The cutoff phenomenon for random walk on random directed graphs

Justin Salez

Joint work with C. Bordenave and P. Caputo
Outline of the talk
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1. The cutoff phenomenon for Markov chains
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2. Random walk on directed graphs
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3. Main results
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1. The cutoff phenomenon for Markov chains
2. Random walk on directed graphs
3. Main results
Markov chain mixing

Any Markov chain with irreducible, aperiodic transition matrix $P$ on a finite state space $X$ converges to its stationary law $\pi = \pi P$.

$\exists \text{Distance to equilibrium at time } t$

$D_{tv}(t) := \max_{x \in X} \| P_t(x, \cdot) - \pi(\cdot) \|_{tv}$

Mixing times ($0 < \epsilon < 1$): $t_{mix}(\epsilon) := \min \{ t \geq 0 : D_{tv}(t) \leq \epsilon \}$
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▷ Distance to equilibrium at time $t$:

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**Theorem** (Aldous-Diaconis ‘86). For any fixed \( 0 < \varepsilon < 1 \),

\[ \frac{t_{\text{MIX}}(\varepsilon)}{n \log n} \xrightarrow[n \to \infty]{} 1. \]
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**Theorem** (Diaconis-Fill-Pitman ‘90). For any fixed \( \lambda \in \mathbb{R} \),

\[ D_{TV}(n \log n + \lambda n + o(n)) \]
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**Theorem** (Diaconis-Fill-Pitman ‘90). For any fixed \( \lambda \in \mathbb{R} \),

\[
D_{TV}(n \log n + \lambda n + o(n)) \xrightarrow{n \to \infty} \Phi(\lambda)
\]

with \( \Phi: \mathbb{R} \to (0, 1) \) decreasing from \( \Phi(-\infty) = 1 \) to \( \Phi(+\infty) = 0 \).
Ubiquity of the cutoff phenomenon

Cutoff has been shown to arise in various contexts, including:

- Card shuffling (Aldous, Diaconis, Shahshahani...)
- Birth-and-death chains (Diaconis, Saloff-Coste...)
- Random walks on finite groups (Chen, Saloff-Coste...)
- Glauber dynamics (Levin, Lubetzky, Luczak, Peres, Sly...)
- Random walks on sparse graphs
  - Random regular graphs (Lubetzky, Sly '10)
  - Ramanujan graphs (Lubetzky, Peres '15)
  - Trees (Basu, Hermon, Peres '15)
  - Random graphs with given degrees (Berestycki, Lubetzky, Peres, Sly '15 and Ben-Hamou, S. '15)

Still, this phenomenon is far from being completely understood. In particular, very few results outside the reversible world...
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In particular, very few results outside the reversible world...
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3. Main results
Random walk on a digraph

\[ X = \{1, \ldots, 6\} \]

\[
\begin{array}{cccccc}
0 & 0 & 0 & 1 & 0 & 0 \\
\frac{1}{4} & 0 & \frac{1}{4} & \frac{1}{4} & 0 & \frac{1}{4} \\
\frac{1}{3} & 0 & \frac{1}{3} & 0 & \frac{1}{3} & 0 \\
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- How long does it take for the walk to mix?
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- How long does it take for the walk to mix?
- What does the stationary distribution \( \pi \) look like?
Motivation: ranking algorithms (credit: the opte project)
Random digraph with given degrees (Cooper-Frieze '04)
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**Goal**: generate a random digraph $G$ on $\mathcal{X} = \{1, \ldots, n\}$ with given in-degrees $\{d_x^-\}_{x \in \mathcal{X}}$ and out-degrees $\{d_x^+\}_{x \in \mathcal{X}}$ (equal sum $m$)
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Simulation: $n = 3 \times 1000, (d^+, d^-) = (3, 2), (3, 4), (4, 4)$
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Distribution of the stationary masses \( \{n\pi(x) : x \in \mathcal{X}\} \)
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A glimpse at the eigenvalues
1. The cutoff phenomenon for Markov chains
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3. Main results
Cutoff and profile

Sparse regime: \(2 \leq d \pm x \leq \Delta\) with \(\Delta\) fixed as \(n \to \infty\)

\[
\mu := \frac{1}{m} \sum_{x \in X} d - x \log d + x, \quad \sigma^2 := \frac{1}{m} \sum_{x \in X} (d - x)^2 (-\mu)\]

**Theorem 1 (cutoff):** set \(t_n = \log n \mu\).

\[
\text{D}^2_{\text{tv}}(\lambda t_n + o(t_n)) \xrightarrow{P} \begin{cases} 1 & \text{if } \lambda < 1 \\ 0 & \text{if } \lambda > 1 \end{cases}
\]

**Theorem 2 (profile):** set \(w_n = \sqrt{\sigma^2 \log n} \mu^{3/2} (\gg \log \log n)\).

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Sensitivity to initial condition

Theorem 3 (vertex irrelevance): previous results unchanged if

$$D_{tv}(t) := \min_{x \in X} \|P_t(x, \cdot) - \pi\|_{tv}$$

What about a more spread-out initial law, e.g. $$\nu(x) := d - x$$?

Theorem 4 (constant-time relaxation): for fixed $$t \geq 0$$,

$$\|\nu P_t - \pi\|_{tv} \leq \sqrt{\Delta^2 \rho t} + o\left(1\right)$$

with $$\rho^2 := \frac{1}{m} \sum_{x \in X} d - x d + x \leq \frac{1}{2}$$

Corollary: $$\pi(x)$$ is determined by the local geometry around $$x$$ only!
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- how does all this depend on the choice of the initial vertex?

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\| \nu P^t - \pi \|_{TV} \leq \frac{\sqrt{\Delta}}{2} \varrho^t + o_P(1) \quad \text{with} \quad \varrho^2 := \frac{1}{m} \sum_{x \in X} \frac{d_x^-}{d_x^+} \leq \frac{1}{2}
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\]

**Corollary:** \( \pi(x) \) is determined by the local geometry around \( x \) only!
Distribution of the stationary masses \( \{n\pi(x): x \in \mathcal{X}\} \)
Distribution of the stationary masses $\{n\pi(x) : x \in \mathcal{X}\}$

\[
d_{\mathcal{W}}(\mathcal{L}, \mathcal{L}') = \sup_{f \in \text{Lip}_1(\mathbb{R})} \left| \int_{\mathbb{R}} f \, d\mathcal{L} - \int_{\mathbb{R}} f \, d\mathcal{L}' \right|
\]
Distribution of the stationary masses \( \{n\pi(x): x \in \mathcal{X}\} \)

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\]

**Theorem 5** (asymptotics for the equilibrium masses):

\[
d_{\mathcal{W}} \left( \frac{1}{n} \sum_x \delta_{n\pi(x)}, \mathcal{L} \right) \xrightarrow{\mathbb{P}} 0 \quad \text{as} \quad n \to \infty
\]

L \in \text{P}_1(\mathbb{R}) determined by the recursive distributional equation

\[
d_{\mathcal{W}} + I_{d_{\mathcal{W}}} - I_{\sum_{k=1}^X \delta_{X_k}} = X_1, \quad \text{in which} \quad (X_k)_{k \geq 1} \text{ are i.i.d and independent of } I, \quad \text{and} \quad P(I = x) = d_{\mathcal{W}} + x_m.
\]
Distribution of the stationary masses \( \{n\pi(x) : x \in \mathcal{X}\} \)

\[
d_{\mathcal{W}}(\mathcal{L}, \mathcal{L}') = \sup_{f \in \text{Lip}_1(\mathbb{R})} \left| \int_{\mathbb{R}} f \, d\mathcal{L} - \int_{\mathbb{R}} f \, d\mathcal{L}' \right|
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**Theorem 5** (asymptotics for the equilibrium masses):

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\]

\( \mathcal{L} \in \mathcal{P}_1(\mathbb{R}) \) determined by the recursive distributional equation

\[
\frac{1}{d_\mathcal{I}^+} \sum_{k=1}^{d_\mathcal{I}^-} X_k \overset{\text{law}}{=} X_1,
\]

in which \((X_k)_{k \geq 1}\) are i.i.d and independent of \(\mathcal{I}\), \(\mathbb{P}(\mathcal{I} = x) = \frac{d_\mathcal{I}^+}{m} \).
Thank you!