,1\}^d \rightarrow \mathbb{C}$

$f = [f_0, f_1, \ldots, f_d]$

where $f_i = f(x)$ for $\sum_j x_j = i$

counting \textit{perfect} matchings:

at every vertex $v \in V$,

$f_v = [0, 1, 0, 0\ldots, 0]$  

the \textit{exact-one} function

holant = $\sum_{\sigma \in \{0,1\}^E} \prod_{v \in V} [0, 1, 0, 0, \ldots, 0] (\sigma |_{E(v)})$
graph $G=(V,E)$  

Boolean symmetric $f : \{0, 1\}^d \to \mathbb{C}$

$f = [f_0, f_1, \ldots, f_d]$

where $f_i = f(x)$ for $\sum_j x_j = i$

counting edge covers:

at every vertex $v \in V$,  
$f_v = [0, 1, 1, 1 \ldots, 1]$

the at-least-one function

holant = $\sum_{\sigma \in \{0,1\}^E} \prod_{v \in V} [0, 1, 1, 1, \ldots, 1](\sigma |_{E(v)})$
graph \( G=(V,E) \) \( \) \( \) \( \) Boolean symmetric \( f : \{0, 1\}^d \to \mathbb{C} \)

\[
f = [f_0, f_1, \ldots, f_d] \]

where \( f_i = f(x) \) for \( \sum_j x_j = i \)

\[
f_v = [1, \mu, 1, \mu, 1, \mu, \ldots] \]

subgraphs world (Jerrum-Sinclair’90):

\[
Z = \sum_{X \subseteq E} \mu^{\text{odd}(X)} \]

\( \text{odd}(X) = \#\text{odd-degree vertices in } X \)

\[
\text{holant} = \sum_{\sigma \in \{0, 1\}^E} \prod_{v \in V} [1, \mu, 1, \mu, \ldots] (\sigma \mid E(v)) \]
graph $G=(V,E)$

counting Eulerian orientations:

$$\text{holant} = \sum_{\sigma \in \{0,1\}^{|F|}} \prod_{v \in V} \left[ \prod_{\deg(v)/2} [0, \ldots, 0, 1, 0, \ldots, 0] \right] (\sigma |_{F(v)}) \prod_{\{e=(e_1,e_2) \in E \}} [\sigma(e_1) \neq \sigma(e_2)]$$

incidence graph $B(V,E,F)$
Treewidth

- $\text{tw}(G)$: *treewidth* of graph $G$
- measures how much a graph is like a tree:
  - $\text{tw}(\text{tree}) = 1$
  - $\text{tw}(k \times k \text{ grid}) = k$
  - $\text{tw}(K_n) = n-1$

- *q*-state spin system on $G$ with $n$ vertices:
  - Courcelle’s Theorem: $f(q, \text{tw})\text{poly}(n)$ time [Courcelle’90]
  - Junction-tree belief propagation: $q^{O(\text{tw})}\text{poly}(n)$ time

- Holant (*tensor networks*) on $G$ with $O(1)$ max-degree: $q^{O(\text{tw})}\text{poly}(n)$
  [Markov, Shi’08] [Arad, Landau’10]
Treewidth

- Every vertex is in some bag.
- Every edge is in some bag.
- If two bags have a same vertex, then all bags in the path between them have that vertex.

- width: max bag size - 1
- treewidth: width of optimal tree decomposition
Separator Decomposition

given a graph $G=(V,E)$
a binary tree $T_G$ of $\leq n$ nodes:

1. Every node is a vertex set $V_i \subseteq V$.
   The root is $V$. Every leaf is $\emptyset$.
2. Every node $V_i$ has a separator $S_i \neq \emptyset$ in $G[V_i]$ separating $V_i$ into $V_j$ and $V_k$.
3. $V_j$ and $V_k$ are children of $V_i$.

$\text{width of } T_G = \max_i \{|S_i|, |\partial V_i|\}$

$\partial V_i : \text{vertex boundary of } V_i \text{ in } G$

$\text{sw}(G) : \text{width of optimal } T_G$

1. $\text{sw}(G) = \Theta(\text{tw}(G))$
2. $T_G$ can be constructed in $2^{O(\text{tw})}\text{poly}(n)$ time
FPT Algorithm for Spin System

\[ \max \{ |S_i|, |\partial V_i| \} = O(\text{tw}(G)) \]

for \( q \)-state spin systems

\[ Z = \sum_{\sigma \in [q]^V} \prod_{e=(u,v) \in E} A_e(\sigma_u, \sigma_v) \]

dynamic programming:

- table size: \( q^{O(\text{tw}(G))}n \)
- running time: \( q^{O(\text{tw}(G))}n \)

for \( \tau \in [q]^{\partial V_i} \):

\[ Z(V_i, \tau) = \sum_{\text{feasible } \sigma \in [q]^{S_i}} Z(V_j, \tau \cup \sigma |_{\partial V_j}) Z(V_k, \tau \cup \sigma |_{\partial V_k}) Z(S_i \cup \partial V_i, \sigma \cup \tau) \]

also works for bounded-degree Holant
classify the computational complexity of

\[ \text{Holant}(\mathcal{G}, \mathcal{F}) \]

in terms of graph family \( \mathcal{G} \) and function family \( \mathcal{F} \)

Classify \( \text{Holant}(\mathcal{G}, \mathcal{F}) \):

- \( \mathcal{G} = \) all graphs: \( 2^{O(n)} \) time
- \( \mathcal{G} = \) graphs with treewidth \( k \): \( 2^{O(k) \text{poly}(n)} \) time
- \( \mathcal{G} = \) planar graphs:
  - PTAS for log-Holant (log-partition function)
  - FPTAS for Holant assuming strong spatial mixing
Pinning & Peering

symmetric $f : [q]^d \rightarrow \mathbb{C}$

fix any $\tau \in [q]^{d-k}$

$\text{Pin}_\tau(f) : [q]^k \rightarrow \mathbb{C}$

Pinning: $\forall \sigma \in [q]^k$

$\text{Pin}_\tau(f)(\sigma) = f(\tau_1, \tau_2, \ldots, \tau_{d-k}, \sigma_1, \sigma_2, \ldots, \sigma_k)$

Peering: equivalent relation between $\tau, \tau' \in [q]^{d-k}$

$\tau \sim \tau'$ if $\text{Pin}_\tau(f) = \text{Pin}_{\tau'}(f)$

$f$ is $r$-regular if $r = \max_{0 \leq k \leq d} \left| \left\{ \text{Pin}_\tau(f) \mid \tau \in [q]^{d-k} \right\} \right|$ not violating $f$
Boolean symmetric $f : \{0, 1\}^d \rightarrow \mathbb{C}$

$\tau \in \{0,1\}^{i+j}$ $\tau$ has $i$ 1’s and $j$ 0’s

$f = [f_0, f_1, \ldots, f_{i-1}, \underbrace{f_i, \ldots, f_{d-j-1}}_{\text{II}}, f_{d-j}, \ldots, f_d]\$

$\text{Pin}_\tau(f)$ where $f_i = f(x)$ for $\Sigma_j x_j = i$

$f$ is $r$-regular if $r = \max_{0 \leq k \leq d} \left| \{ \text{Pin}_\tau(f) \mid \tau \in \{q\}^{d-k} \} \right|$ (not violating $f$)

- 2-state spin systems: $\text{EQ}=[1,0,...,0,1]$ is 2-regular, $A_e$ is 3-regular
- matchings & PMs: $[1,1,0,...,0]$ and $[0,1,0,...,0]$ are 2-regular
- edge covers: $[0,1,1,...,1]$ is 2-regular
- subgraphs world: $[1, \mu, 1, \mu, \ldots]$ is 2-regular
- Eulerian orientations: $[0, \ldots, 0, 1, 0, \ldots, 0]$ is $(d/2+1)$-regular
symmetric \( f : \{0, 1, 2\}^d \to \mathbb{C} \)

\[ f_{ij} = f(x) \]

where \( x \) has \( i \) 0's and \( j \) 1's

- all \( d \)-ary symmetric function is at most \( \binom{d + q - 1}{q - 1} \)-regular
- bounded-degree Holant is \( O(1) \)-regular
For any constant domain size $q \geq 2$, Holant of any graph $G = (V, E)$ with $r$-regular symmetric signatures can be computed in time:

- $r^{O(n)}$ where $n = |V|$
- $r^{O(tw(G))} n + 2^{O(tw(G))} \text{Poly}(n)$

Theorem

- extendable to asymmetric signatures (under proper assumption for evaluating asymmetric functions with unbounded arity);
- implications in approximate counting on planar graphs:
  - PTAS for log-holant
  - FPTAS for holant assuming strong spatial mixing
A Simple $r^{O(n)}$-time Algorithm

given a Holant instance: $G=(V,E)$ \quad $V = \{v_1, v_2, \ldots, v_n\}$

all $f_{v_i} : [q]^{\deg(v_i)} \to \mathbb{C}$

are $\leq r$-regular

**f is $r$-regular if** $r = \max_{0 \leq k \leq d} |\{\text{Pin}_\tau(f) \mid \tau \in [q]^{d-k}\}|$

**dynamic programming:**

- table size: $r^n n$
- running time: $r^n q^n = r^{O(n)}$

holant($G, \{f_{v_1}, \ldots, f_{v_n}\}$) = \sum_{\text{feasible}} f_{v_n}(\sigma) \cdot \text{holant}(G \setminus \{v_n\}, \{\text{new } f_{v_1}^\sigma, \ldots, f_{v_i}^\sigma\})

where $f_{v_i}^\sigma = \begin{cases} f_{v_i} & v_i \not
\text{Pin}_{\sigma_j}(f_{v_i}) & v_i \text{ is } j\text{’s nbr. of } v_n \end{cases}$
Oracles

given a Holant instance: \( G = (V,E) \quad V = \{v_1, v_2, \ldots, v_n\} \)

\[
\begin{array}{cc}
  f_{v_1} & f_{v_2} \\
  f_{v_i} & f_{v_n}
\end{array}
\]

all \( f_{v_i} : [q]^{\deg(v_i)} \rightarrow \mathbb{C} \)

are \( \leq r \)-regular

fix \( f_{v_i} \) and arity \( k \): there are \( \leq r \) distinct \( \text{Pin}_{\tau}(f_{v_i}) : [q]^k \rightarrow \mathbb{C} \)
denoted as \( f_{i,1}^k, f_{i,2}^k, \ldots, f_{i,r}^k \)

Evaluation oracle: Given \( (i, j, k) \) and \( \sigma \in [q]^k \), returns \( f_{i,j}^k(\sigma) \)

Pinning oracle: Given \( (i, j, k) \) and \( \tau \in [q]^{k-l} \), returns \( j' \) that \( f_{i,j'}^l = \text{Pin}_{\tau}(f_{i,j}^k) \)

For symmetric function \( f : [q]^d \rightarrow \mathbb{C} \) with constant \( q \),
these oracles can be implemented using:

- Poly\((d)\) preprocessing time
- Poly\((d)\) space
- O\((1)\) query time

for Holant with finitely many signatures: this can be “hard-wired” into the algorithm
An FPT Algorithm

\[
\max \{ |S_i|, |\partial V_i| \} = O(tw(G))
\]

for Holant problems:

\[
\text{holant} = \sum_{\sigma \in [q]^E} \prod_{v \in V} f_v(\sigma |_{E(v)})
\]

each \( v \in S_i \) has unbounded degree but only has 3 classes of edges

**Peering:** \( \tau \sim \tau' \) if \( \text{Pin}_\tau(f) = \text{Pin}_{\tau'}(f) \)

**Observation:**

\( f(\rho \sigma \tau) \) is determined by the equivalent classes for \( \rho, \sigma, \tau \)

enumerate equivalent classes of local configurations!
An FPT Algorithm

each \( v \in S_i \) has unbounded degree but only has 3 classes of edges

\[ \tau \sim \tau' \text{ if } \text{Pin}_\tau(f) = \text{Pin}_{\tau'}(f) \]

\( f(\rho\sigma\tau) \) is determined by the equivalent classes for \( \rho, \sigma, \tau \)

\[
\text{Peer}_\tau(f)(\sigma) = \begin{cases} 
1 & \text{Pin}_\sigma(f) = \text{Pin}_\tau(f) \\
0 & \text{o.w.}
\end{cases}
\]

\( f \) is \( r \)-regular \( \longrightarrow \) \( \text{Peer}_\tau(f) \) is \( \leq r \)-regular

locally enumerate equivalent classes \( \rho, \sigma, \tau \)
An FPT Algorithm

each \( v \in S_i \) has **unbounded** degree
but only has **3 classes** of edges

\( \tau \sim \tau' \) if \( \text{Pin}_\tau(f) = \text{Pin}_{\tau'}(f) \)

\( f(\varrho_{\sigma_{\tau}}) \) is determined by the equivalent classes for \( \varrho, \sigma, \tau \)

\[
\text{Peer}_\tau(f)(\sigma) = \begin{cases} 
1 & \text{Pin}_\sigma(f) = \text{Pin}_\tau(f) \\
0 & \text{o.w.}
\end{cases}
\]

Holant \((V_i, \text{boundary signatures Peer}_\pi(f_v) \text{ at } v \in \partial V_i)\)

\[
= \sum_{\text{equiv. classes } \rho_v, \sigma_v, \tau_v \text{ at every } v \in S_i} \text{Holant} \left( V_j, \text{old boundary signatures of } V_j \text{ and Peer}_{\rho_v}(f_v) \text{ at } v \in S_i \right) \cdot \text{Holant} \left( V_k, \text{old boundary signatures of } V_k \text{ and Peer}_{\tau_v}(f_v) \text{ at } v \in S_i \right) \\
\cdot \text{Holant} \left( S_i \cup \partial V_i, \text{old boundary signatures of } V_i \text{ and Peer}_{\sigma_v}(f_v) \text{ at } v \in S_i \right) \cdot \prod_{v \in S_i} f_v(\rho_v, \sigma_v, \tau_v)
\]
An FPT Algorithm

Each $v \in S_i$ has unbounded degree but only has 3 classes of edges

$\tau \sim \tau'$ if $\text{Pin}_\tau(f) = \text{Pin}_{\tau'}(f)$

$f(\varphi \sigma \tau)$ is determined by the equivalent classes for $\varphi, \sigma, \tau$

$\text{Peer}_\tau(f)(\sigma) = \begin{cases} 1 & \text{Pin}_\sigma(f) = \text{Pin}_\tau(f) \\ 0 & \text{o.w.} \end{cases}$

**Peering oracle:** Given $i$, (equivalent class) $\tau$, returns $\text{Peer}_\tau(f_{v_i})$

**Pinning oracle & Evaluation oracle:** for $\text{Peer}_\tau(f_{v_i})$

**Evaluation oracle:** Given $i$, (equivalent classes) $\varphi, \sigma, \tau$, returns $f_{v_i}(\varphi \sigma \tau)$

For symmetric functions, can be implemented using $\text{Poly}(n)$ preprocessing time, $\text{Poly}(n)$ space, and $O(1)$ query time
Theorem
For any constant domain size $q \geq 2$, Holant of any graph $G = (V, E)$ with $r$-regular signatures can be computed in time:

- $r^{O(n)}$ where $n = |V|$ assuming Peering, Pinning, & Evaluation oracles.
- $r^{O(tw(G))} |V| + 2^{O(tw(G))} \text{Poly}(|V|)$

Asymmetric $f : [q]^d \to \mathbb{C}$

fix any $\tau \in [q]^{d-k}$ and $S \in \binom{[d]}{d-k}$

$$\text{Pin}_{S, \tau}(f) = f(\tau_1, \ldots, \tau_2, \ldots, \tau_3, \ldots, \tau_{d-k}, \ldots)$$

at positions in $S$

$f$ is $r$-regular if $r = \max_{0 \leq k \leq d} \max_{S \in \binom{[d]}{d-k}} |\{\text{Pin}_{S, \tau}(f) \mid \tau \in [q]^{d-k}\}|$
Planar Decomposition

**Baker’s Decomposition** (Baker’93):
For all $k$, a planar graph $G$ can be decomposed into subgraphs $G_1, ..., G_k$ each of treewidth $O(k)$.

**Jerrum-Goldberg-McQuillan’12**:
For spin systems on planar graphs, if always $\text{holant} \geq \exp(\Omega(n))$ then $\exists$ PTAS for log-holant.
Correlation Decay

strong spatial mixing (SSM): \[ \forall \sigma_B \in [q]^B \]

\[
| \Pr(\sigma(e) = c \mid \sigma_A) - \Pr(\sigma(e) = c \mid \sigma_A, \sigma_B) | \leq \text{poly}(|V|) \exp(-\Omega(t))
\]

SSM: sufficiency of local information for approx. of \[ \Pr(\sigma(e) = c \mid \sigma_A) \]

efficiency of local computation

Theorem (Demaine-Hajiaghayi’04)
For apex-minor-free graphs, treewidth of \( t \)-ball is \( O(t) \).
Correlation Decay

**strong spatial mixing (SSM):** \( \forall \sigma_B \in [q]^B \)

\[
|\Pr(\sigma(e) = c | \sigma_A) - \Pr(\sigma(e) = c | \sigma_A, \sigma_B)| \leq \text{poly}(|V|) \exp(-\Omega(t))
\]

**SSM:** sufficiency of local information for approx. of \( \Pr(\sigma(e) = c | \sigma_A) \)

for planar graphs

for self-reducible problems

efficiency of local computation

FPTAS for Holant
Conclusion

• A class of Holant problems behave like bounded-degree #CSP:
  • \(2^{O(n)}\)- and \(2^{O(tw)}\)Poly\((n)\)-time algorithms.

• Regularity implies efficient DP algorithms, but:
  • is not robust to holographic transformation;
  • does not cover inversion-based algorithms.

• Open question: a Holant problem with Boolean-domain symmetric signatures (and is easy for decision) with \(n^{\Omega(tw)}\) lower bound:
  • must have unbounded degree, must be irregular.
Thank you!