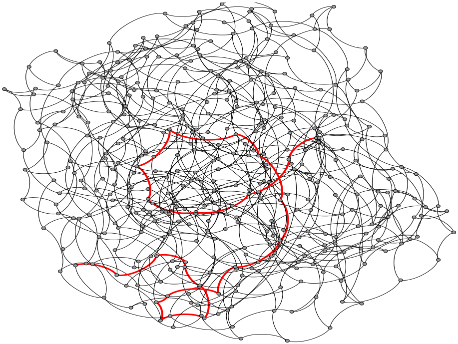
The cutoff phenomenon for random walk on random directed graphs

Justin Salez



Joint work with C. Bordenave and P. Caputo

Outline of the talk

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▷ **Mixing times** $(0 < \varepsilon < 1)$:

$$t_{\mathrm{MIX}}(\varepsilon) := \min\{t \geq 0 \colon \mathrm{D}_{\mathrm{TV}}(t) \leq \varepsilon\}$$

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Theorem (Diaconis-Fill-Pitman '90). For any fixed $\lambda \in \mathbb{R}$,

$$\mathrm{D}_{\mathrm{TV}}(n \log n + \lambda n + o(n)) \xrightarrow[n\to\infty]{} \Phi(\lambda)$$

with $\Phi \colon \mathbb{R} \to (0,1)$ decreasing from $\Phi(-\infty) = 1$ to $\Phi(+\infty) = 0$.

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In particular, very few results outside the **reversible** world...

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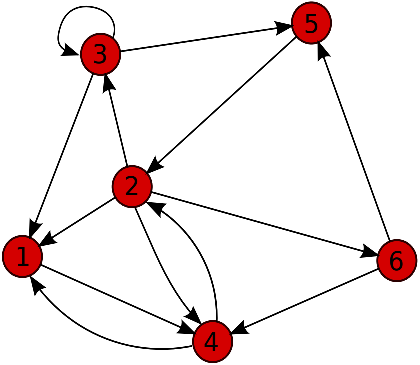
3. Main results

Random walk on a digraph



$$\mathcal{X} = \{1, \ldots, 6\}$$

0	0	0	1	0	0
$\frac{1}{4}$	0	$\frac{1}{4}$	$\frac{1}{4}$	0	$\frac{1}{4}$
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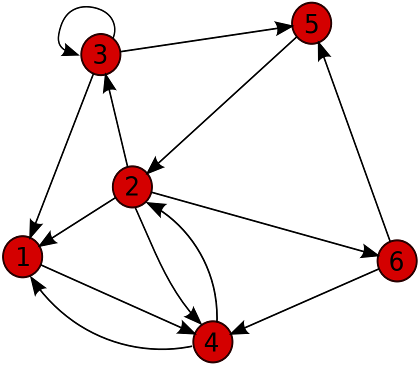

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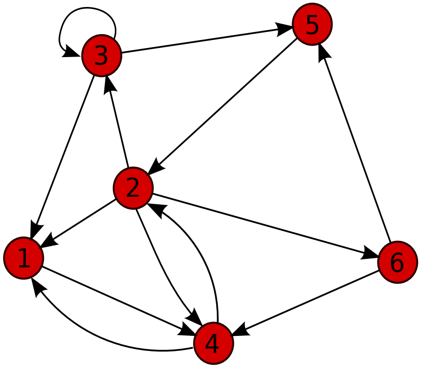

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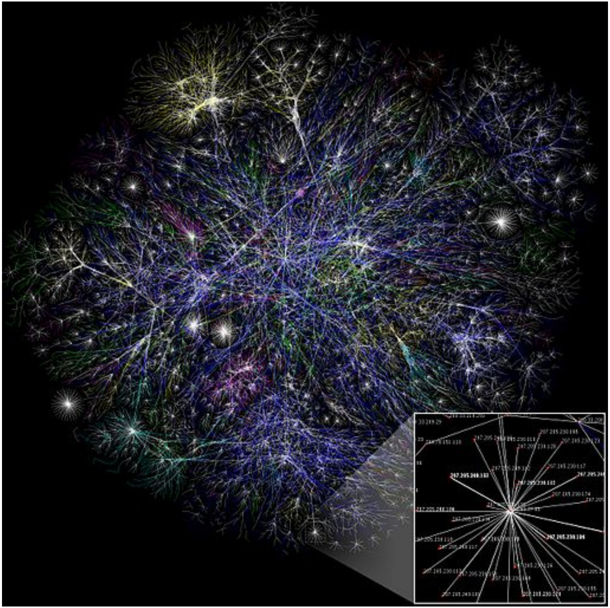
- How long does it take for the walk to mix?
- What does the stationary distribution π look like?

Motivation: ranking algorithms (credit: the opte project)



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Goal: generate a random digraph G on $\mathcal{X} = \{1, \ldots, n\}$ with given in-degrees $\{d_x^-\}_{x \in \mathcal{X}}$ and out-degrees $\{d_x^+\}_{x \in \mathcal{X}}$ (equal sum m)

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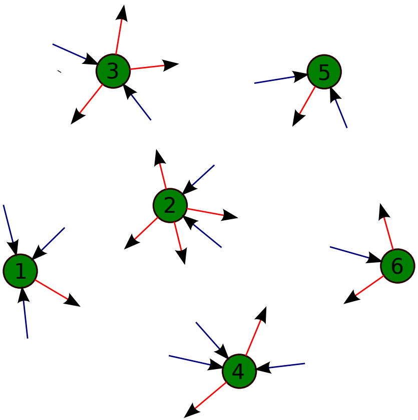

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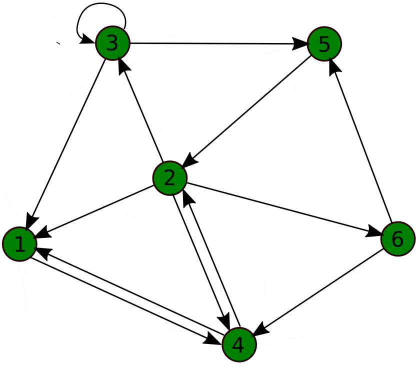

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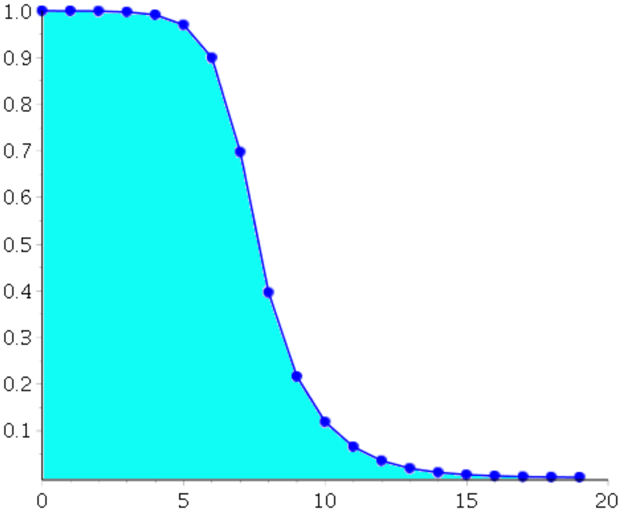

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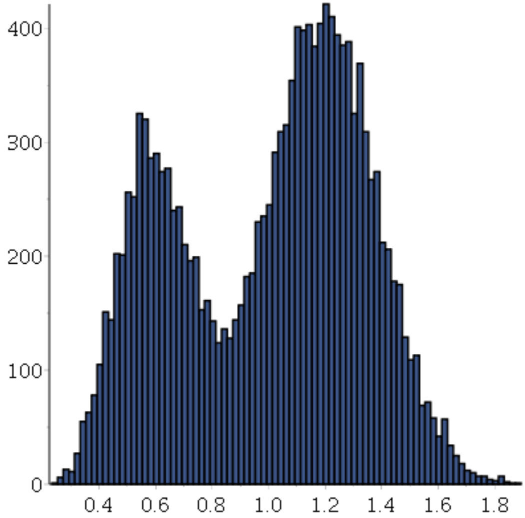
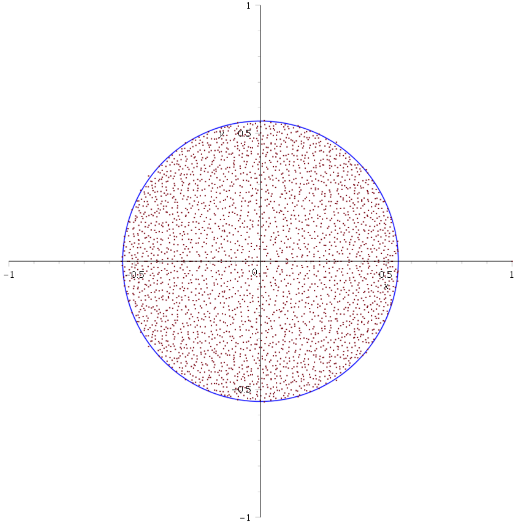


A glimpse at the eigenvalues



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$$\mathrm{D_{TV}}(t_n + \lambda w_n + o(w_n)) \xrightarrow[n\to\infty]{\mathbb{P}} \Phi(\lambda) = \frac{1}{\sqrt{2\pi}} \int_{\lambda}^{\infty} e^{-\frac{u^2}{2}} \, \mathrm{d}u$$

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$$\|\nu P^t - \pi\|_{\mathrm{TV}} \;\leq\; \frac{\sqrt{\Delta}}{2}\varrho^t + o_{\mathbb{P}}(1) \;\text{ with }\; \varrho^2 := \frac{1}{m}\sum_{x\in\mathcal{X}}\frac{d_x^-}{d_x^+} \;\leq\; \frac{1}{2}$$

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Corollary: $\pi(x)$ is determined by the local geometry around x only!

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Theorem 5 (asymptotics for the equilibrium masses):

$$d_{\mathcal{W}} \left(\frac{1}{n} \sum_{x} \delta_{n\pi(x)}, \mathcal{L} \right) \xrightarrow[n \to \infty]{\mathbb{P}} 0.$$

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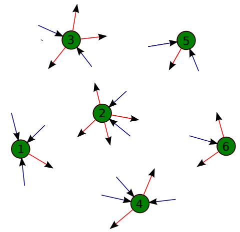
$\mathcal{L} \in \mathcal{P}_1(\mathbb{R})$ determined by the **recursive distributional equation**

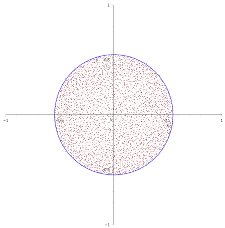
$$\frac{1}{d^+_{\mathcal{I}}} \sum_{k=1}^{d^-_{\mathcal{I}}} X_k \stackrel{\text{law}}{=} X_1,$$

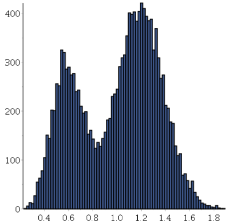
in which $(X_k)_{k\geq 1}$ are i.i.d and independent of $\mathcal{I}$, $\mathbb{P}(\mathcal{I} = x) = \frac{d^+_x}{m}$.

Thank you!

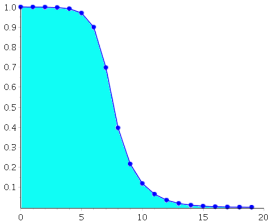


3
5
2
1
6
4





400
300
200
100
0
0.4
0.6
0.8
1.0
1.2
1.4
1.6
1.8



1.0
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0
5
10
15
20