Approximating 2-State Spin Systems

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Based on joint work with Pinyan Lu,

and with Leslie Ann Goldberg

Simons Institute

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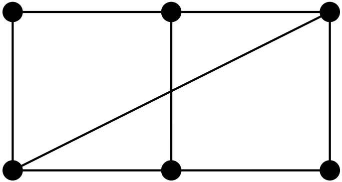
Ising Model

Edge interaction

	0	1
0	β	1
1	1	β



Configuration $\sigma : V \to \{0, 1\}$

$$w(\sigma) = \beta^{mono(\sigma)}$$

$$\pi(\sigma) \sim w(\sigma)$$

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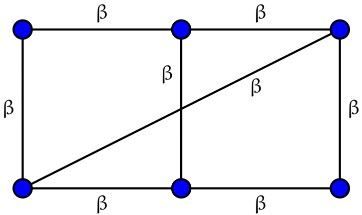

Configuration $\sigma : V \to \{0, 1\}$

$$w(\sigma) = \beta^8$$

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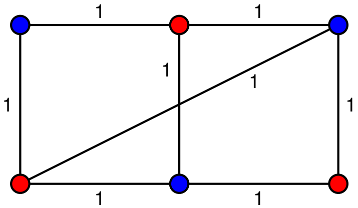

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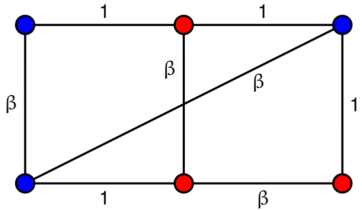

Partition function (normalizing factor):

$$Z_G(\beta) = \sum_{\sigma:V\to\{0,1\}} w(\sigma)$$

where $w(\sigma) = \beta^{mono(\sigma)}$, $mono(\sigma)$ is the number of monochromatic edges under σ.

2-State Spin System

Edge:

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1	1	β

Vertex:

0	1
1	1

More generally, three parameters β, γ, and λ.

$$w(\sigma) = \beta^{m_0(\sigma)}\gamma^{m_1(\sigma)}\lambda^{n_0(\sigma)}$$

$m_0(\sigma)$: # of $(0,0)$ edges;

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$n_0(\sigma)$: # of 0 vertices.

$$Z_G(\beta,\gamma,\lambda) = \sum_{\sigma:V\to\{0,1\}} w(\sigma)$$

2-State Spin System

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Examples

- Ising model: $\begin{bmatrix} \beta & 1 \\ 1 & \beta \end{bmatrix}$ (no field)

$$Z_G(\beta) = \sum_{\sigma : V \to \{0,1\}} \beta^{mono(\sigma)}$$

- Hardcore gas model: $\begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}$ and $\begin{bmatrix} \lambda \\ 1 \end{bmatrix}$ (Weighted independent set)

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Approximate Counting

- Exact evaluating Z is **#P**-hard unless $\beta\gamma = 1$ or $\beta = \gamma = 0$ or $\lambda = 0$.
- Approximate the partition function Z.
  - Fully Polynomial-time Randomized Approximation Scheme (FPRAS) and FPTAS:
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Ferromagnetic and Anti-ferromagnetic

Edge Interaction

$$\begin{bmatrix} \beta & 1 \\ 1 & \gamma \end{bmatrix}$$

- If $\beta\gamma = 1$, then the 2-spin system is trivial.
- Ferromagnetic Ising: $\beta\gamma > 1$.

  Neighbours tend to have the same spin.
- Anti-ferromagnetic Ising: $\beta\gamma < 1$.

  Neighbours tend to have different spins.

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Outline

1. Anti-ferromagnetic 2-Spin Systems
2. Ferromagnetic 2-Spin Systems
3. Complex weighted Ising models

   (approximation of $|Z|$)

Anti-ferromagnetic Systems

Anti-ferromagnetic Systems

For antiferro systems,

$$\text{FPTAS for } Z \quad \Leftrightarrow \quad \text{Correlation decays}$$

Computational Transition

Approximate counting weighted independent sets (Hardcore model)

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FPTAS **NP**-hard

$\lambda_c(\Delta) = \frac{(\Delta-1)^{\Delta-1}}{(\Delta-2)^{\Delta}}$ Activity λ

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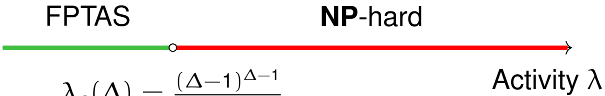

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- Algorithm: [Weitz 06]
- Hardness: [Sly 10] [Galanis, Štefankovič, Vigoda 12] [Sly Sun 14]

Uniqueness Transition

- $\lambda_c(\Delta)$: uniqueness threshold of Gibbs measures in $\mathbb{T}_\Delta$.
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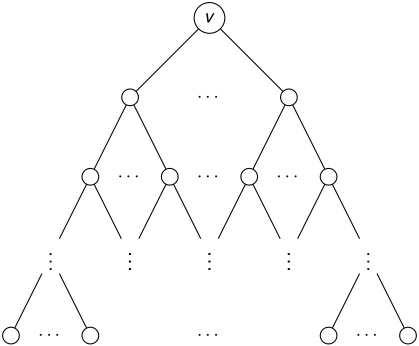

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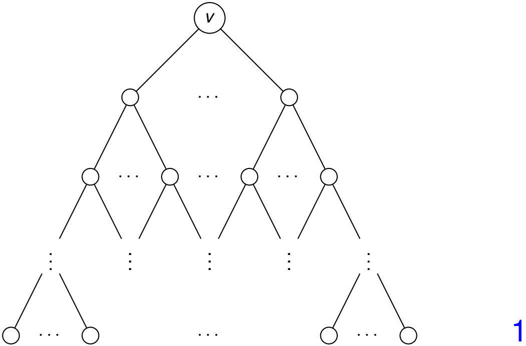

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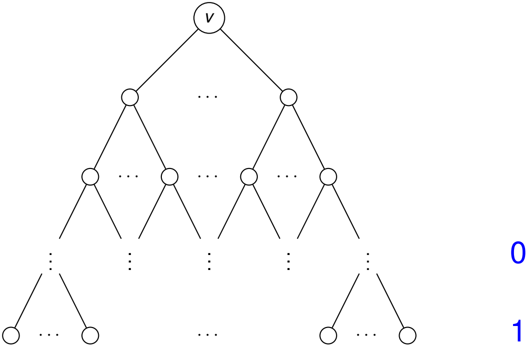

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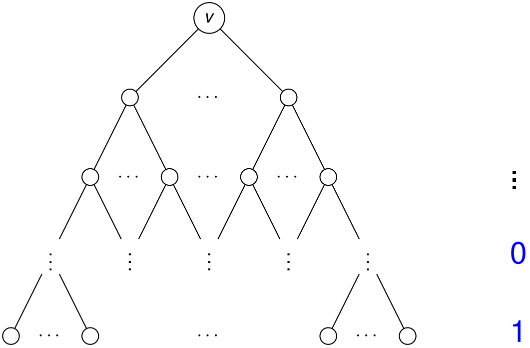

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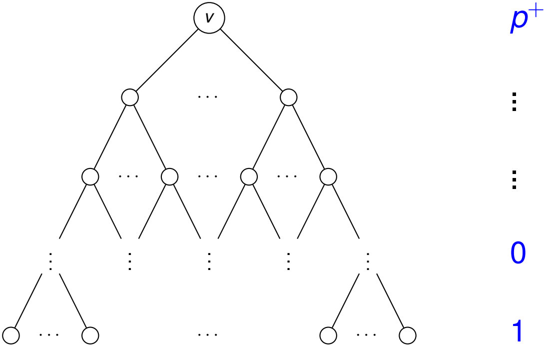

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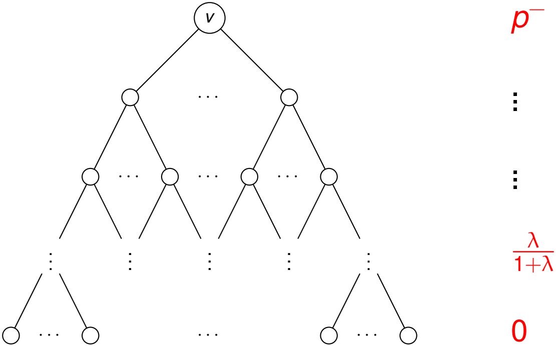

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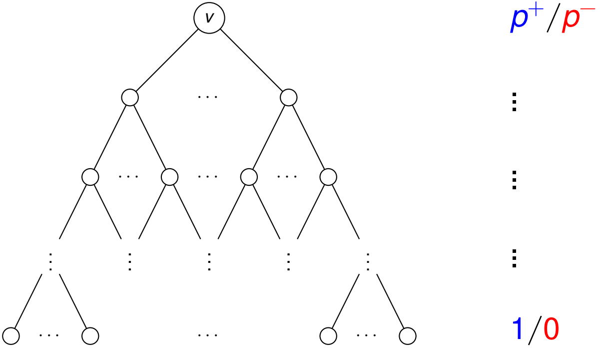

Does $|p^+ - p^-|$ go to 0 or not?

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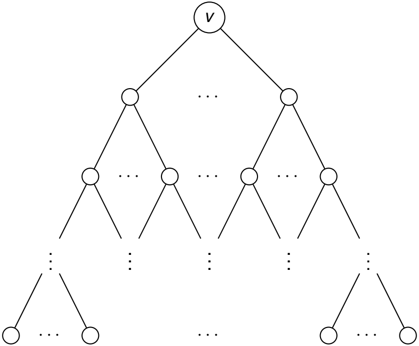

p^+/p^-

$\vdots$

$1/0$

$$|p^+ - p^-| \to 0 \quad \Leftrightarrow \quad \lambda \leqslant \lambda_c(\Delta).$$

Uniqueness Transition (cont.)

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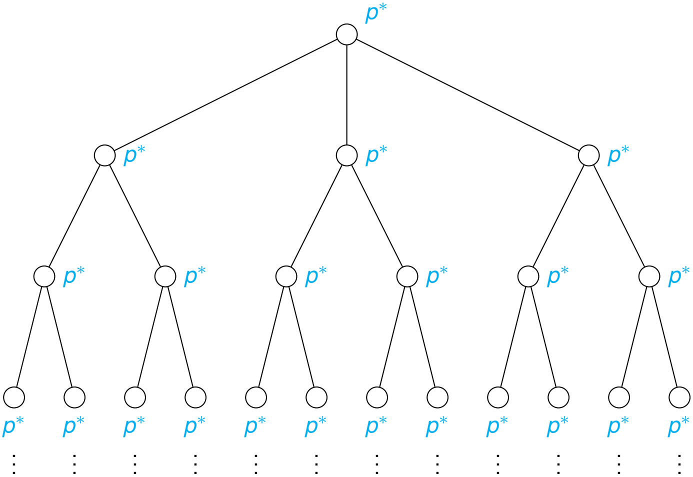

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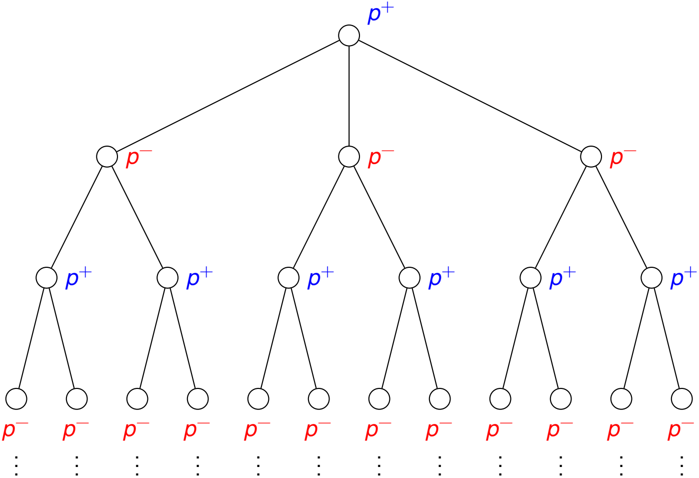

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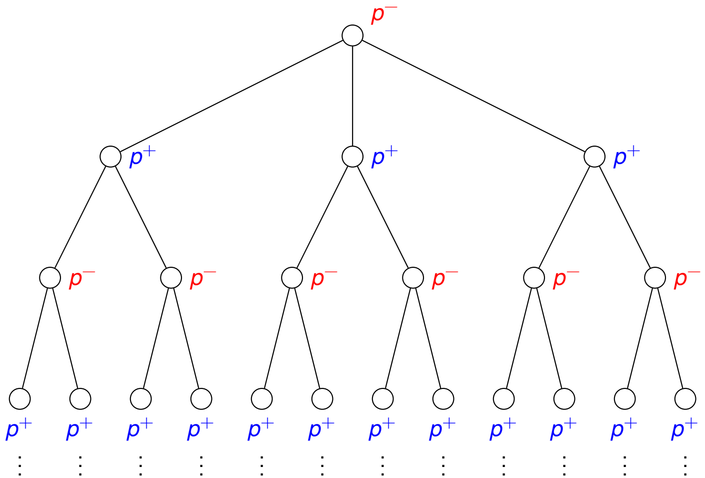

Weak and Strong Spatial Mixing

- WSM: Let σ_Λ and τ_Λ be two partial configurations on Λ,

$$|p_v^{\sigma_\Lambda} - p_v^{\tau_\Lambda}| \leqslant \exp(-\Omega(\mathrm{dist}(v, \Lambda)))$$

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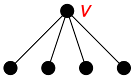
- SSM $\Rightarrow$ WSM $\Leftrightarrow$ Uniqueness
- SSM in $\mathbb{T}_\Delta \Rightarrow$ FPTAS in graphs of degree $\leqslant \Delta$ [Weitz 06]

Breaking Cycles

Goal: calculate marginal probabilities using tree recursions.

Replace a vertex of degree d with d copies.

$$R_v = \frac{\Pr(v = 0)}{\Pr(v = 1)}$$



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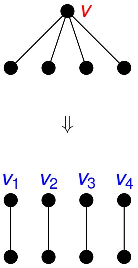

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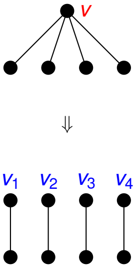

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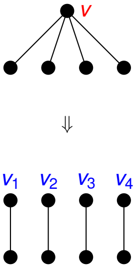
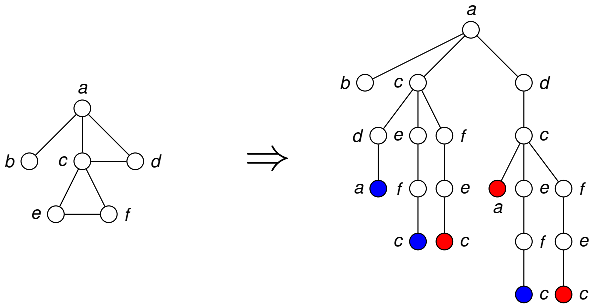


Each term $\frac{\Pr(0011)}{\Pr(0111)}$ can be viewed as the marginal ratio of v_i conditioned on a certain configuration of other v_j's.

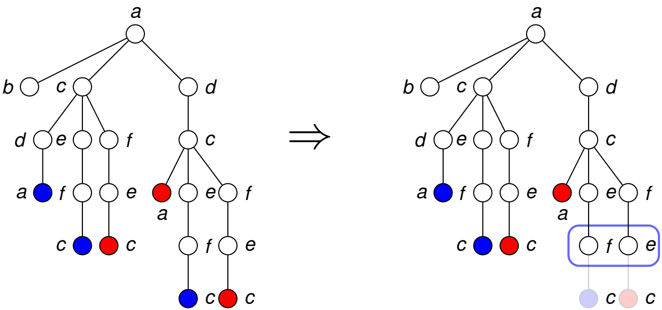
Self-Avoiding Walk (SAW) Tree

- SAW tree is essentially the tree of self-avoiding walks originating at v except that the vertices closing a cycle are also included in the tree.
  - Cycle-closing vertices are fixed according to the rule in the last slide.
- Do the tree recursion to calculate p_v.



- However, SAW tree is of exponential size in general.
  - Truncate the recursion within logarithmic depth.
  - SSM bounds the error.

    Non-uniqueness leads to constant error.



Classification of Antiferro 2-Spin Systems

The implication

$$\text{Uniqueness} \Rightarrow \text{SSM}.$$

is established for all anti-ferromagnetic 2-spin systems ($\beta\gamma < 1$).

[Sinclair, Srivastava, Thurley 12] , [Li, Lu, Yin 12,13]

Hence, for any anti-ferromagnetic 2-spin system,

$$\text{Uniqueness} \Leftrightarrow \text{SSM} \Leftrightarrow \text{FPTAS}.$$

(For general graphs, we require uniqueness to hold for all integer degrees.)

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Ferromagnetic 2-Spin Systems

Ferromagnetic Ising

Ferro ($\beta > 1$) Ising without field:

Edge: $\begin{bmatrix} \beta & 1 \\ 1 & \beta \end{bmatrix}$ Vertex: $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$

For fixed Δ:

Uniqueness in $\mathbb{T}_\Delta$ Non-uniqueness

1 $\beta_c(\Delta) = \frac{\Delta}{\Delta-2}$ β

FPRAS [Jerrum, Sinclair 93]

1 (General graphs) β

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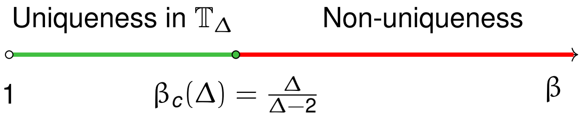

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Markov chain in the "subgraphs" world:

fast mixing for any $\beta = \gamma > 1$ and $\lambda_v \geqslant 1$ (or $\leqslant 1$) for all $v \in V$.

(even if uniqueness or SSM fails) [Jerrum, Sinclair 93]

- Extended to $\lambda_v \leqslant \frac{\gamma}{\beta}$ (if $\beta \leqslant \gamma$)

  [Goldberg, Jerrum, Paterson 03] , [Liu, Lu, Zhang 14] .

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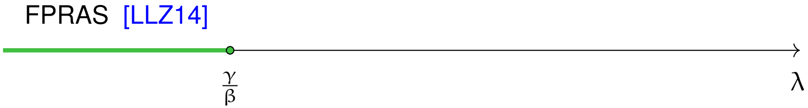

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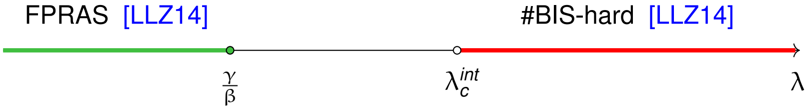

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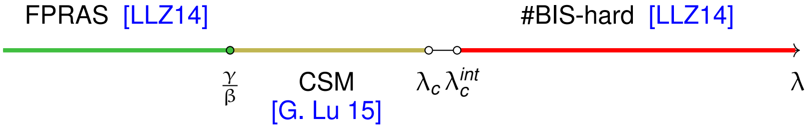

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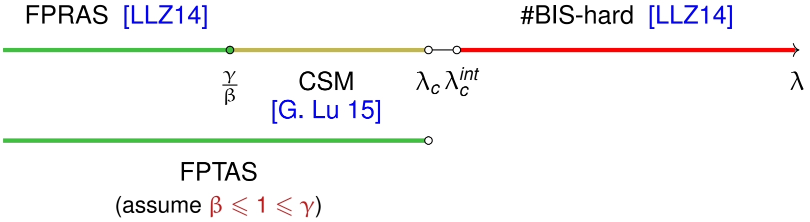

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Instead of worst case configurations in SSM, we only allow partial configurations that are dominated by the product measure of isolated vertices ($p_v \leqslant \frac{\lambda}{1+\lambda}$).

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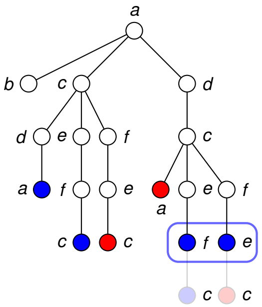

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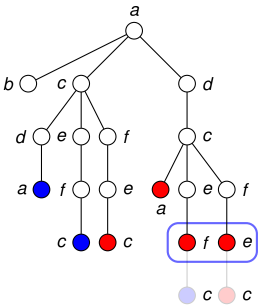

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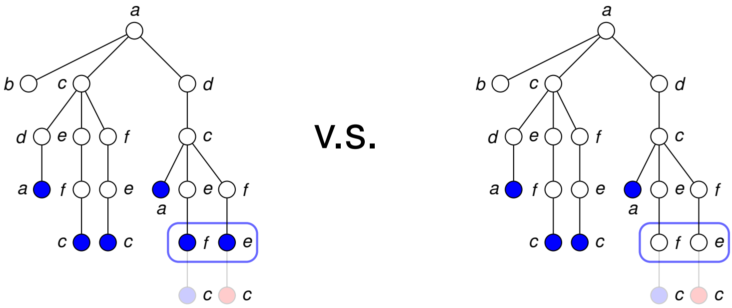

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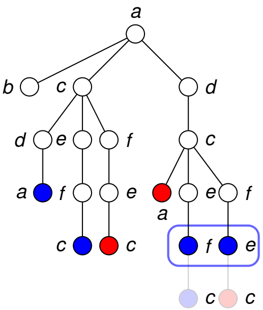

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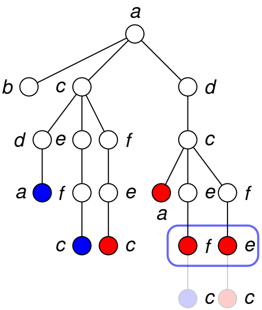

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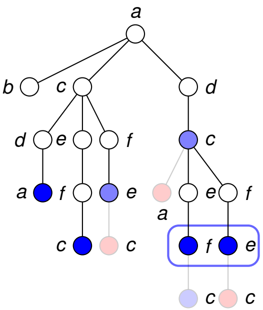

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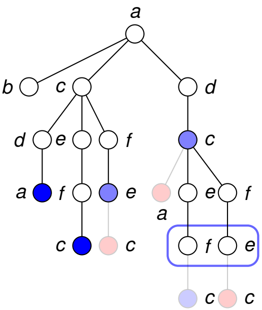

CSM $\Rightarrow$ SSM

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If $\beta > 1$, then pruning fails.

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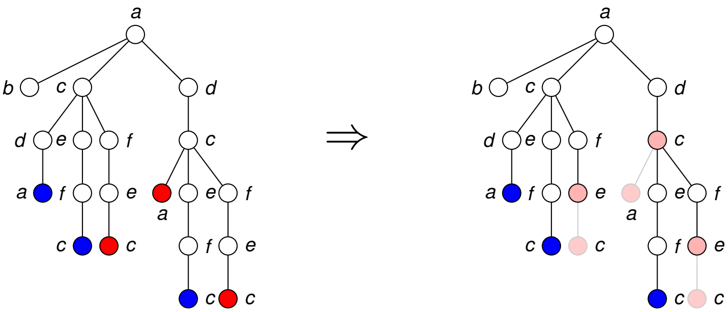

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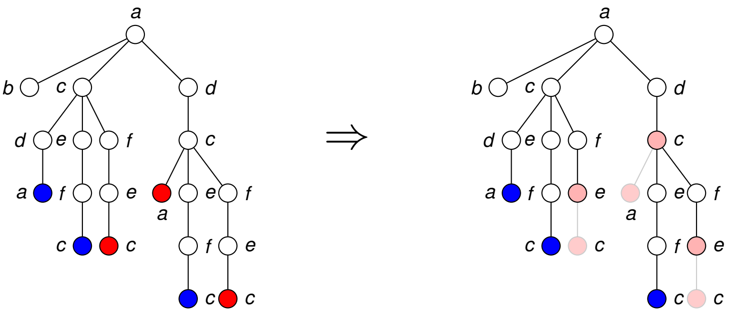

However, if $\lambda_v \leqslant \lambda_c$, then $p_v \leqslant \frac{\lambda}{1+\lambda}$ for any graph G.

FPTAS without SSM?

The Exact Threshold?

Our result is tight up to an integrality gap.

However, neither λ_c nor λ_c^{int} is the right bound.

- There exists a small interval beyond λ_c where FPTAS still exists.
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$$Z_G(\beta) = \sum_{\sigma:V\to\{0,1\}} \beta^{mono(\sigma)}$$

Exact evaluation of $Z_G(\beta)$:

- #**P**-hard unless $\beta = 0, \pm 1, \pm \mathtt{i}$. [Jaeger, Vertigan, Welsh 90]

Lemma (Fuiji, Morimae 13)

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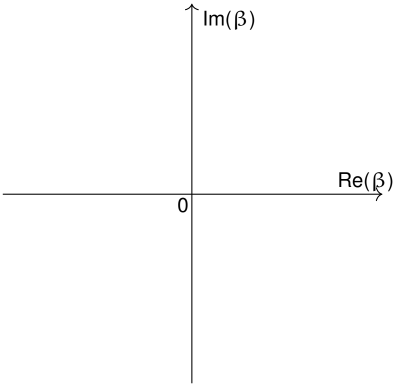

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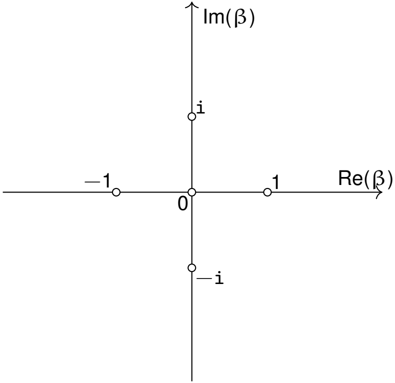

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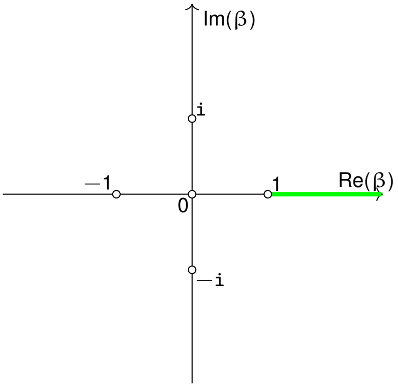

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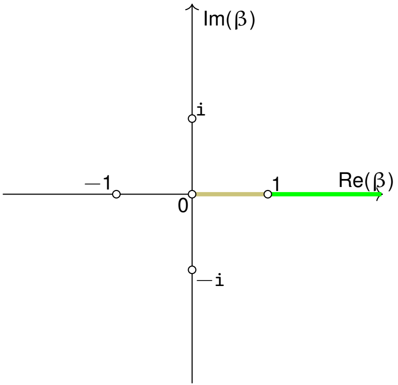

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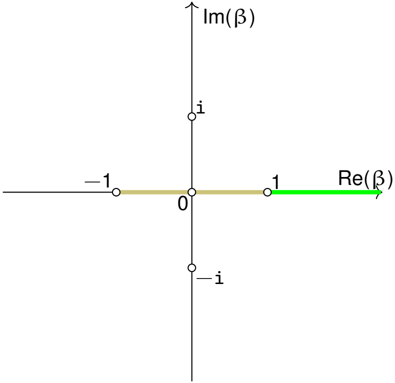

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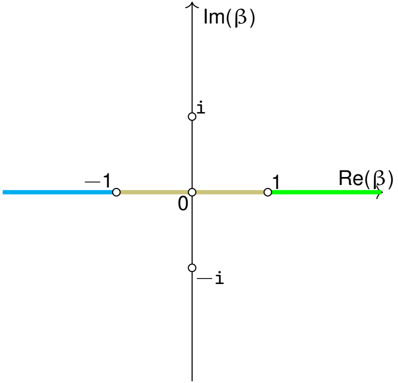

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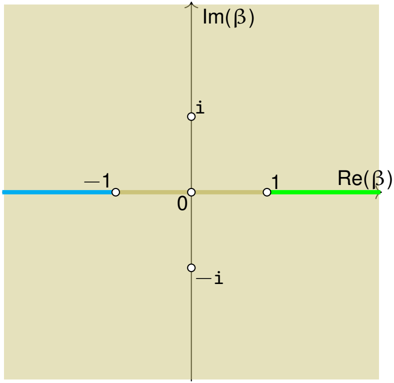

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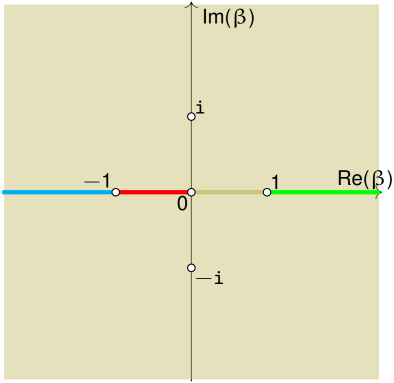

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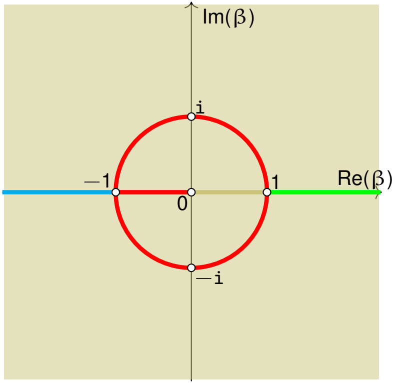

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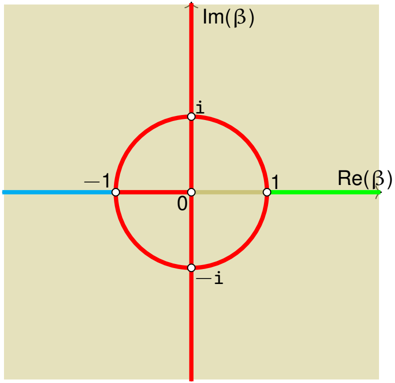

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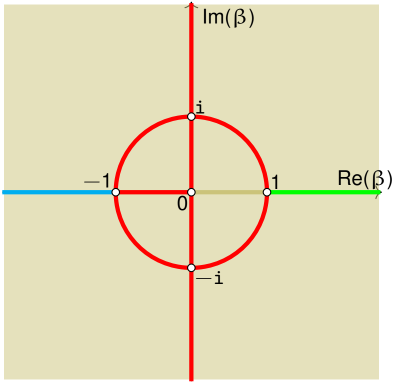

- If $\beta = re^{\mathrm{i}\theta}$ where $\theta = \frac{p\pi}{2q}$, p and q are two co-prime positive integers and p is odd, #**P**-hard. [GG14]

#P-hardness

If $Z_G(\beta) = 0$, even the approximation requires the exact answer.

We relax our problem so that if $Z_G(\beta) = 0$, we accept any return.

Our hardness results hold for these relaxed versions.

We reduce #MINIMUM CARDINALITY (s, t)-CUT [Provan, Ball 83] to approximating $|Z_G(\beta)|$ for any $\beta \in (-1, 0)$.

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  Notice that $f(x)$ is a linear function in x.
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Bisection with an Oracle of Approximating Norms

The oracle returns $|f(x)|$ up to some constant K. Call the approximation $g(x)$. We recursively shrink the interval containing x_0.

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- At least one of the cases is true, so we can shrink the interval by $\frac{2}{3}$.

Divide the interval into more subintervals so that we don't need an exact evaluation of $|f(x)|$ at x_0.

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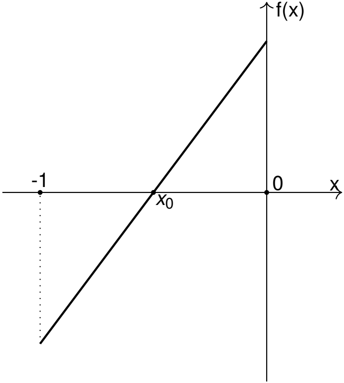

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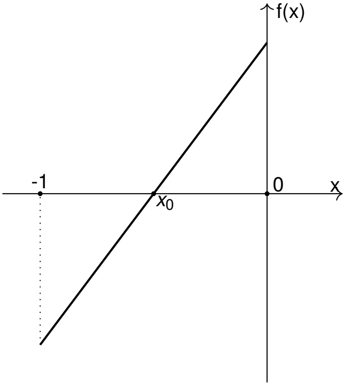

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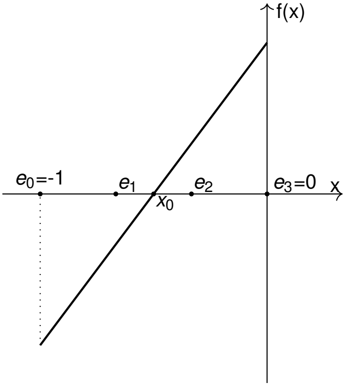

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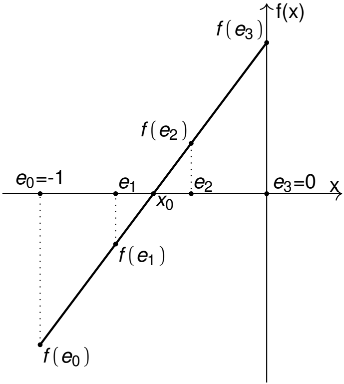

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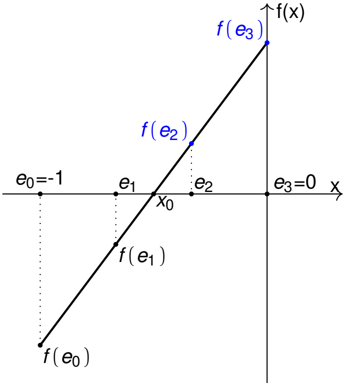

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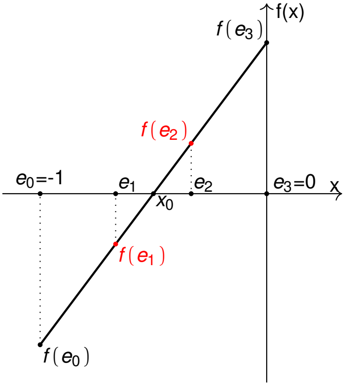

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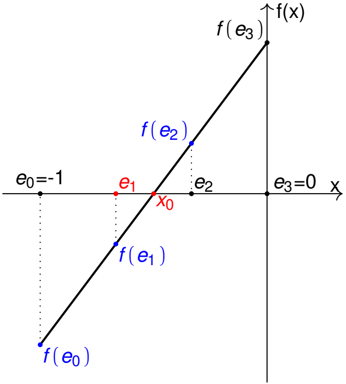

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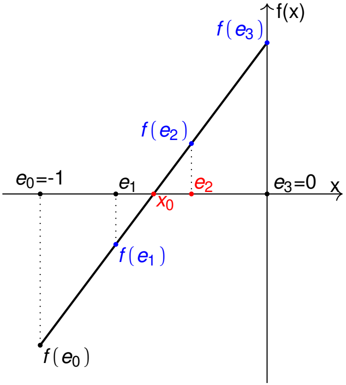

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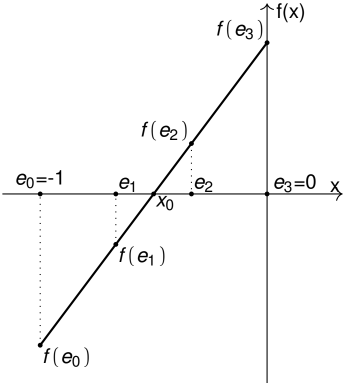

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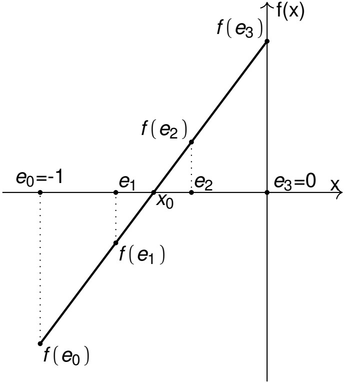

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Complex Ising with Fields

Edge weight β, external field λ:

$$Z_G(\beta;\lambda) \;=\; \sum_{\sigma:V\to\{0,1\}} w(\sigma)$$

where $w(\sigma) = \beta^{m(\sigma)}\lambda^{c_1(\sigma)}$, $m(\sigma)$ is the number of monochromatic edges under σ, and $c_1(\sigma)$ is the number of "blue" vertices.

Theorem

Let β and λ be two roots of unity. Then the following holds:

- If $\beta = \pm 1$, or $\beta = \pm\mathtt{i}$ and $\lambda \in \{1, -1, \mathtt{i}, -\mathtt{i}\}$, $Z_G(\beta;\lambda)$ can be computed exactly in polynomial time.
- Otherwise $|Z_G(\beta;\lambda)|$ is #**P**-hard to approximate.

Hardness results of approximating $\arg(Z_G)$:

- Given an oracle computing the sign of Tutte polynomial at $(-e^{2\pi i/5}, -e^{8\pi i/5})$ over planar graphs, all problems in **BQP** can be solved classically in polynomial time.

  [Bordewich, Freedman, Lovász, Welsh 05]

- To determine this sign is #**P**-hard over general graphs.

  [Goldberg, G. 14]

Open Questions

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Ferro 2-spin systems:

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Thank You!