(An algorithmic perspective to the)

Decay of Correlation in Spin Systems

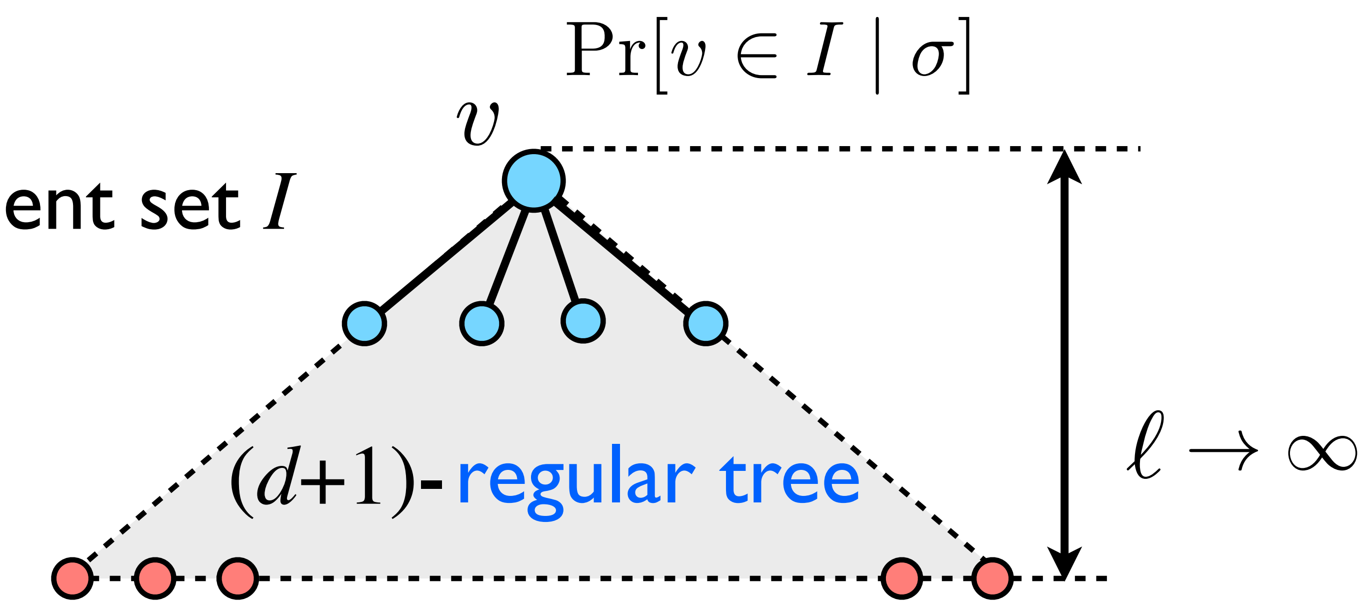
Yitong Yin
Nanjing University

Decay of Correlation

hardcore model:

random independent set I

$$\mu(I) \propto \lambda^{|I|}$$



σ: fixing leaves to be *occupied/unoccupied* by I

Decay of correlation: $\Pr[v \in I \mid \sigma]$ does not depend on σ when $l \to \infty$

$$\text{iff} \quad \lambda \le \lambda_c = \frac{d^d}{(d-1)^{(d+1)}}$$

counting total weights $\lambda^{|I|}$ of all I.S. in graphs with max-degree $\le d+1$

- $\lambda < \lambda_c \Rightarrow$ FPTAS [Weitz 06]
- $\lambda > \lambda_c \Rightarrow$ no FPRAS unless NP=RP

[Sly10] [Galanis Štefankovič Vigoda 12] [Sly Sun 12]

Spin System

undirected graph $G = (V, E)$ fixed integer $q \geq 2$

configuration $\sigma \in [q]^V$

weight: $w(\sigma) = \prod_{\{u,v\} \in E} A(\sigma_u, \sigma_v) \prod_{v \in V} b(\sigma_v)$

$A : [q] \times [q] \to \mathbb{R}_{\geq 0}$ symmetric q×q matrix (symmetric binary constraint)

$b : [q] \to \mathbb{R}_{\geq 0}$ q-vector (unary constraint)

partition function: $Z_G = \sum_{\sigma \in [q]^V} w(\sigma)$

Gibbs distribution: $\mu_G(\sigma) = \frac{w(\sigma)}{Z_G}$

undirected graph $G=(V,E)$ fixed integer $q\geq 2$

configuration $\sigma\in[q]^V$

weight: $w(\sigma)=\prod_{\{u,v\}\in E}A(\sigma_u,\sigma_v)\prod_{v\in V}b(\sigma_v)$

- 2-spin model: $q=2$, $\sigma\in\{0,1\}^V$

$$A=\begin{bmatrix}A_{00} & A_{01}\\ A_{10} & A_{11}\end{bmatrix}=\begin{bmatrix}\beta & 1\\ 1 & \gamma\end{bmatrix}\ \text{edge activities}\qquad b=\begin{bmatrix}b_0\\ b_1\end{bmatrix}=\begin{bmatrix}\lambda\\ 1\end{bmatrix}\ \text{external field}$$

  - hardcore model: $\beta=0,\gamma=1$

$$Z_G=\sum_{\substack{I:\text{ independent}\\ \text{sets in } G}}\lambda^{|I|}$$

  - Ising model: $\beta=\gamma$

- multi-spin model: general $q\geq 2$
  - Potts model:
  - q-coloring: $\beta=0$

$$A=\begin{bmatrix}\beta & & 1\\ & \beta & \\ 1 & \ddots & \\ & & \beta\end{bmatrix}\qquad b=\begin{bmatrix}1\\ \vdots\\ 1\end{bmatrix}$$

Models

- spin systems:
  - Ising model, Potts model, q-coloring
  - hardcore model → hypergraph matchings
- monomer-dimer (NOT a spin system) → hypergraph matchings (generalization)

$$Z_G = \sum_{M\text{: matchings in } G} \lambda^{|M|}$$

- Holant problem defined by the (weighted-)EQ, the At-Most-One constraint, and any binary constraints
- The recursion of marginal probabilities is the same as a recursion on the tree of self-avoiding walks.

correlation decay on trees ⟺ (hopefully) tractability of approximate counting

Gibbs Measure

undirected graph $G=(V,E)$ configuration $\sigma \in [q]^V$

Gibbs distribution: $\mu(\sigma) = \mu_G(\sigma) = \dfrac{w(\sigma)}{Z_G}$

by the chain rule: denoted $V = \{v_1, v_2, \ldots, v_n\}$

$$Z_G = \frac{w(\sigma)}{\mu(\sigma)} = \frac{w(\sigma)}{\prod_{i=1}^{n} \Pr_{X\sim\mu_G}[X_{v_i} = \sigma_{v_i} \mid \forall j < i : X_{v_j} = \sigma_{v_j}]}$$

marginal probability: $\mu_v^\sigma(x) = \Pr_{X\sim\mu_G}[X_v = x \mid X_S = \sigma]$

where $v\in V, x\in[q]$, boundary condition $\sigma\in[q]^S$ on $S\subset V$

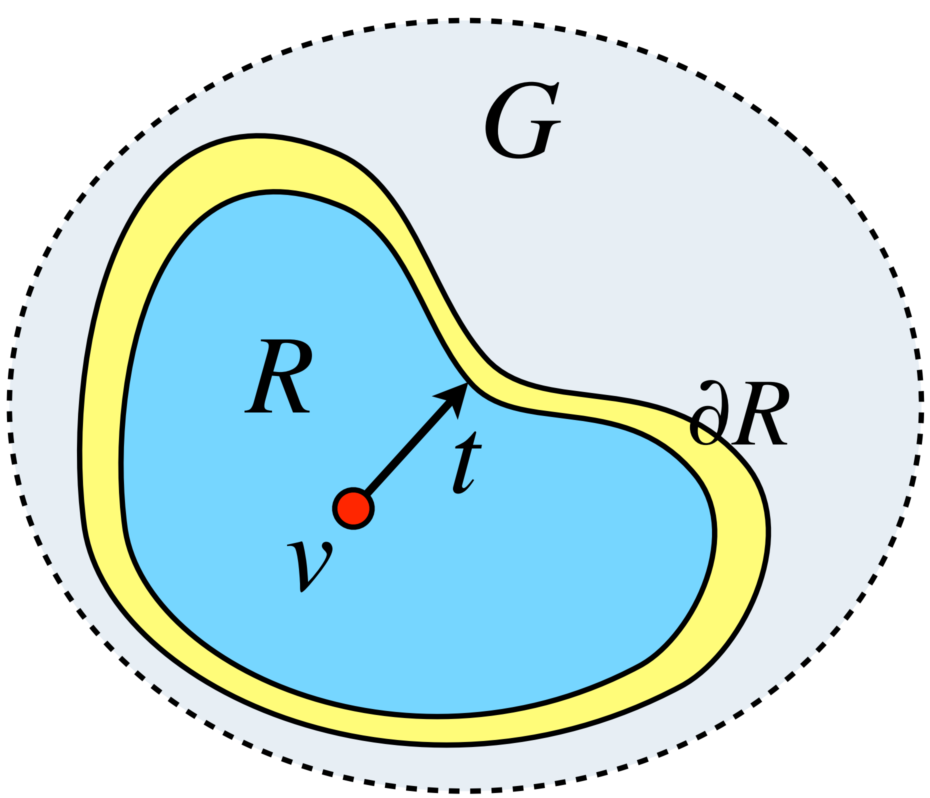
approximately computing $\mu_v^\sigma(x)$ within $\mu_v^\sigma(x) \pm \frac{1}{n}$ in time $\mathrm{Poly}(n)$ ⇒ FPTAS for Z_G

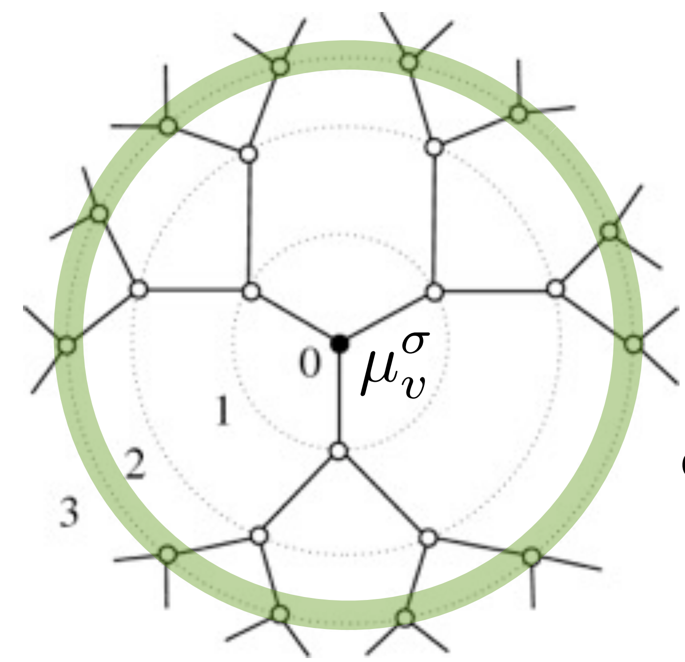
Spatial Mixing (Decay of Correlation)

μ_v^σ : marginal distribution at vertex v conditioning on σ

weak spatial mixing (WSM) at rate $\delta(\ )$:

$$\forall \sigma, \tau \in [q]^{\partial R} : \qquad \|\mu_v^\sigma - \mu_v^\tau\|_{TV} \leq \delta(t)$$





on infinite graphs:

WSM ⟺ uniqueness of infinite-volume Gibbs measure

$\lambda \leq \lambda_c = \frac{d^d}{(d-1)^{(d+1)}}$ ⟺ WSM of hardcore model on infinite $(d+1)$-regular tree

Spatial Mixing (Decay of Correlation)

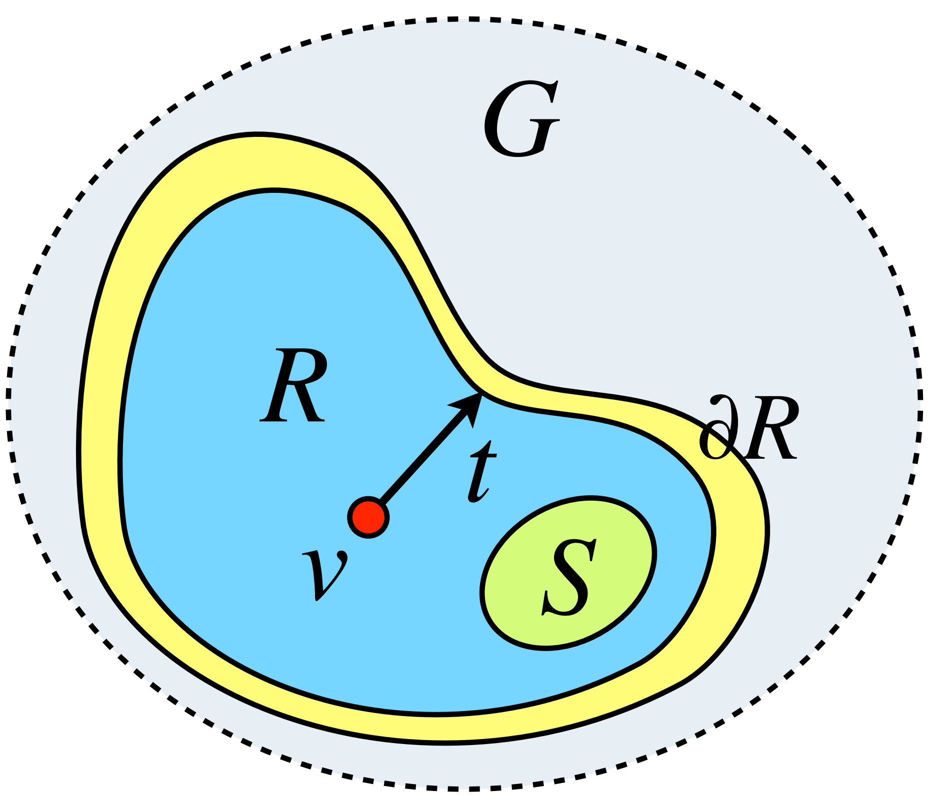
μ_v^σ : marginal distribution at vertex v conditioning on σ

weak spatial mixing (WSM) at rate $\delta(\ )$:

$$\forall \sigma, \tau \in [q]^{\partial R}: \qquad \|\mu_v^\sigma - \mu_v^\tau\|_{TV} \le \delta(t)$$

strong spatial mixing (SSM) at rate $\delta(\ )$:

$$\forall \sigma, \tau \in [q]^{\partial R}, \forall \rho \in [q]^S: \quad \|\mu_v^{\sigma\cup\rho} - \mu_v^{\tau\cup\rho}\|_{TV} \le \delta(t)$$



SSM

marginal prob. $\mu_v^\rho(x)$

is well approximated

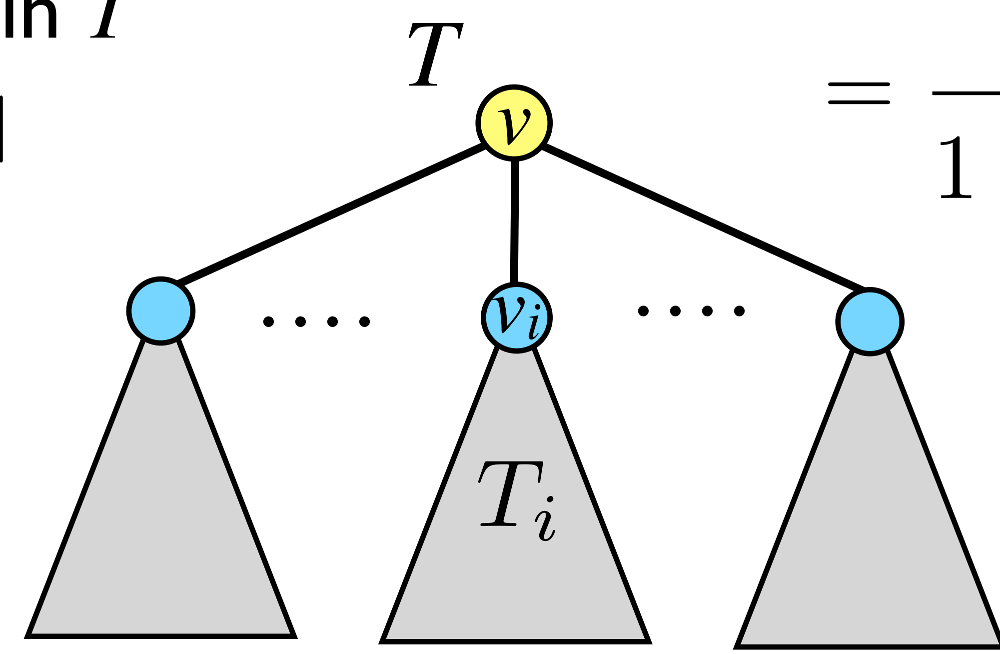
by the *local information*

Tree Recursion

hardcore model:

independent set I in T

$$\mu(I) \propto \lambda^{|I|}$$



$$p_T = \Pr[v \text{ is occupied }]$$

$$= \frac{\lambda \prod_{i=1}^{d}(1-p_i)}{1+\lambda \prod_{i=1}^{d}(1-p_i)}$$

$$p_i = \Pr[v_i \text{ is occupied }]$$ in T_i

occupancy ratio: $R_T = \frac{p_T}{1-p_T}$

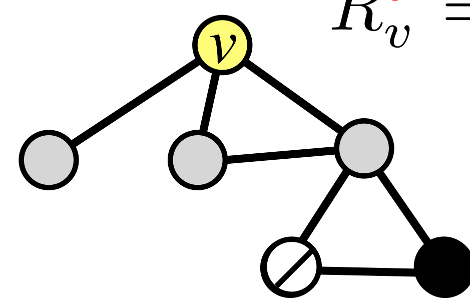
$$R_T = \lambda \prod_{i=1}^{d} \frac{1}{1+R_i}$$

Tree Recursion

hardcore model:

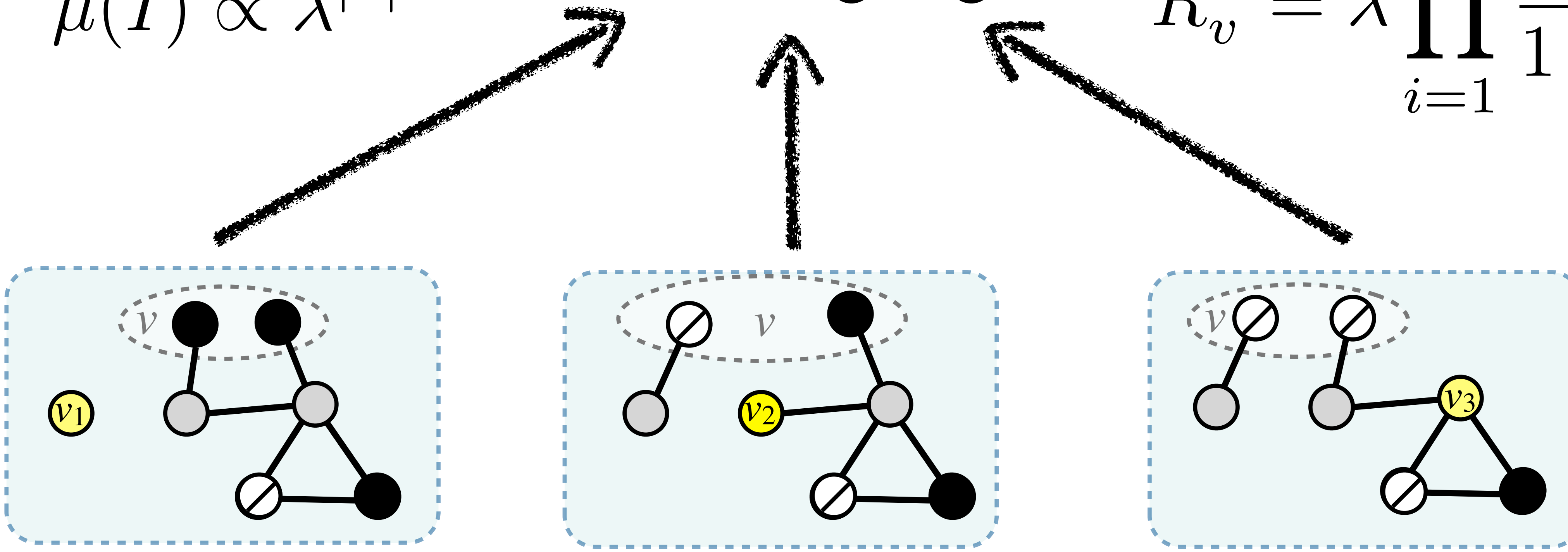
independent set I in G

$$\mu(I) \propto \lambda^{|I|}$$



$$R_v^{\sigma} = \frac{\Pr[v \text{ is occupied } \mid \sigma]}{1 - \Pr[v \text{ is occupied } \mid \sigma]}$$

$$R_v^{\sigma} = \lambda \prod_{i=1}^{d} \frac{1}{1 + R_i^{\sigma}}$$



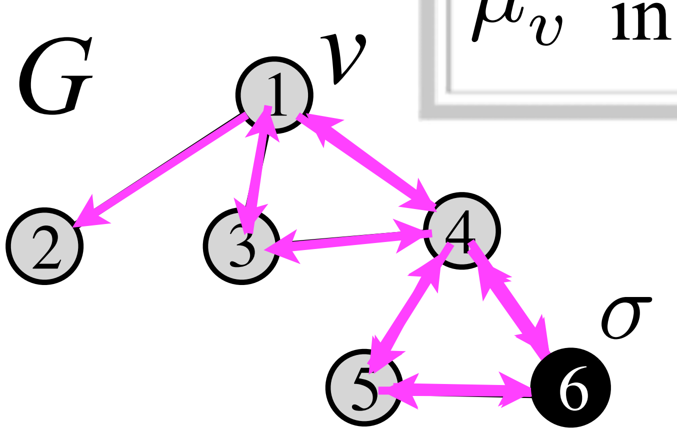
$$R_1^{\sigma} = \frac{\Pr[v_1 \text{ is occupied } \mid \sigma]}{1 - \Pr[v_1 \text{ is occupied } \mid \sigma]} \quad R_2^{\sigma} = \frac{\Pr[v_2 \text{ is occupied } \mid \sigma]}{1 - \Pr[v_2 \text{ is occupied } \mid \sigma]} \quad R_3^{\sigma} = \frac{\Pr[v_3 \text{ is occupied } \mid \sigma]}{1 - \Pr[v_3 \text{ is occupied } \mid \sigma]}$$

● : occupied ⊘ : unoccupied

Self-Avoiding Walk Tree

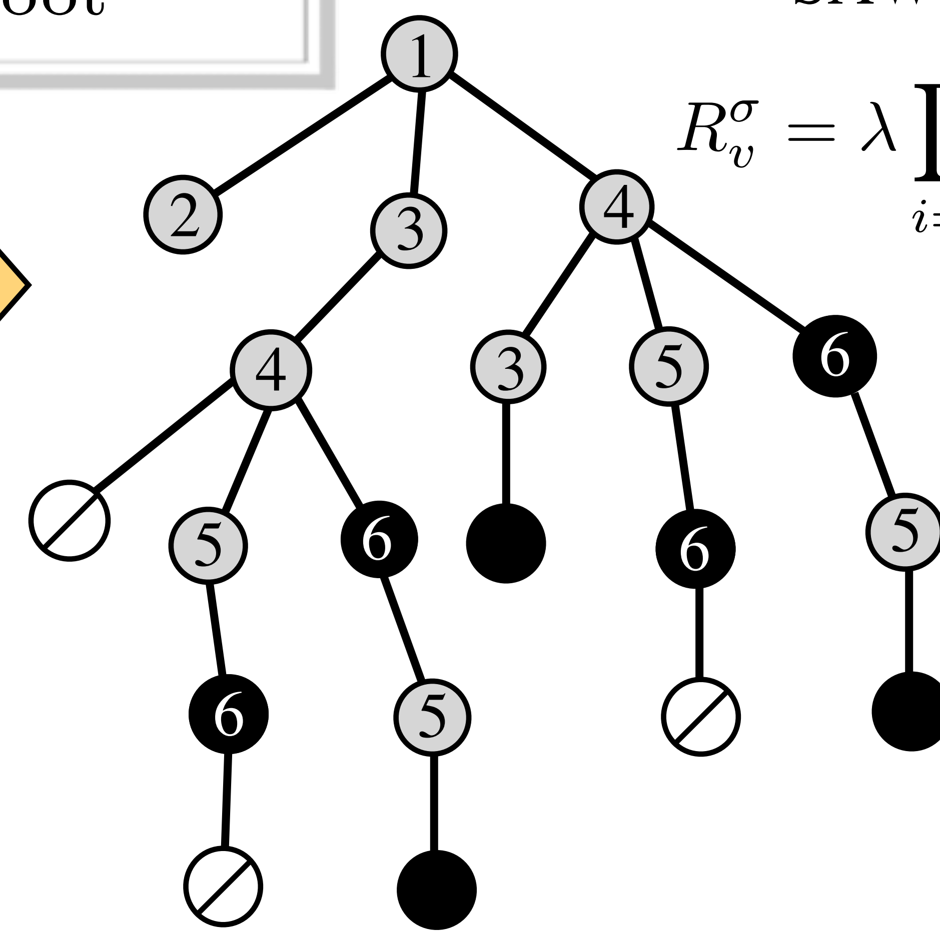
(Godsil 1981; Weitz 2006)

$$\mu_v^\sigma \text{ in } G = \mu_{\text{root}}^\sigma \text{ in } T$$



$$T = T_{\text{SAW}}(G, v)$$

$$R_v^\sigma = \lambda \prod_{i=1}^{d} \frac{1}{1 + R_i^\sigma}$$



SSM in trees:

- SSM in graphs
- efficient approximation of marginals

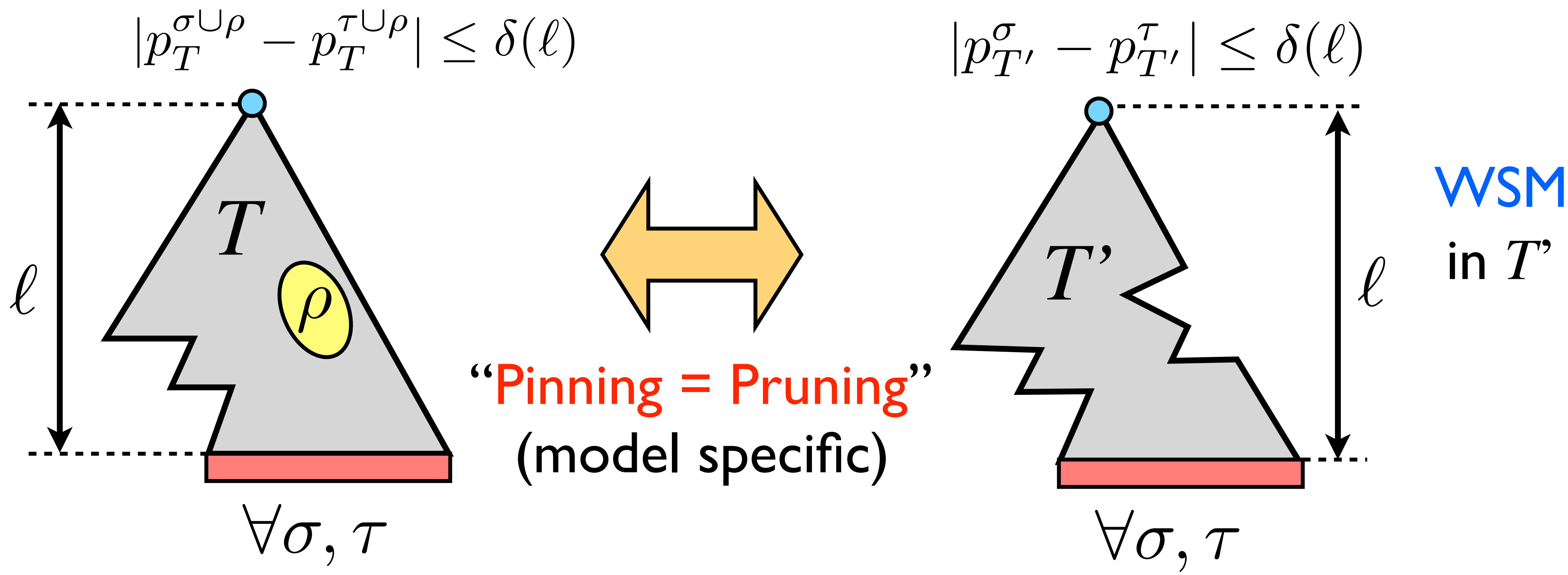
hold for 2-spin model, monomer-dimer, hypergraph matchings

⊘ if cycle closing edge > cycle starting edge

● if cycle closing edge < cycle starting edge

SSM in tree

hardcore model: independent set I of weight $w(I) = \lambda^{|I|}$



Goal: WSM in $(d+1)$-regular tree $\left(\lambda < \lambda_c = \frac{d^d}{(d-1)^{d+1}}\right)$

⇨ SSM in all trees of max-deg ≤ d +1

Weitz's approach:

> $(d+1)$-regular tree is the extremal case for WSM among all trees of max-deg ≤ d +1

hardcore model: independent set I of weight $w(I) = \lambda^{|I|}$

$$R_\ell^{\pm}(\vec{\lambda}) = \lambda \prod_{i=1}^{d} \frac{1}{1 + R_{\ell-1}^{\mp}(\vec{\lambda}_i)}$$

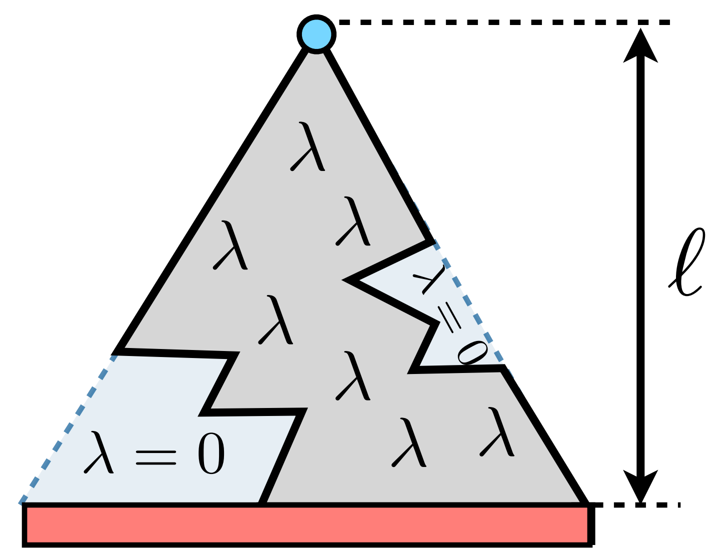
$\vec{\lambda}$: vector of nonuniform λ

$\mathbb{T}$: $(d+1)$-regular tree

$$R_\ell^{+}(\vec{\lambda}) := \sup_{\sigma \text{ at level } \ell} R_{\mathbb{T}}^{\sigma}(\vec{\lambda})$$

$$R_\ell^{-}(\vec{\lambda}) := \inf_{\sigma \text{ at level } \ell} R_{\mathbb{T}}^{\sigma}(\vec{\lambda})$$

$R_\ell^{+}(\lambda), R_\ell^{-}(\lambda)$: for uniform λ



all 0s, all 1s

> $(d+1)$-regular tree is the extremal case for WSM among all trees of max-deg $\leq d+1$

(through the lens of log-of-ratio)

Induction on l with hypothesis:

$$|\log R_\ell^{+}(\vec{\lambda}) - \log R_\ell^{-}(\vec{\lambda})| \leq |\log R_\ell^{+}(\lambda) - \log R_\ell^{-}(\lambda)|$$

$$|\log(1 + R_\ell^{+}(\vec{\lambda})) - \log(1 + R_\ell^{-}(\vec{\lambda}))| \leq |\log(1 + R_\ell^{+}(\lambda)) - \log(1 + R_\ell^{-}(\lambda))|$$

hardcore model: independent set I of weight $w(I) = \lambda^{|I|}$

> $(d{+}1)$-regular tree is the extremal case for WSM among all trees of max-deg $\le d+1$

uniqueness threshold for the infinite $(d{+}1)$-regular tree:

[Weitz 07]: $$\lambda_c = \frac{d^d}{(d-1)^{d+1}}$$

$\lambda < \lambda_c$: SSM at exponential rate on trees of max-deg $\le d+1$

SAW tree ⇒
- SSM at exponential rate on graphs of max-deg $\le d+1$
- FPTAS for graphs of max-deg $\le d+1$

$\lambda = \lambda_c$: SSM at polynomial rate on graphs of max-deg $\le d+1$

[Sly 10] [Galanis Štefankovič Vigoda 12] [Sly Sun 12]:

$\lambda > \lambda_c$: no FPRAS for $(d{+}1)$-regular graphs unless NP=RP

Problem 1: Approximability of the hardcore model when $\lambda=\lambda_c$.

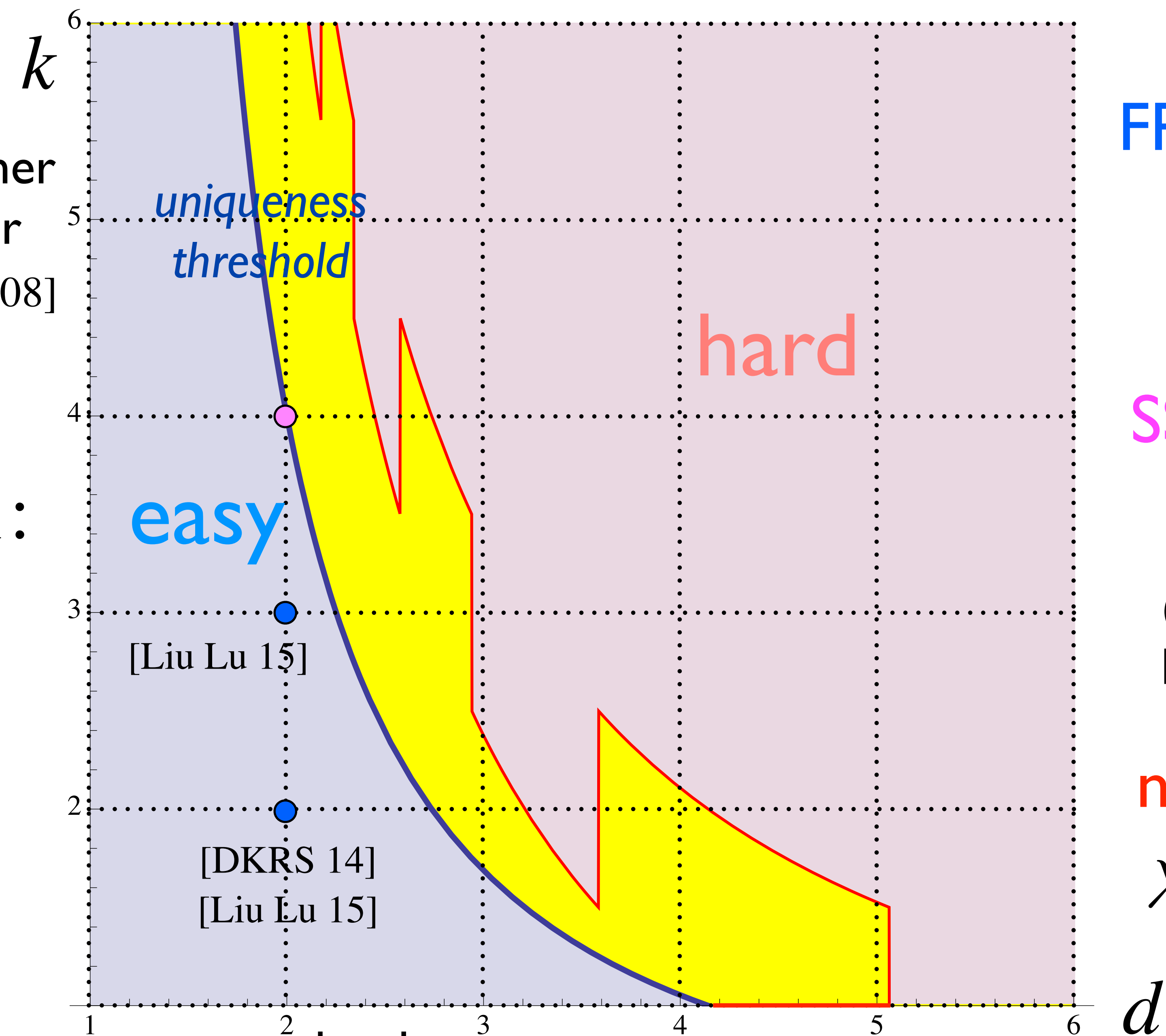
Weitz's approach works for hypergraph matchings:

hypergraph $H=(V,E)$, where $E\subseteq 2^V$: an $M\subseteq E$ is a matching if all edges in M are disjoint

(duality)
- matchings of hypergraphs with activity λ of max-deg $\le(k+1)$ and max-edge-size $\le(d+1)$
- independent sets of hypergraphs with activity λ of max-deg $\le(d+1)$ and max-edge$\le(k+1)$

monomer -dimer [BGKNT 08]

$\lambda = 1$:



hardcore [Weitz 07] [Sly 10]

FPTAS / SSM at exp rate:

$$\lambda < \lambda_c = \frac{d^d}{k(d-1)^{(d+1)}}$$

SSM at polynomial rate:

$$\lambda = \lambda_c$$

(e.g. matchings in 3-uniform hypergrpahs of max-deg 5)

no FPRAS unless NP=RP:

$$\lambda > \frac{2k+1+(-1)^k}{k+1}\lambda_c \approx 2\lambda_c$$

Problem 2: Transition of approximability for hypergraph matchings.

The Potential Method

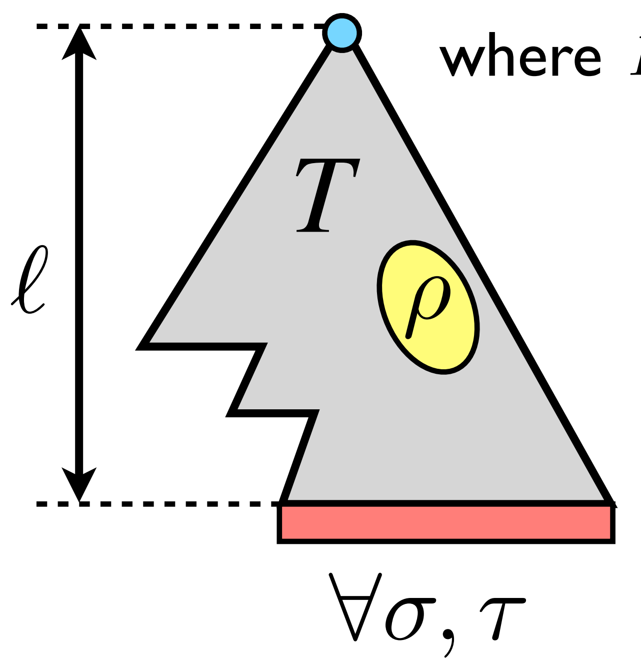
hardcore model: independent set I of weight $w(I) = \lambda^{|I|}$

recursion: $R_T^{\sigma\cup\rho} = \lambda \prod_{i=1}^{d} \frac{1}{1 + R_i^{\sigma\cup\rho}}$

SSM:

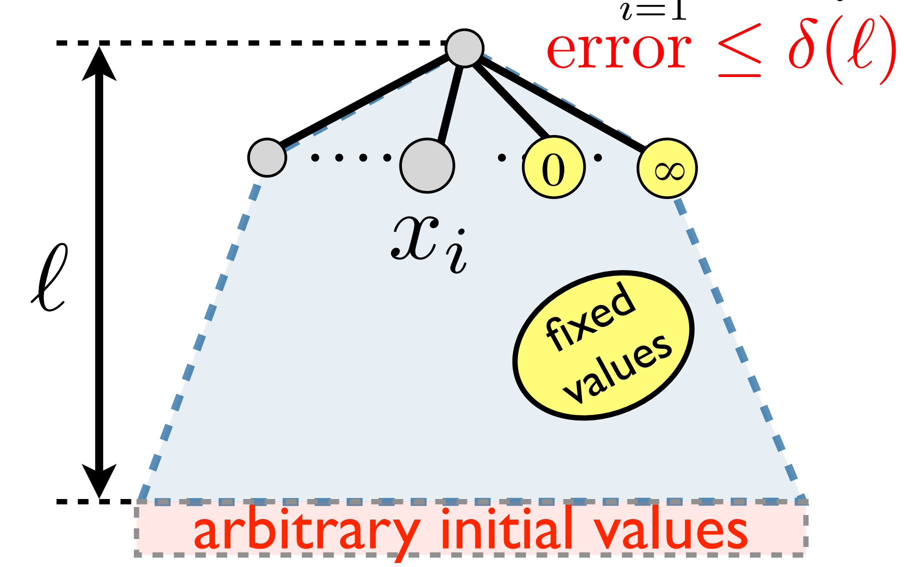
$|R_T^{\sigma\cup\rho} - R_T^{\tau\cup\rho}| \le \delta(\ell)$

where $R = \frac{p}{1-p}$



dynamical system:

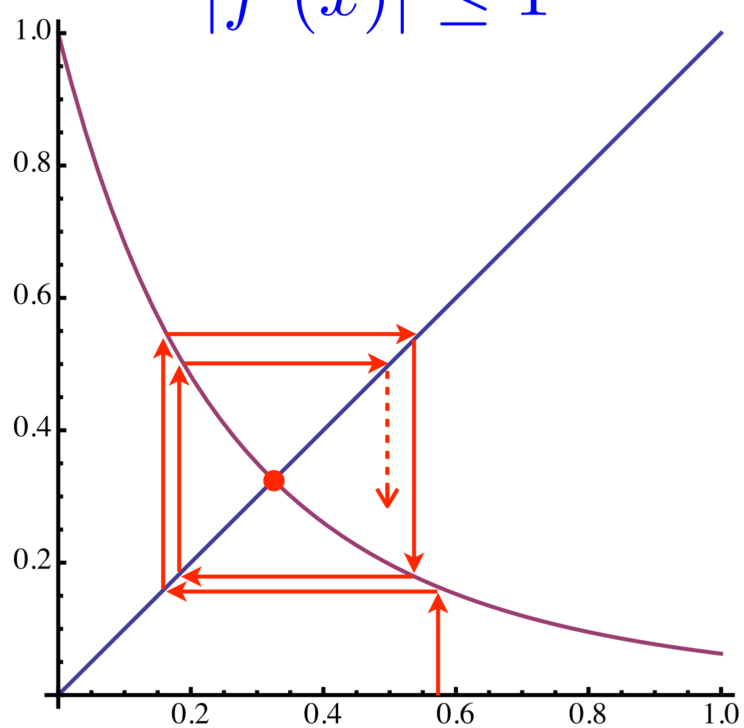
$f(\vec{x}) = \lambda \prod_{i=1}^{d} \frac{1}{1 + x_i}$



symmetric version: $f(x) = \dfrac{\lambda}{(1+x)^d}$

unique fixed point: $\hat{x} = f(\hat{x})$

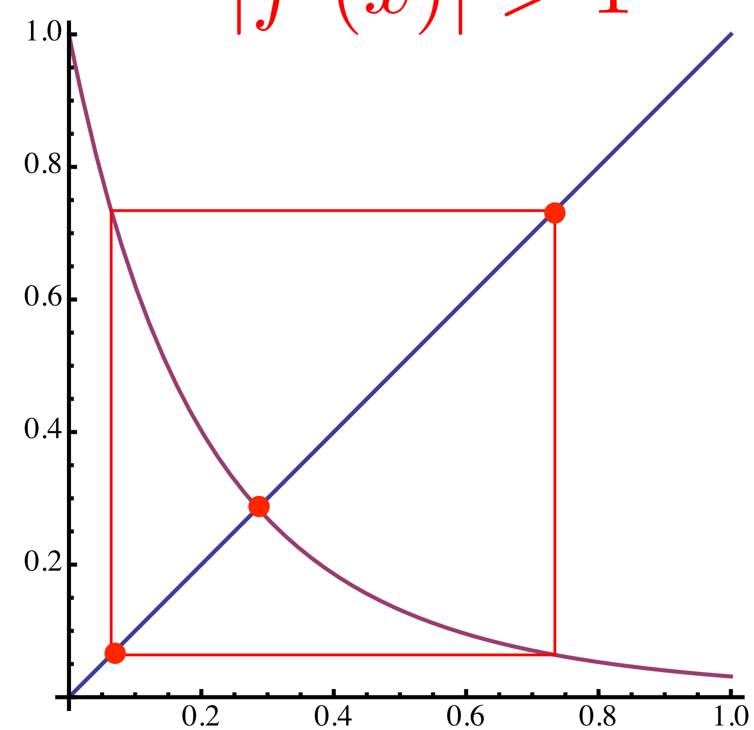
$|f'(\hat{x})| \leq 1$



$$\lambda \leq \lambda_c = \frac{d^d}{(d-1)^{(d+1)}}$$

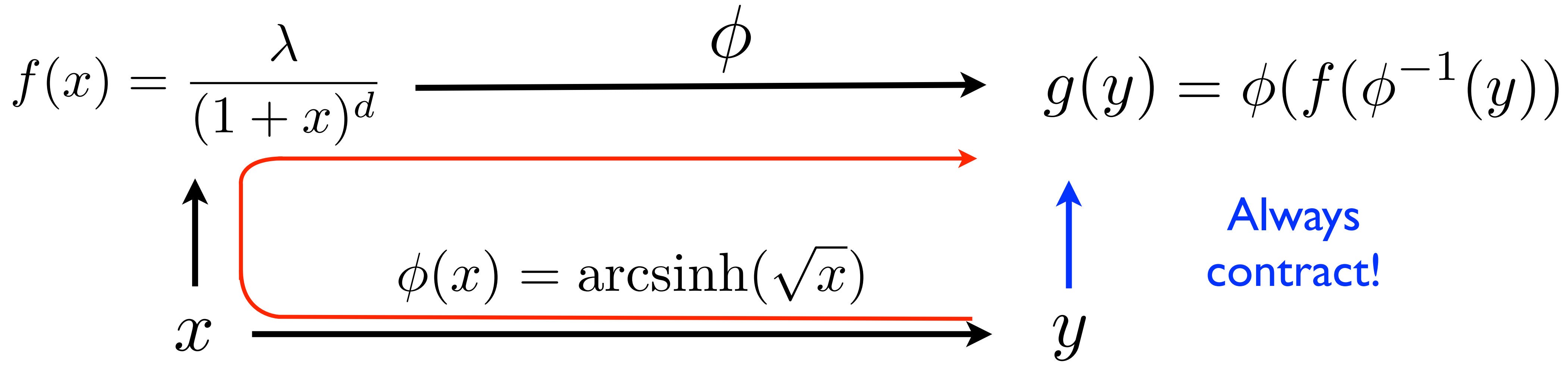
$\Leftrightarrow$ $|f'(\hat{x})| \leq 1$

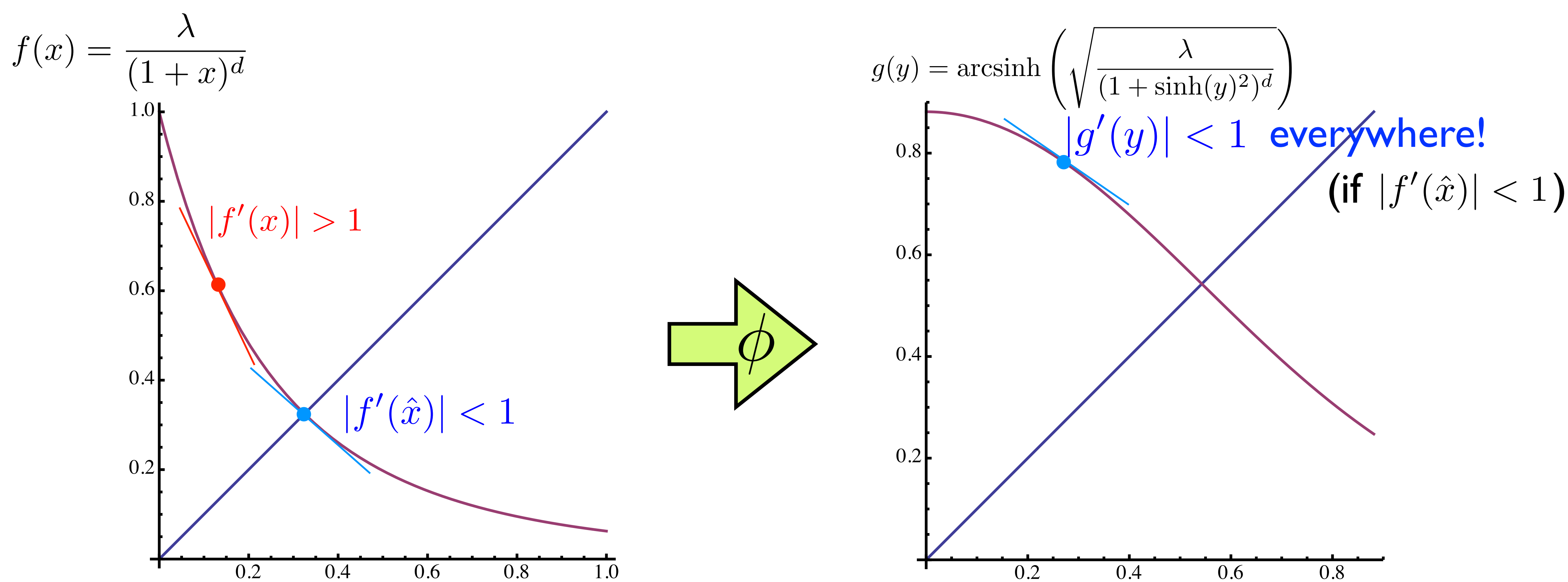
$|f'(\hat{x})| > 1$



$\exists x^- < \hat{x} < x^+$

$$\begin{cases} x^+ = f(x^-) \\ x^- = f(x^+) \end{cases}$$





$$|g'(y)| = \frac{\phi'(f(x))}{\phi'(x)}|f'(x)| = \sqrt{\frac{df(x)}{1+f(x)}}\sqrt{\frac{dx}{1+x}} \le \sqrt{|f'(\hat{x})|}$$

where $x = \phi^{-1}(y)$

by choosing

$$\phi'(x) = \frac{1}{2\sqrt{x(1+x)}}$$

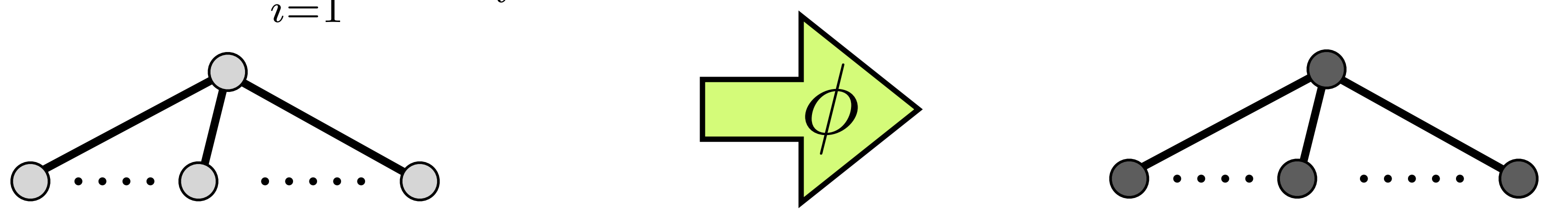
(assuming $|f'(\hat{x})| < 1$)

original:

$$f(\vec{x}) = \lambda \prod_{i=1}^{d} \frac{1}{1+x_i}$$

potential:

$$g(\vec{y}) = \phi(f(\phi^{-1}(y_1), \ldots, \phi^{-1}(y_d)))$$



x_i $\quad y_i = \phi(x_i)$ $\quad y_i$

$$\epsilon_i = |y_i - y_i'|$$

Mean Value Theorem:

$$\epsilon = |g(\vec{y}) - g(\vec{y}')| = \left|\nabla g(\vec{\xi}) \cdot \vec{\epsilon}\right| = \sum_{i=1}^{d} \left|\frac{\partial f(\vec{x})}{x_i}\right| \frac{\Phi(f(\vec{x}))}{\Phi(x_i)} \epsilon_i$$

(where $\xi_i = \phi(x_i)$, denote $\Phi(x) = \phi'(x)$)

recall: $\left(\left|\frac{\partial f(\vec{x})}{x_i}\right| = \frac{f(\vec{x})}{1+x_i}\right)$

$$\leq f(\vec{x})\Phi(f(\vec{x})) \sum_{i=1}^{d} \frac{1}{(1+x_i)\Phi(x_i)} \max_i \epsilon_i \leq \sqrt{\frac{f(\vec{x})}{1+f(\vec{x})}} \sum_{i=1}^{d} \sqrt{\frac{x_i}{1+x_i}} \max_i \epsilon_i$$

(choose $\Phi(x) = \phi'(x) = \frac{1}{\sqrt{x(1+x)}}$)

$$\leq \sqrt{\frac{df(x)}{1+f(x)}} \sqrt{\frac{dx}{1+x}} \max_i \epsilon_i \quad \leq \sqrt{|f'(\hat{x})|} \max_i \epsilon_i$$

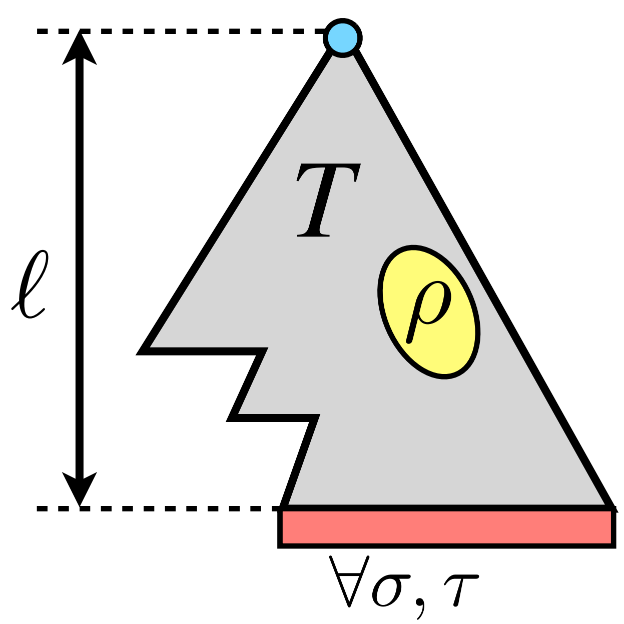
(concavity) (where $f(x) = \frac{\lambda}{(1+x)^d}$) (assuming $|f'(\hat{x})| < 1$)

The Potential Method

hardcore model: independent set I of weight $w(I) = \lambda^{|I|}$

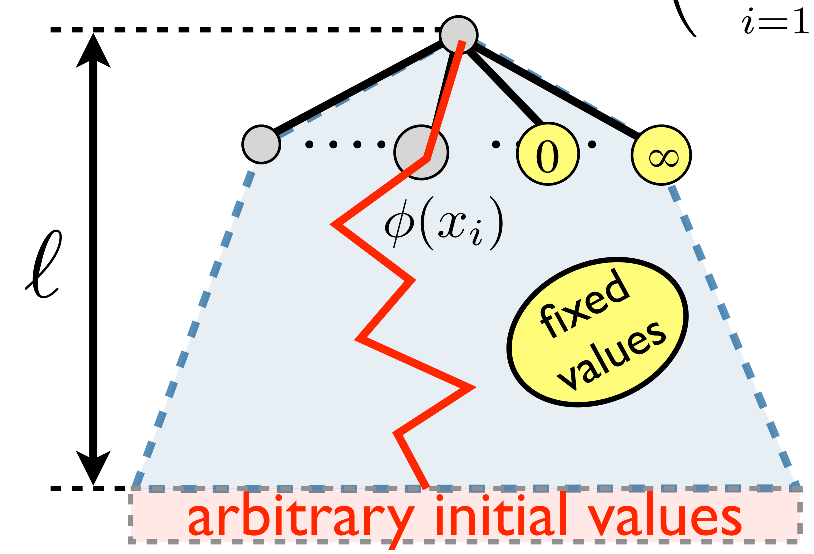
SSM:

$|R_T^{\sigma \cup \rho} - R_T^{\tau \cup \rho}| \leq \delta(\ell)$



dynamical system for potentials:

$$\phi(f(\vec{x})) = \phi\left(\lambda \prod_{i=1}^{d} \frac{1}{1+x_i}\right)$$



$$\delta(\ell) \leq \frac{\phi(\lambda)}{\phi'(\lambda)} \cdot \alpha^{\ell}$$

$$\epsilon = \left|\phi\left(R_T^{\sigma \cup \rho}\right) - \phi\left(R_T^{\tau \cup \rho}\right)\right|$$

$$\leq \alpha \max_i \epsilon_i \quad \leq \alpha^{\ell} \cdot \epsilon_{\text{initial}}$$

uniqueness: $\alpha < 0.999$

The Potential Method

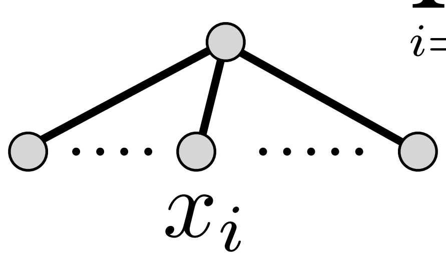
antiferromagnetic 2-spin:

$$Z_G = \sum_{\sigma\in\{0,1\}^V} \prod_{\{u,v\}\in E} A(\sigma_u,\sigma_v) \prod_{v\in V} b(\sigma_v)$$

where $A = \begin{bmatrix} A_{00} & A_{01} \\ A_{10} & A_{11} \end{bmatrix} = \begin{bmatrix} \beta & 1 \\ 1 & \gamma \end{bmatrix} \qquad b = \begin{bmatrix} b_0 \\ b_1 \end{bmatrix} = \begin{bmatrix} \lambda \\ 1 \end{bmatrix}$

antiferromagnetic: $\beta\gamma < 1$

$$f(\vec{x}) = \lambda \prod_{i=1}^{d} \frac{\beta x_i + 1}{x_i + \gamma}$$



let $\phi(x) = \int \frac{1}{\sqrt{x(\beta x+1)(x+\gamma)}} \mathrm{d}x$

so $\Phi(x) = \phi'(x) = \frac{1}{\sqrt{x(\beta x+1)(x+\gamma)}}$

decay factor (in the potential world):

$$\alpha = \sum_{i=1}^{d} \left| \frac{\partial f(\vec{x})}{x_i} \right| \frac{\Phi(f(\vec{x}))}{\Phi(x_i)} \le \sqrt{\frac{df(x)}{(\beta f(x)+1)(f(x)+\gamma)}} \sqrt{\frac{dx}{(\beta x+1)(x+\gamma)}} \le \sqrt{|f'(\hat{x})|}$$

(where $f(x) = \lambda \left(\frac{\beta x + 1}{x+\gamma} \right)^d$)

partition function of *anti-ferromagnetic* 2-spin system with parameter (β,γ,λ) on graphs with max-degree $\leq\Delta$

uniqueness: WSM on all d-regular trees for $d\leq\Delta$

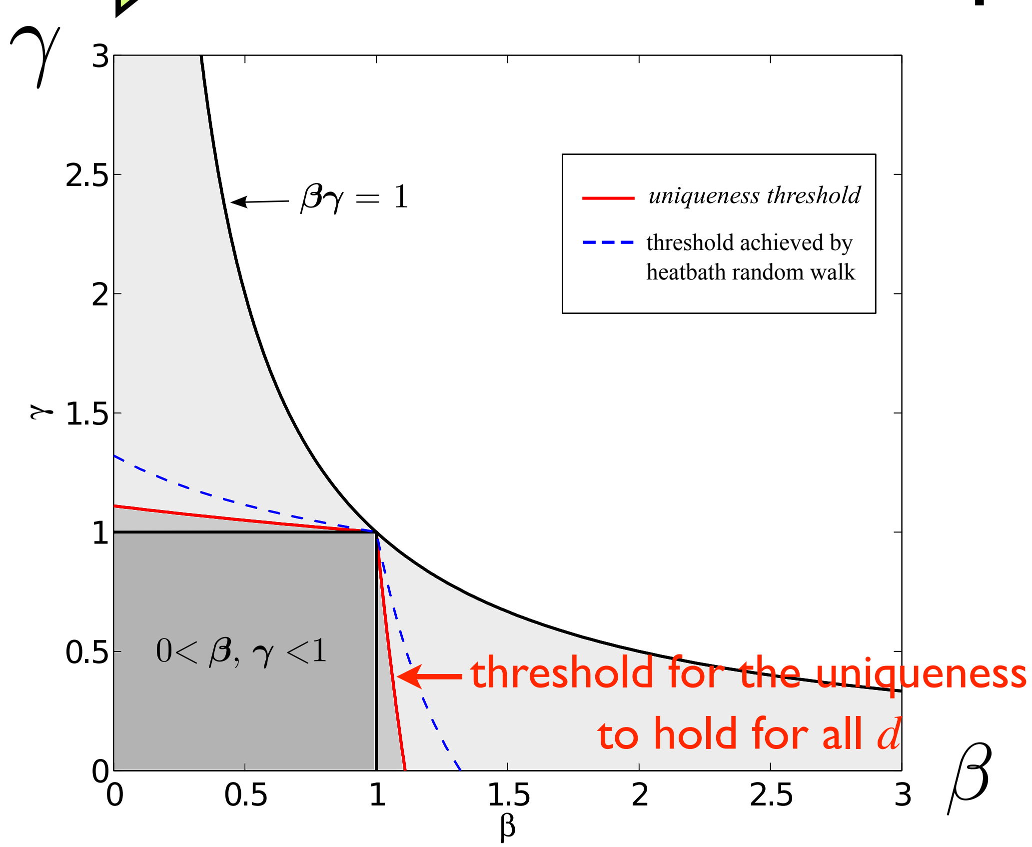
non-uniqueness: no WSM on a d-regular tree with $d\leq\Delta$

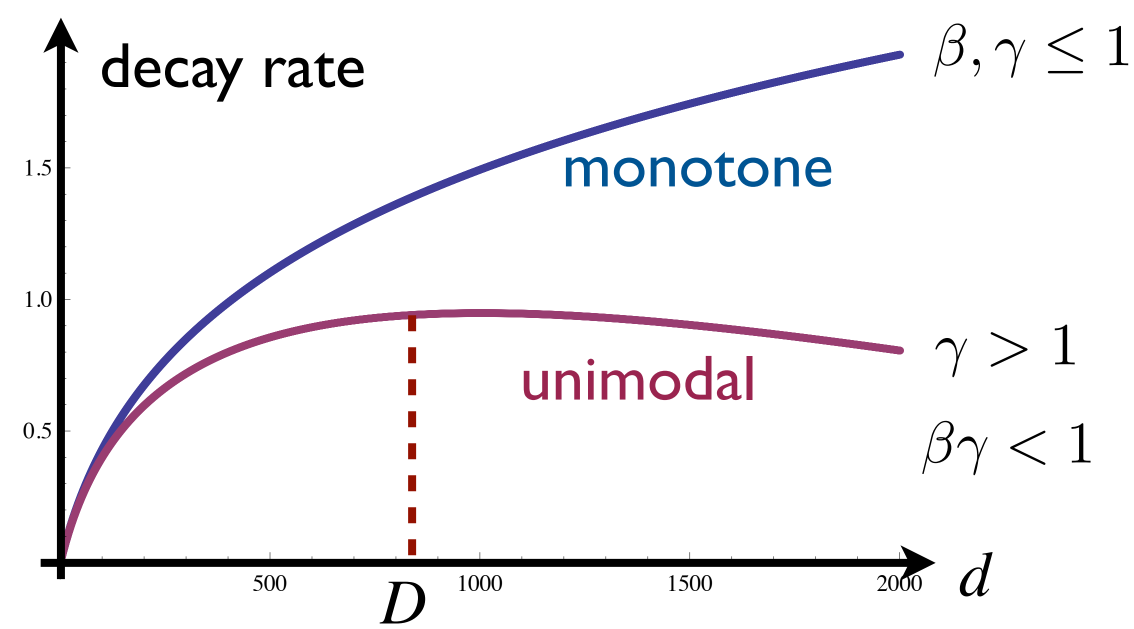
[Li Lu Y. 12; 13]: (β,γ,λ) in the interior of uniqueness regime

⇨ FPTAS for graphs with max-degree $\leq\Delta$

[Sly Sun 12]: (β,γ,λ) in the interior of non-uniqueness regime

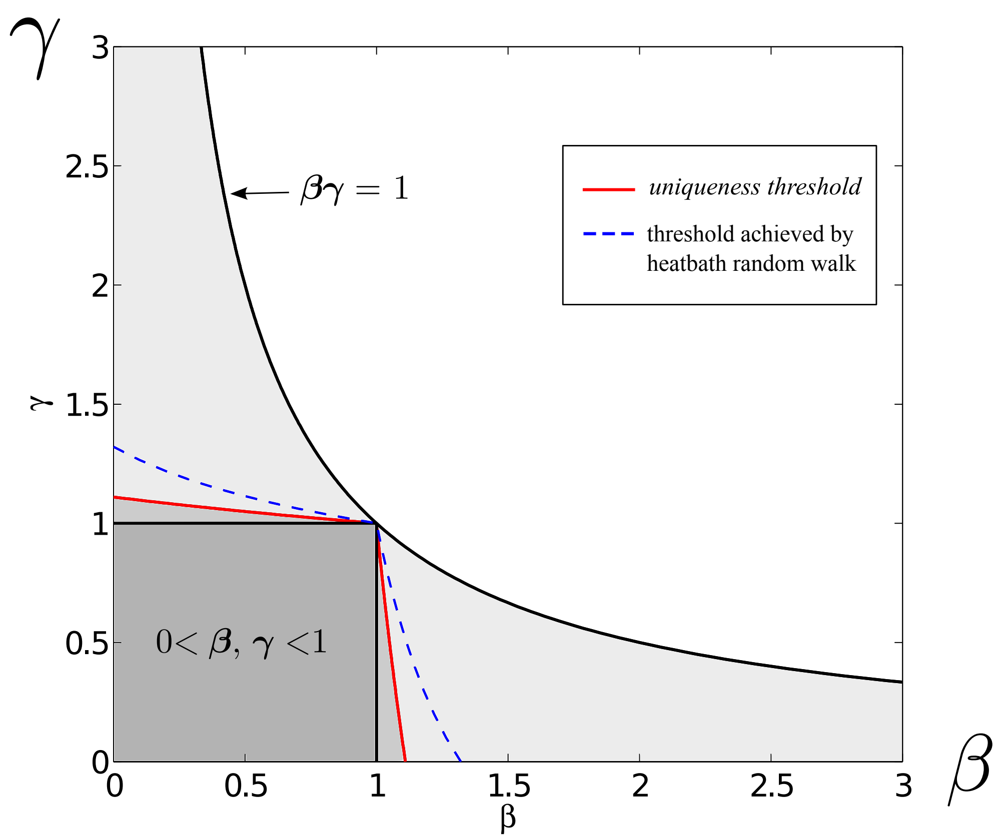
⇨ no FPRAS for the problem unless NP=RP





the extremal case of ssm/wsm is no longer the Δ-regular tree

Ferromagnetic 2-spin



ferromagnetic 2-spin: $\beta\gamma > 1$

$$Z_G = \sum_{\sigma \in \{0,1\}^V} \prod_{\{u,v\} \in E} A(\sigma_u, \sigma_v) \prod_{v \in V} b(\sigma_v)$$

$$A = \begin{bmatrix} A_{00} & A_{01} \\ A_{10} & A_{11} \end{bmatrix} = \begin{bmatrix} \beta & 1 \\ 1 & \gamma \end{bmatrix} \quad b = \begin{bmatrix} b_0 \\ b_1 \end{bmatrix} = \begin{bmatrix} \lambda \\ 1 \end{bmatrix}$$

- Transition of approximability is still open.
- [Jerrum Sinclair 93] [Goldberg Jerrum Paterson 03]: FPRAS for ferro Ising model, or ferro 2-spin with $\lambda \leq \sqrt{\gamma/\beta}$
- Tractable when there is no decay of correlation!

(or IS there?)

Primitive Spatial Mixing

Primitive Spatial Mixing (PSM) at rate $\delta(\ )$:

For rooted trees T_1, T_2 which are identical in the first l levels, the marginal distributions at the respective roots have:

$$\|\mu_{T_1} - \mu_{T_2}\|_{TV} \leq \delta(l)$$

weaker than WSM/SSM:

- no fixed vertices
- no boundary condition
- initial values must be "realizable"

Belief Propagation

2-spin model on G=(V,E) with parameter (β,γ,λ)

loopy Belief Propagation:

$$R_{v\to u}^{(t)} = \lambda \prod_{w\in N(u)\setminus\{v\}} \frac{\beta R_{u\to w}^{(t-1)} + 1}{R_{u\to w}^{(t-1)} + \gamma}$$

with initial values $R_{v\to u}^{(0)}$ for all edge orientations

Weak Spatial Mixing on trees

⇓

convergence of loopy BP on graphs

Belief Propagation

2-spin model on $G=(V,E)$ with parameter (β,γ,λ)

loopy Belief Propagation:

$$R_{v\to u}^{(t)} = \lambda \prod_{w\in N(u)\setminus\{v\}} \frac{\beta R_{u\to w}^{(t-1)}+1}{R_{u\to w}^{(t-1)}+\gamma}$$

with initial values $R_{v\to u}^{(0)}$ for all edge orientations

Primitive Spatial Mixing on trees

⇓

convergence of loopy BP on graphs

(if initial values are chosen wisely)

Primitive Spatial Mixing

Primitive Spatial Mixing (PSM) at rate $\delta(\ )$:

For rooted trees T_1, T_2 which are identical in the first l levels, the marginal distributions at the respective roots have:

$$\|\mu_{T_1} - \mu_{T_2}\|_{TV} \leq \delta(l)$$

weaker than WSM/SSM:

- no fixed vertices
- no boundary condition
- initial values must be "realizable"

Problem 3: The approximability of ferromagnetic 2-spin systems is captured by the primitive spatial mixing on trees.

[Guo Lu 15]: $\lambda < \left(\frac{\gamma}{\beta}\right)^{\frac{\sqrt{\beta\gamma}}{\sqrt{\beta\gamma}-1}}$ ⇒ PSM on all trees

if further $\beta \leq 1$ (pinning are realizable) ⇒ FPTAS

q-Coloring

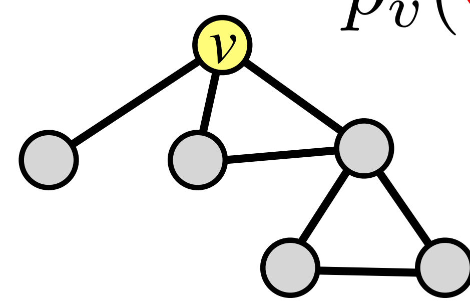
proper q-coloring of graph $G(V, E)$ with max-degree $\leq\Delta$

- [Jonasson 02]: WSM on Δ-regular tree iff $q\geq \Delta+1$
- [Galanis Štefankovič Vigoda 13]: when $q<\Delta$, no FPRAS unless NP=RP, even for triangle-free graphs
- tractable threshold $q\geq\alpha\Delta+\beta$:
  - randomized MCMC algorithms: $\alpha = 11/6$ [Vigoda 99]
  - correlation-decay based algorithms: $\alpha>2.58\sim$ [Lu Y. 13]
  - SSM-only threshold: $\alpha>1.763\sim$ [Goldberg Martin Paterson 04] [Gamarnik Katz Misra 13]

Problem 4: Transition of approximability for q-colorings.

q-Coloring

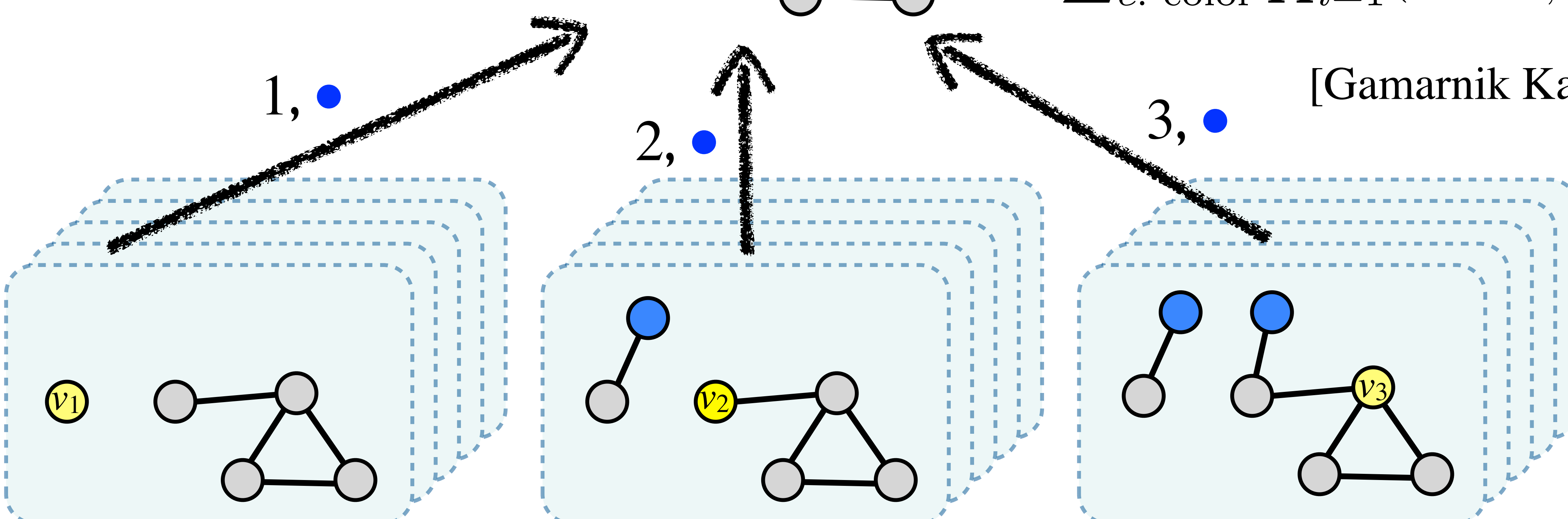
q colors:
$\{\color{red}{\bullet}, \color{blue}{\bullet}, \color{orange}{\bullet}, \color{green}{\bullet}, \color{magenta}{\bullet}\}$



$$p_v(\color{red}{\bullet}) = \Pr[v\text{'s color is } \color{red}{\bullet}]$$

$$= \frac{\prod_{i=1}^{d}(1 - p_{i,\color{red}{\bullet}}(\color{red}{\bullet}))}{\sum_{c:\text{ color}} \prod_{i=1}^{d}(1 - p_{i,c}(c))}$$

[Gamarnik Katz 07]

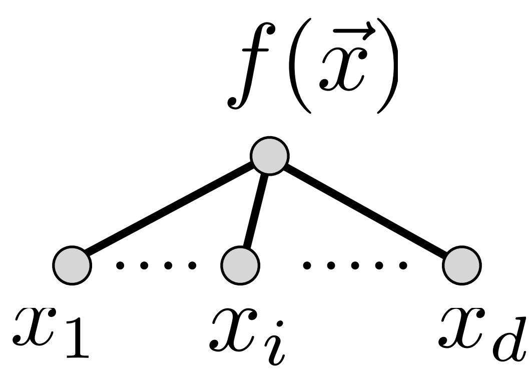


recursion $F : \underbrace{[0,1]^q \times [0,1]^q \times \cdots \times [0,1]^q}_{d} \to [0,1]^q$

Problem 4': Threshold for the SSM for q-colorings.

Correlation Decay in different norms

dynamical system:



propagation of errors:

(up to the translation to potentials)

$$\epsilon \leq \sum_{i=1}^{d} \alpha_i(\vec{x})\epsilon_i = \langle \vec{\alpha}(\vec{x}), \vec{\epsilon} \rangle$$

for the hardcore model:

$$f(\vec{x}) = \lambda \prod_{i=1}^{d} \frac{1}{1 + x_i}$$

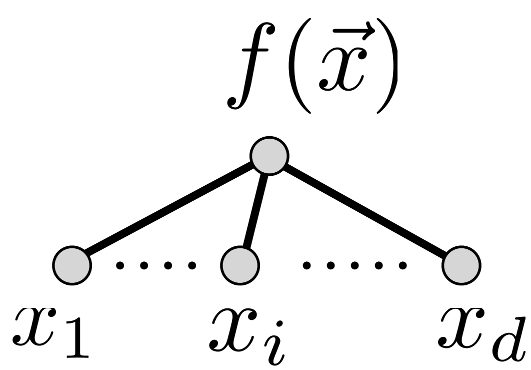
translated to potentials $\phi(x) = \operatorname{arcsinh}(\sqrt{x})$

$$\epsilon \leq \sum_{i=1}^{d} \sqrt{\frac{df(\vec{x})}{1 + f(\vec{x})}} \sqrt{\frac{dx_i}{1 + x_i}} \epsilon_i$$

$$= \alpha_i(\vec{x})$$

Correlation Decay in different norms

dynamical system:



propagation of errors:

(up to the translation to potentials)

$$\epsilon \le \sum_{i=1}^{d} \alpha_i(\vec{x})\epsilon_i = \langle \vec{\alpha}(\vec{x}), \vec{\epsilon} \rangle$$

worst path

$$\le \|\vec{\alpha}(\vec{x})\|_1 \cdot \|\vec{\epsilon}\|_\infty \le \alpha \cdot \|\vec{\epsilon}\|_\infty$$

if *ideally*: $\epsilon \le \alpha \sum_{i=1}^{d} \epsilon_i$ or generally $\epsilon^p \le \alpha \sum_{i=1}^{d} \epsilon_i^p$ for $p \ge 1$

- Decay of correlation in terms of # of self-avoiding walks.
  - $p=1$: *aggregate* SSM ⇒ optimal mixing time for monotone systems
  - $p \ge 1$: SSM and FPTAS in terms of connective constant (a notion of average degree)

Aggregate Spatial Mixing

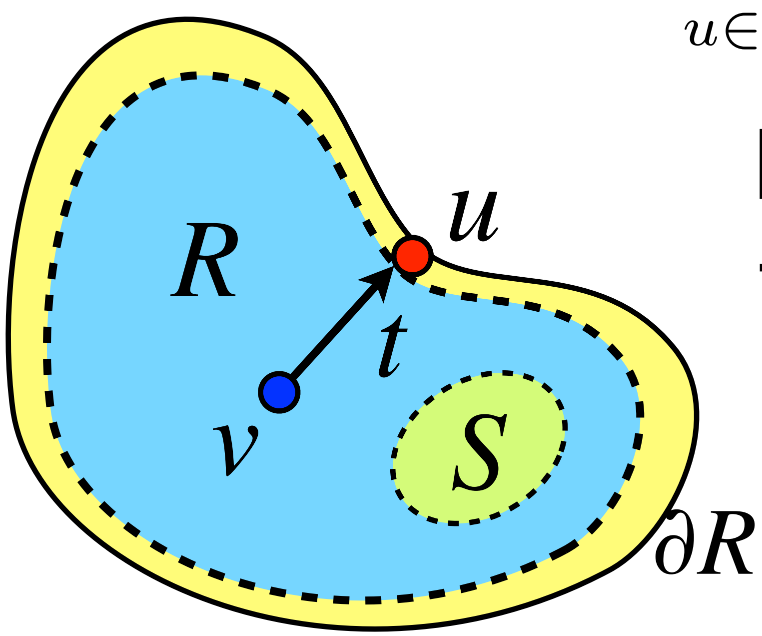
μ_v^σ : marginal distribution at vertex v conditioning on σ

aggregate weak spatial mixing (aWSM) at rate $\delta(\ )$:

$$\sum_{u\in\partial R}\sup_{\substack{\sigma,\tau\in[q]^{\partial R}\\ \text{differ at } u}}\|\mu_v^\sigma-\mu_v^\tau\|_{TV}\le\delta(t)$$

aggregate strong spatial mixing (aSSM) at rate $\delta(\ )$:

$$\forall\rho\in[q]^S:\ \sum_{u\in\partial R}\sup_{\substack{\sigma,\tau\in[q]^{\partial R}\\ \text{differ at } u}}\|\mu_v^{\sigma\cup\rho}-\mu_v^{\tau\cup\rho}\|_{TV}\le\delta(t)$$



[Mossel Sly 13]:

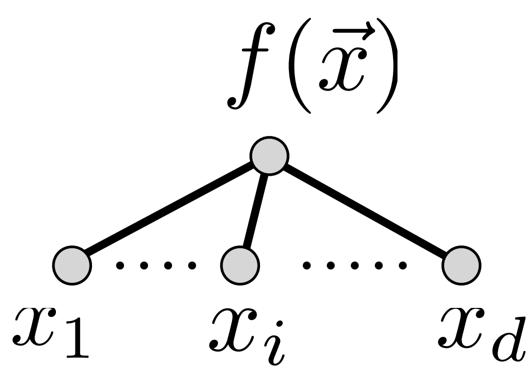
for *monotone* systems (where *censoring* works)
- ferro 2-spin
- anti-ferro 2-spin on bipartite graphs

ASSM ⇒ mixing time of Glauber dynamics

$\tau_{\text{mix}}=O(n\log n)$

Correlation Decay in different norms

dynamical system:



propagation of errors:

(up to the translation to potentials)

$$\epsilon \leq \alpha \sum_{i=1}^{d} \epsilon_i$$

with $\alpha < \frac{1}{d}$

⇒ ASSM

For ferro 2-spin on graphs with max-degree ≤d+1:

- $d < \frac{\sqrt{\beta\gamma}+1}{\sqrt{\beta\gamma}-1}$ ⇒ ASSM ⇒ $\tau_{\text{mix}} = O(n \log n)$
- for Ising without field: this is the uniqueness threshold
- for general 2-spin systems: *strictly* stronger than the uniqueness condition

Connective Constants

[Madras Slade 1996]

$\mathsf{SAW}(v,\ell)$: set of self-avoiding walks of length l starting from v

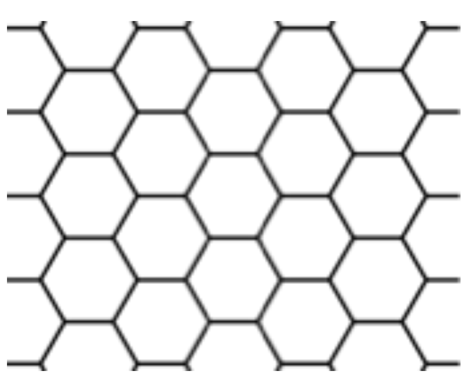
connective constant for an infinite graph G:

$$\Delta_{\mathsf{con}}(G) = \sup_{v \in V} \lim_{\ell \to \infty} \sup |\mathsf{SAW}(v,\ell)|^{1/\ell}$$

connective constant for a family $\mathcal{G}$ of finite graphs is $\leq \Delta_{\mathsf{con}}$
if $\exists C>0$ such that $\forall\ G(V,E) \in \mathcal{G}$

$$\forall \ell : \quad |\mathsf{SAW}(v,\ell)| \leq |V|^C \Delta_{\mathsf{con}}^{\ell}$$

for $G(n, d/n)$: $\Delta_{\mathsf{con}} < (1+\varepsilon)\ d$ w.h.p.

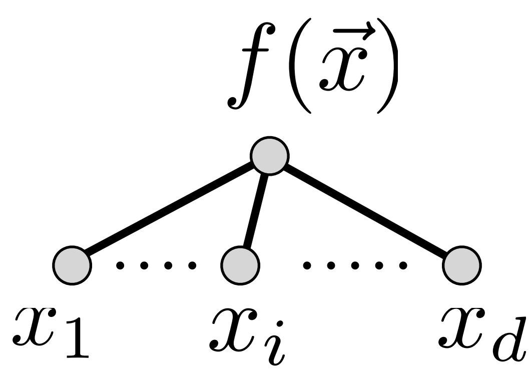


$\Delta_{\mathsf{con}} = \sqrt{2+\sqrt{2}}$ for honeycomb lattice

[Duminil-Copin Smirnov 12]

Correlation Decay in different norms

dynamical system:



propagation of errors:

(up to the translation to potentials)

$$\epsilon \le \sum_{i=1}^{d} \alpha_i(\vec{x})\epsilon_i = \langle \vec{\alpha}(\vec{x}), \vec{\epsilon} \rangle$$

Hölder's inequality

$$\le \|\vec{\alpha}(\vec{x})\|_{\frac{p}{p-1}} \cdot \|\vec{\epsilon}\|_p$$

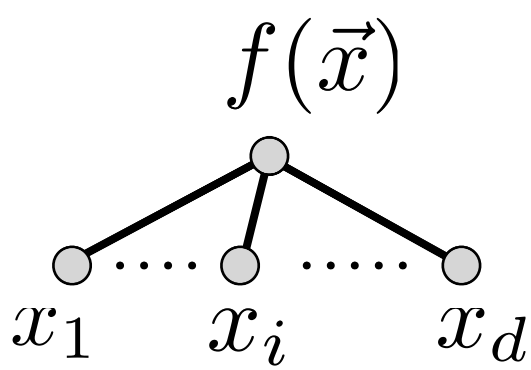
for $p \ge 1$

$$\epsilon^p \le \underbrace{\|\vec{\alpha}(\vec{x})\|^p_{\frac{p}{p-1}}}_{\le \alpha} \sum_{i=1}^{d} \epsilon_i^p$$

SSM if $\Delta_{\mathrm{con}} < 1/\alpha$

Correlation Decay in different norms

dynamical system:



propagation of errors:

(up to the translation to potentials)

$$\epsilon \le \sum_{i=1}^{d} \alpha_i(\vec{x})\epsilon_i = \langle \vec{\alpha}(\vec{x}), \vec{\epsilon} \rangle$$

Hölder's inequality

$$\le \|\vec{\alpha}(\vec{x})\|_{\frac{p}{p-1}} \cdot \|\vec{\epsilon}\|_p$$ for $p \ge 1$

for the hardcore model (with proper potential function):

$$\epsilon \le \sum_{i=1}^{d} \sqrt{\frac{df(\vec{x})}{1+f(\vec{x})}} \sqrt{\frac{dx}{1+x}} \epsilon_i = \langle \vec{\alpha}(\vec{x}), \vec{\epsilon} \rangle$$

$$\epsilon^p \le \|\vec{\alpha}(\vec{x})\|^p_{\frac{p}{p-1}} \sum_{i=1}^{d} \epsilon_i^p$$

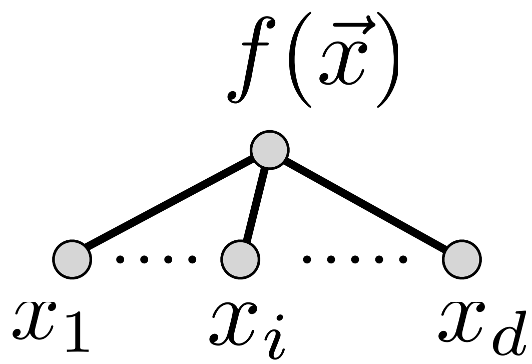
choose $p = \dfrac{1}{1 - \frac{\Delta_c - 1}{2} \ln\left(1 + \frac{1}{\Delta_c - 1}\right)}$

$$\|\vec{\alpha}(\vec{x})\|^p_{\frac{p}{p-1}} \le \frac{1}{\Delta_c}$$

where $\Delta_c = \Delta_c(\lambda)$ satisfies $\lambda = \dfrac{\Delta_c^{\Delta_c}}{(\Delta_c - 1)^{\Delta_c + 1}}$

Correlation Decay in different norms

dynamical system:



propagation of errors:

(up to the translation to potentials)

$$\epsilon^p \leq \alpha \sum_{i=1}^{d} \epsilon_i^p$$

SSM

if $\Delta_{\text{con}} < 1/\alpha$

[Srivastava Sinclair Štefankovič Y. 15]

FPTAS for family of graphs with bounded Δ_{con} :

- Hardcore model: $\Delta_{\text{con}} < \Delta_{\text{c}}(\lambda)$
- Ising without field: $\Delta_{\text{con}} < \Delta_{\text{c}}(\beta)$
- monomer-dimer model: any finite Δ_{con}

uniqueness condition in terms of Δ_{con}

[Jerrum Sinclair 89]: FPRAS for all graphs

[Bayati Gamarnik Katz Nair Tetali 07]: FPTAS for constant degree

Problem 5: FPTAS for matchings in general graphs.

Open Problems

- hardcore model at the uniqueness threshold
- transition of approximability of hyper-matchings
- PSM capturing the approximability of ferro 2-spin
- transition of approximability of q-coloring
- deterministically approximately counting matchings in general graphs

PSM ≤ WSM ≤ SSM ≤ ASSM

How to establish the "correct" correlation decay and relate it to approximate counting ?

Thank you!

Any questions?