Glauber Dynamics of Lattice Triangulations on Thin Rectangles

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Lattice triangulations: basic facts

Definition
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\begin{center}
\begin{tikzpicture}
\draw (0,0) grid (5,4);
\draw (0,0) -- (4,4);
\draw (0,1) -- (4,3);
\draw (0,2) -- (4,2);
\end{tikzpicture}
\end{center}
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• $\Omega(m, n)$ the set of all triangulations of $R_{m,n}$
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- $m = 1$: 
  $\#\Omega(1, n) = \binom{2n}{n}$. Equivalence with lattice paths

(a) A one dimensional lattice triangulation

(b) The associated lattice path
An important difference w.r.t. spin systems

- The middle point of each (random) edge is a given (deterministic) point in the half-integer lattice;

- Assigning an edge $\sigma_x \iff$ assigning a “spin $s_x$”.

- For a spin system on a graph interaction is **local**: the law of $s_x$ is determined given the neighbors.

- An edge $\sigma_x$ has 4 neighboring edges whose midpoints can be, however, **very far** from $x$.

- Lack of locality/geometry.
Sampling lattice triangulations

Flip moves: an edge is \textit{flippable} if it is the diagonal of a parallelogram.
Sampling lattice triangulations

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![Diagram](image)

Flip graph on $\Omega(m, n)$ is **connected**.
Sampling lattice triangulations

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Flip graph on $\Omega(m, n)$ is \textbf{connected}.

\textbf{Markov chain reversible w.r.t. uniform distribution:}
- pick a midpoint $x$ \textbf{u.a.r.}
- flip $\sigma_x$ with $\text{Prob} = 1/2$ if flippable.
Weighted triangulations and Glauber dynamics

Consider the Gibbs distribution on $\Omega(m, n)$

$$
\mu(\sigma) = \frac{\lambda |\sigma|}{Z}, \quad |\sigma| = \sum_{x \in \Lambda_{m,n}} |\sigma_x|
$$

where $|\sigma_x| = \|\sigma_x\|_1$. 
Weighted triangulations and Glauber dynamics

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where $|\sigma_x| = \|\sigma_x\|_1$.

**Glauber chain**: pick u.a.r. a midpoint $x \in \Lambda_{m,n}$. If the edge $\sigma_x$ is flippable to edge $\sigma'_x$ then flip it with probability

$$\frac{\mu(\sigma')}{\mu(\sigma')} + \mu(\sigma) = \frac{\lambda |\sigma'_x|}{\lambda |\sigma'_x| + \lambda |\sigma_x|}.$$

Reversible w.r.t. $\mu$. 

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Simulations suggest a phase transition (here $n = m = 50$):

- $\lambda = 1$: $T_{\text{mix}} = \Theta(mn(n+m))$
- $\lambda = 1.1$: $T_{\text{mix}} = \exp(\Omega(mn(n+m)))$
- $\lambda = 0.9$: $T_{\text{mix}} = \text{poly}(m,n)$
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Conjecture
Main results for any $m, n$

Theorem (Rapid mixing for small $\lambda$)

There exists $\lambda_0 > 0$ such that, for all $\lambda < \lambda_0$ and any possible set of constraint edges, $T_{\text{mix}} = O(mn(m + n))$. 
Main results for any $m, n$

**Theorem (Rapid mixing for small $\lambda$)**

There exists $\lambda_0 > 0$ such that, for all $\lambda < \lambda_0$ and any possible set of constraint edges, $T_{\text{mix}} = O(mn(m+n))$.

**Theorem (Slow mixing for $\lambda > 1$)**

For all $\lambda > 1$ and without constraint edges

\[ T_{\text{mix}} \geq \exp(c(m+n)). \]
Rapid mixing for small $\lambda$

Path coupling (Bubley-Dyer 1997) + exponential metric [inspired by S. Greenberg, A. Pascoe, D. Randall ’09].

**Exponential metric:** Fix $\alpha > 1$, and for $\sigma, \tau$ differing only at $x$ set

$$\Delta(\sigma, \tau) = \begin{cases} 
\alpha^2 - 1 & \text{if } |\sigma_x| = |\tau_x| = 2 \text{ (unit diagonals)} \\
|\alpha^{\sigma_x} - \alpha^{\tau_x}| & \text{otherwise}.
\end{cases}$$
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\end{cases}
$$

**Lemma**

*For $\lambda < \lambda_0 = 1/8$, $\alpha = 8$, there is a coupling such that*

$$
\mathbb{E}_{\sigma, \tau} [\Delta(\sigma', \tau')] \leq \Delta(\sigma, \tau) \left(1 - \frac{1}{2|\Lambda_{n,m}|}\right).
$$
Torpid mixing for $\lambda > 1$

**Definition (Exponential Bottleneck)**

A set $A \subset \Omega(m, n)$ such that $\mu(A) \leq 1/2$ and

$$\frac{\mu(\partial A)}{\mu(A)} \leq e^{-c(m+n)}.$$

Here $\partial A = \{(\sigma, \sigma') : \sigma \in A, \sigma' \notin A, \sigma \leftrightarrow \sigma'\}$.

**Lemma**

*Exponential bottleneck* $\Rightarrow$

$$T_{\text{mix}} = \Omega(\exp[c(n + m)]), \quad c > 0.$$
The Herringbone bottleneck

- $A$ is the set of all Herringbone triangulations.
- Orientation in 1D layers oscillates $+/-$.

- $\sigma \in \partial A$ iff an internal edge is vertical.
- For $\lambda > 1$, $\sigma \in \partial A$ is exponentially unlikely (in $\max(n, m)$) given $A$. 
Optimal bounds on $T_{\text{mix}}$ for $m = 1$

**Theorem**

- $\lambda < 1$: $T_{\text{mix}} = \Theta(n^2)$ (*path coupling + exponential metric*)
- $\lambda > 1$: $T_{\text{mix}} = \exp(\Omega(n^2))$ (*1 layer bottleneck*)
- $\lambda = 1$: $T_{\text{mix}} \sim n^3 \log n$ (*e.g. coupling, D.B. Wilson ’01*)
Optimal bounds for thin rectangles ($m = \text{const}, \; n \gg 1$)

Theorem

- $\lambda < 1$: $T_{\text{mix}} = \Theta(n^2)$
- $\lambda > 1$: $T_{\text{mix}} = \exp(\Omega(n^2/m))$
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**Theorem**

- \(\lambda < 1:\; T_{\text{mix}} = \Theta(n^2)\)
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**A poly(n) bound on** \(T_{\text{mix}}\) **for** \(\lambda = 1\) **is still missing**
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- **Lower bound** for \(\lambda > 1\): (slightly) improved version of the Herringbone Bottleneck.
Optimal bounds for thin rectangles \((m = \text{const}, \ n \gg 1)\)

**Theorem**

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- **Lower bound** for \(\lambda > 1\): (slightly) improved version of the Herringbone Bottleneck.

- **Upper bound** for \(\lambda < 1\):
  - In time \(O(n^2)\) the chain enters the set \(\tilde{\Omega}\) of “short” \((O(\log n))\) triangulations. Main tool: Lyapunov function (A. Stauffer ’15).
  - Mixing time bounds \(O(n^{(1+o(1))})\) of restricted chain in \(\tilde{\Omega}\) via Log-Sobolev bounds + improved canonical paths arguments.
Exponential tails of edge length ($m$ fixed).

**Lemma (No constraint edges)**

Fix $\lambda < 1$. There exist $c_1, c_2$ such that, for any $t \geq c_1 n^2$ and any $\ell \geq 1$,

$$\sup_{\sigma} \sup_{x \in \Lambda_{n,m}} \mathbb{P}_\sigma(|\sigma_x(t)| \geq \ell) \leq c_1 \exp(-c_2 \ell)$$

**Lemma (Constraint edges $\tau$)**

Fix $\lambda < 1$. Let $\bar{\sigma}_x$ the ground state of $\sigma_x$ in the presence of constraint edges $\tau$. There exist $c_1, c_2$ such that, for any $t \geq c_1 n^2$, any $\ell \geq 1$ and any $x$,

$$\sup_{\sigma} \mathbb{P}(\bigcup_y \{\sigma_y(t) \cap \bar{\sigma}_x \neq \emptyset\} \cap \{|\sigma_y(t)| \geq |\bar{\sigma}_x| + \ell\}) \leq c_1 \exp(-c_2 \ell)$$
Coupling in presence of constraint edges

- Let $R$ be a $k \times m$ rectangle inside $R_{n,m}$.
- Let $\tau, \tau'$ be constraint edges not intersecting $R$.

**Lemma**

Fix $\lambda < 1$ and $m$. There exists $c$ and $k_0$ together with a coupling of $\mu^\tau, \mu^{\tau'}$ such that, if $k \geq k_0$, with probability at least $1 - \exp[-ck]$ there exist $\epsilon k$ common vertical crossings of unit edges in $R$. 
Back to thin rectangles: $T_{\text{mix}} = O(n^2)$ for any $\lambda < 1$

Step 1: Burn-in phase.

For some $T = c(\lambda)n^2$, uniformly in the initial condition and w.h.p.

$$\sigma(t) \in \tilde{\Omega}, \quad t \in [T, T + n^{10}].$$

$\tilde{\Omega}$ is the set of triangulations with at most $O(\log n)$ edges.

- The restricted chain to $\tilde{\Omega}$ is irreducible with reversible measure $\tilde{\mu} := \mu(\cdot \mid \tilde{\Omega})$.

- Because of structural properties $\tilde{\mu}, \mu$ well coupled.

- Sufficient to prove $\tilde{T}_{\text{mix}} = o(n^2)$. 
Step 2: spatial mixing in $\tilde{\Omega}$

Lemma (Spatial mixing)

The relative density of the marginals on the left block (light gray) of $\tilde{\mu}$ conditioned on two arbitrary (short) triangulations in the right block (dark gray) is exponentially (in $|J_c|$) close to one if $|J_c| = \Omega(\text{polylog}(n))$. 

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Step 3: Log-Sobolev constant in $\tilde{\Omega}$

**Figure:** The rectangle $\Lambda$ decomposed into two almost-halves $\Lambda_1, \Lambda_2$ with $\Lambda_1 \cap \Lambda_2 \equiv \Omega(\log n) \times m$ rectangle.

Spatial mixing implies **quasi-factorization** of the entropy:
Step 3: Log-Sobolev constant in $\tilde{\Omega}$

Figure: The rectangle $\Lambda$ decomposed into two almost-halves $\Lambda_1, \Lambda_2$ with $\Lambda_1 \cap \Lambda_2 \equiv \Omega(\log n) \times m$ rectangle.

Spatial mixing implies quasi-factorization of the entropy:

$$\text{Ent}_\Lambda(f^2) \leq (1 + n^{-\varepsilon})\tilde{\mu} \left[ \text{Ent}_{\Lambda_1}(f^2|\Lambda \setminus \Lambda_1) + \text{Ent}_{\Lambda_2}(f^2|\Lambda \setminus \Lambda_2) \right].$$

Multiscale analysis of the Log-Sobolev constant.
Notation

- Dirichlet form:

\[ \mathcal{E}(f, f) = \frac{1}{2n} \sum_{\sigma, \sigma' \in \tilde{\Omega}} \tilde{\mu}(\sigma) \rho(\sigma, \sigma')(f(\sigma) - f(\sigma'))^2. \]
Notation

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- Entropy:
  \[ \text{Ent}(f^2) = \tilde{\mu}[f^2 \log(f^2/\tilde{\mu}[f^2])]. \]
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• Logarithmic Sobolev constant

\[ c_S(n) := \sup_f \frac{\text{Ent}(f^2)}{E(f, f)}, \]

• \( \tilde{T}_{\text{mix}} \leq C \log n \times c_S(n). \)
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- \( \tilde{T}_{\text{mix}} \leq C \log n \times c_S(n). \)

Theorem

\[ c_S(n) \leq n^{1+o(1)} \Rightarrow \tilde{T}_{\text{mix}} = O(n^{1+o(1)}). \]
High level overview

- Quasi-factorization of the entropy $c_S(2^n) \leq (1 + n - \epsilon) \times 2 \times c_S(n/2)$
- The factor 2 comes from double counting the flips in the overlapping region.
- A random averaging of the location of the overlap block reduces it to $(1 + 1/\log_2 n)$.
- Sloppy notation: boundary edges are there $\Rightarrow c_S(n) \leq \text{const} \times c_S(\text{polylog}(n))$. 

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Quasi-factorization of the entropy ⇒

\[ c_S(2n) \leq (1 + n^{-\epsilon}) \times 2 \times c_S(n/2) \]
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Quasi-factorization of the entropy $\Rightarrow$

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$$\Rightarrow c_S(n) \leq \text{const} \times c_S(\text{polylog}(n)).$$
Bounds on $c_S$ on scale $L_n = \text{polylog}(n)$

- Exponential bound $c_S(L_n) = O(\exp(cL_n))$ not difficult but not sufficient.
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- **Important feature**: on each scale $L_j = 2^{-j} n$ the Log-Sobolev problem involves *constraint edges inherited* from the conditioning on scale $L_{j-1}, \ldots, L_0$. 
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- Straightforward "bootstrapping" i.e.

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  c_S(n) \leq O(c_S(L_n)) = O(\exp(cL_n)) \\
  \Downarrow \\
  c_S(L_n) = O(\exp(O(\text{polylog}(L_n))))
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not feasible.
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  \[
  \downarrow
  \]
  \[
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  \]
  not feasible.
- Need e.g. a $O(poly(L_n))$ bound on $c_S(L_n)$ by different means.
A \textit{poly}(L_n) \text{ upper bound on } c_S(L_n)

- \( c_S(L_n) = O(n \times T_{\text{rel}}(L_n)) \)

- \( T_{\text{rel}}(L_n) \leq \mathcal{C} \) (congestion rate)

\[
\mathcal{C} := \max_{\eta \sim \eta'} \sum_{\sigma, \sigma': \Gamma_{\sigma, \sigma'} \ni (\eta, \eta')} \frac{\mu(\sigma)\mu(\sigma')}{\mu(\eta)p(\eta, \eta')} |\Gamma_{\sigma, \sigma'}|
\]

where, for any \( \sigma, \sigma' \in \tilde{\Omega} \), \( \Gamma_{\sigma, \sigma'} \) is a path in \( \tilde{\Omega} \) from \( \sigma \) to \( \sigma' \).

- Typically \( \mathcal{C} = O(\exp(cL_n)) \). We need \( O(\text{poly}(L_n)) \).
An improved canonical paths argument

- Reversible ergodic Markov chain $X_t$ on $\mathcal{X}$. 

Lemma (Canonical paths with burn-in time)

$$T_{rel} \leq 6T^2\rho + 3C(X^\prime)\rho^2$$
An improved canonical paths argument

- Reversible ergodic Markov chain $X_t$ on $\mathcal{X}$.

- $\mathcal{X}' \subset \mathcal{X}$ be such that for any pair $x, y \in \mathcal{X}'$ it is possible to define a canonical path $\Gamma_{x,y}$ entirely contained in $\mathcal{X}'$. Let $C(\mathcal{X}')$ be the associated congestion rate.

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- Fix time $T$ and let

$$\rho = \min_{x \in \mathcal{X}} \mathbb{P}_x(X_T \in \mathcal{X}')$$
An improved canonical paths argument

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**Lemma (Canonical paths with burn-in time)**

$$T_{rel} \leq \frac{6T^2}{\rho} + \frac{3C(\mathcal{X}')}{\rho^2}$$
Back to thin rectangles

Theorem
Consider the original triangulation chain on $n \times m$ rectangle with (possibly) boundary edges sticking in but not longer than $n/4$. Then

$$T_{\text{rel}} = O(poly(n)).$$

Corollary (The needed $poly(Ln)$ bound)
For the restricted chain on $\tilde{\Omega}$ on $L_n \times m$ rectangle

$$T_{\text{rel}}(L_n) = O(poly(L_n)) = O(polylog(n)).$$
Sketch of proof

Define \( \Omega' \subset \Omega \) as follows:

- any edge does not exceed by more than \( O(\log n) \) its minimal allowed (by the boundary edges) length.
- for any \( x \neq y \), if \( \sigma_y \) crosses the ground state edge \( \bar{\sigma}_x \) at \( x \) then \( |\sigma_y| \leq |\bar{\sigma}_x| + O(\log n) \).

**Lemma**

Fix \( T = cn^2 m \). Then \( \Omega' \) satisfies the hypotheses of the \textit{“canonical paths with burn-in lemma”} with

\[
\rho = \min_{\sigma} \mathbb{P}(\sigma(T) \in \Omega') \geq 1/2
\]

and congestion rate \( C' = O(poly(n)) \) for a suitable choice of canonical paths in \( \Omega' \).
Key feature of the set $\Omega'$

- Pb: Given $\sigma, \eta$ construct path between them.

- In principle, to flip $\sigma_x$ to $\eta_x$ one may need to reshuffle edges in $\sigma$ with midpoints very far from $x$.

- If $\sigma, \eta \in \Omega'$ “very far” is not more than $O(\log n)$.

- It is possible to construct the path by processing the slabs left-to-right without never changing more than 2 slabs at a time.