Finitary Coloring

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(Proper) $q$-coloring of graph $G$: labelling of vertices with colors $1, \ldots, q$ giving adjacent vertices different colors.

E.g. $\mathbb{Z}^2$
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Can we color with no "central authority" - each vertex is an identical "independent agent"?
Random stationary colorings?

I.i.d. colors clearly impossible.
Approximate independence?
Trivial tails? Decay of correlations? Mixing?

E.g. $\mathbb{Z}^2$

2-coloring (but long-range correlations)

with prob 1/2

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**Definition:** a process \( X=(X_v)_{v \in \mathbb{Z}^d} \) is a **finitary factor of an iid process** (ffiid) if:

1. \( X = f(U) \) for \( U=(U_v) \) an iid process on \( \mathbb{Z}^d \)
2. \( f \) is translation-equivariant:
   \[ f(T \cdot u) = T \cdot f(u) \] for any translation \( T \)
3. \( f \) is finitary:
   \( X_0 \) is determined by \( U \) on \([-R,R]^d\) for some random \( R < \infty \),
   (the **coding radius**)

I.e.: \( \exists \ r = r(u) \) s.t.

\[ u = u' \text{ on } [-r,r]^d \Rightarrow f(u)_0 = f(u')_0. \]

\( R := r(U) \)
Question: does there exist an ffiid coloring? If so, how small can we make coding radius $R$?

Not with $q=2$ colors on $\mathbb{Z}^d$.
Not with $R \equiv 0$ (i.i.d.).

Application:
Network of machines.
Colors represent updating schedules/communication frequencies (neighbors must not conflict).
Can the machines choose colors locally, in distributed fashion? How locally?
E.g.

∃ an i.i.d 4-coloring of $\mathbb{Z}^2$ with $P(R>r) \leq e^{-cr}$:

Label each vertex black/white indep. w.p. $\frac{1}{2}$

Percolation theory:

$\frac{1}{2} < p_c^{\text{site}}(\mathbb{Z}^2) \Rightarrow$ black/white clusters finite and diam(cluster at 0) has exp tails

Checkerboard

white clusters in red/blue
black clusters in green/yellow

(starting from NE corner)
∃ an ffl iid 4-coloring of $\mathbb{Z}^2$ with $P(R>r) \leq e^{-cr}$:

Label each vertex black/white indep. w.p. $\frac{1}{2}$

Percolation theory:
$\frac{1}{2} < p_{c_{\text{site}}}^\mathbb{Z}^2 \Rightarrow$ black/white clusters finite and diam(cluster at 0) has exp tails

Checkerboard white clusters in red/blue, black clusters in green/yellow

(isometry-equivariant version: use argmax of iid $U[0,1]$ rvs)
Can we do better?  $R$ bounded?

Other $d$?

Other numbers of colors?
**Theorem:** For $\mathbb{Z}^d$, $d \geq 2$: $d=1$

\[ \exists \text{ ffiid 3-coloring with } P(R>r) < r^{-a} \]

Any ffiid 3-coloring has $E(R^2) = \infty$ (power law)

\[ \exists \text{ ffiid 4-coloring with } P(R>r) < 1/c_r \]

Any ffiid $q$-coloring has $P(R>r) > 1/c_r$ (tower law)

$(a, c, C \in (0, \infty)$ depending on $d, q)$
Proofs...
3-coloring $\mathbb{Z}^2$ with power law tails:

Draw / or \ w.p. $\frac{1}{2}$ in each square
3-coloring $\mathbb{Z}^2$ with power law tails:

On even sub-lattice, see critical bond percolation (clusters finite, power law tails)

On odd sub-lattice, see dual perc. config.

Each cluster is surrounded by a cluster and vice versa. We’ll give each cluster a color.
Adjacency graph of clusters:

\[\text{surrounds}\]
Adjacency graph of clusters:

- Color red w.p. $\frac{1}{2}$
- Delete red if parent also red
Adjacency graph of clusters:
Adjacency graph of clusters:

fill in layers

Russo-Seymour-Welsh

$P(R > r) < r^{-a} \quad (a \ll 1)$
3-coloring $Z^d$ with power law tails:

Hierarchical construction of partition of $Z^d$ with tree structure, power tails

Color each cell with a checkerboard:
Each “special” cell chooses a checkerboard colouring, tries to force descendants to use it.
Proof of lower bound $E(R^2) = \infty$
for 3-coloring on $\mathbb{Z}^2$:
in fact, any stationary 3-coloring
has slowly decaying (power law) correlations.

0 2 1 2 0 1 6 5 4 5 6 7
2 1 0 1 2 0 5 4 3 4 5 6
1 2 1 0 1 2 4 5 4 3 4 5
2 0 2 1 2 0 5 6 5 4 5 6
colouring

height change around contour must be 0
height change around contour must be 0

Lemma: \((Y_i)_{i \in \mathbb{Z}}\) ±1-valued, stationary, right-tail-trivial, not deterministic
\[ \limsup_{n \to \infty} \text{Var} \sum_{1}^{n} Y_i = \infty \text{ a.s. } \]

fast decay of correlations \(\Rightarrow\) changes along sides approx independent

\(\Rightarrow\) contradiction.
Tower colouring on \(\mathbb{Z}\) with **some** \# of colors:

Reduction \(2^n\)-labelling \(\rightarrow\) \((2n+1)\)-labelling

(essentially Cole, Vishkin, 1986)

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1st diff. digit

\( (> , 2) \) \( (< , 1) \) \( (> , 2) \) \* \* \( (> , 3) \)

Get a color-clash only where original sequence had one.

Doing this \(k\) times, starting from i.i.d., get 
\( P(\text{clash}) < 1/\text{tower}(ck) \), \(6+1\) colors, \(R \leq \lfloor k/2 \rfloor\).
“Stitch together” these almost-colorings for different $k$ to get $\text{ffiid}$ 6-coloring of $\mathbb{Z}$

\[ \Rightarrow 6^d\text{-coloring on } \mathbb{Z}^d \]

Reduce # colors to degree+1 by elimination:

\[
\begin{array}{cccccccc}
5 & 3 & 6 & 2 & 1 & 6 & 2 & 1 \\
5 & 3 & 1 & 2 & 1 & 3 & 2 & 1 \\
5 & 3 & 1 & 2 & 1 & 3 & 2 & 1 \\
5 & 3 & 1 & 2 & 1 & 3 & 2 & 1 \\
\end{array}
\]

\[\downarrow \quad \downarrow \quad \downarrow \quad \downarrow \quad \downarrow \quad \downarrow \quad \downarrow \quad \downarrow \]

\[
\begin{array}{cccccccc}
5 & 3 & 1 & 2 & 1 & 3 & 2 & 1 \\
5 & 3 & 1 & 2 & 1 & 3 & 2 & 1 \\
5 & 3 & 1 & 2 & 1 & 3 & 2 & 1 \\
5 & 3 & 1 & 2 & 1 & 3 & 2 & 1 \\
\end{array}
\]
Reduction to 4 colors on $\mathbb{Z}^d$: long-range colorings, maximal indep sets, local modifications $\Rightarrow$ tower-ffiid 2-labelling with bounded clusters.
Proof of lower bound \( P(R>r) > 1/\text{tower}(Cr) \)

**Key step:** (essentially M. Naor 1991; arguably Ramsey 1930)

For \((U_i)\) i.i.d., \(f: R^r \rightarrow \{1, \ldots, q\}\),

\[
\mathbb{P}[f(U_1, \ldots, U_r) = f(U_2, \ldots, U_{r+1})] > \frac{1}{2^{2^2 \cdots (4q)}}
\]

(essentially tight!)
Proof that for \((U_i)\) i.i.d. and \(f: \mathbb{R}^r \rightarrow \{1, \ldots, q\}\),

\[
P[f(U_1, \ldots, U_r) = f(U_2, \ldots, U_{r+1})] > 0:
\]

Induction on \(r\). \(r = 1\) easy (i.i.d.).

\[r \geq 2:\]

\[S(u_1, \ldots, u_{r-1}): = \{a: f(u_1, \ldots, u_{r-1}, U_r) = a \text{ wpp}\}.
\]

\(S\) takes \(\leq 2^q\) values; induction \(\Rightarrow S(U_1, \ldots, U_{r-1}) = S(U_2, \ldots, U_r)\) wpp.

So \(\exists a, A \text{ s.t. wpp:}\)

\[
S(U_1, \ldots, U_{r-1}) = A = S(U_2, \ldots, U_r) \\
\cup \\
f(U_1, \ldots, U_r) = a = f(U_2, \ldots, U_{r+1})
\]
Beyond colouring:

A shift of finite type is a subset of \( \{1,...,q\}^\mathbb{Z}_d \) determined by insisting that all \( k \)-boxes lie in some given \( A \subset \{1,...,q\}^{[0,k]^d} \).

E.g.: q-coloring on \( \mathbb{Z} \): \( A = \{(x,y): x \neq y\} \)
Theorem: For $d=1$ and any shift of finite type $S$, either:

1. there is no ffiid process in $S$ ("periodicity" obstruction) (e.g. 2-colouring)

2. $\exists$ ffiid process in $S$ with $P(R>r) < 1/\text{tower}(cr)$
   any ffiid process in $S$ has $P(R>r) > 1/\text{tower}(Cr)$
   (e.g. 3-colouring)

or

3. some constant sequence lies in $S$
   (so ffiid with "$R=0"$
   (e.g. $q=1$, $S=\{1\}^\mathbb{Z}$)
For a shift of finite type in $d \geq 2$, can have:

No ffiid process (e.g. 2-col)

Power law ffiid process (e.g. 3-col)

Tower law ffiid process (e.g. 4-col)

Constant ($R=0$) process possible
   (e.g. "no restriction")

Q: Is any other behaviour possible?
Beyond finitary factors: Process $X = (X_i)_{i \in \mathbb{Z}}$ is a $k$-block factor if $X_i = g(U_{i+1}, \ldots, U_{i+k})$, $(U_i) \text{ iid}$ (Ffiid process with $R \leq k \iff (2k+1)$-block factor).

Theorem $\Rightarrow$ no block factor colourings.

Stationary process $X$ is $k$-dependent if

$(\ldots, X_{-2}, X_{-1}) \perp \! \! \perp (X_k, X_{k+1}, \ldots)$

$k$-block factor $\Rightarrow$ stationary, $(k-1)$-dependent $\iff$? (Ibragimov, Linnik, 1965)

No! (Aaronson, Gilat, Keane, de Valk, 1989)

Longstanding Q: “Natural” counterexample?

Coloring leads to an answer!

Thm (H., Liggett): $\exists$ 1-dependent 4-coloring!