Thresholds in random CSPs

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Counting complexity and phase transitions
Simons Institute, Berkeley
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Plan for the talk

Introduction: random $k$-SAT model

Threshold conjecture, Friedgut’s theorem

Statistical physics viewpoint of random CSPs

Replica symmetry (RS) vs. replica symmetry breaking (RSB)

One-step replica symmetry breaking (1RSB)

Graphical models for clusters
Credits (a non-exhaustive list)


*(combinatorial cluster model)* Alfredo Braunstein, Elitza Maneva, Marc Mézard, Elchanan Mossel, Giorgio Parisi, Alistair Sinclair, Martin Wainwright, Riccardo Zecchina

*(upper bound)* Silvio Franz, Francesco Guerra, Michele Leone, Dmitry Panchenko, Michel Talagrand, Fabio Toninelli

*(lower bound)* Dimitris Achlioptas, Amin Coja-Oghlan, Jian Ding, Cris Moore, Assaf Naor, Konstantinos Panagiotou, Yuval Peres, Allan Sly, Daniel Vilenchik
Random CSPs; and the random $k$-SAT model
The boolean satisfiability (SAT) problem:
The SAT problem

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\[ x_1, x_2, \ldots, x_n \]

variables \( x_i \) taking values in \{TRUE, FALSE\} \equiv \{+, -\}
The SAT problem

The boolean satisfiability (SAT) problem:

set of clauses:

$n$ variables $x_i$ taking values in $\{\text{TRUE}, \text{FALSE}\} \equiv \{+, -\}$
The boolean satisfiability (SAT) problem:

- **Variables**: $x_1, x_2, \ldots, x_n$
- **Clauses**: Each clause constrains a (small) subset of variables
- **Decision**: Determine if there exists any variable assignment satisfying all clauses.
The boolean satisfiability (SAT) problem:

- set of clauses: each clause constrains a (small) subset of variables
- $n$ variables $x_i$ taking values in \{TRUE, FALSE\} \equiv \{+, -\}

Computational question: decide if there exists any variable assignment $x \in \{+, -\}^n$ satisfying all clauses.
Constraint satisfaction problems

SAT is a constraint satisfaction problem (CSP).

A general CSP is a set of variables subject to some constraints: the question is to decide whether there exists some variable assignment satisfying all constraints.

For a large class of CSPs, including SAT, best known algorithms have exponential runtime on worst-case instances, motivating interest in average-case behavior.

One direction is to investigate the typical behavior for models of random CSPs, as the system size becomes large. This line of research has been pursued since the 1980s.
Formal definition of $k$-SAT

A $k$-SAT problem is specified by a boolean formula

\[
\text{clause of width } k = 4
\]

\[
\begin{align*}
( & +x_1 \text{ OR } +x_3 \text{ OR } -x_5 \text{ OR } -x_7 ) \\
\text{AND} ( & -x_1 \text{ OR } -x_2 \text{ OR } +x_5 \text{ OR } +x_6 ) \\
\text{AND} ( & -x_3 \text{ OR } +x_4 \text{ OR } -x_6 \text{ OR } +x_7 )
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Assign variables $x_i \in \{+, -\}$ to satisfy all clauses.
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Equivalently, a factor graph with colored edges:
Formal definition of $k$-SAT

A $k$-SAT problem is specified by a boolean formula

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\begin{align*}
\text{clause of width } k = 4 \\
(p \lor x_3 \lor \neg x_5 \lor \neg x_7) \\
\text{AND } (-x_1 \lor \neg x_2 \lor +x_5 \lor +x_6) \\
\text{AND } (-x_3 \lor +x_4 \lor \neg x_6 \lor +x_7)
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**Equivalently**, a factor graph with colored edges:

- **Blue** edge affirms
- **Yellow** edge negates
Random $k$-SAT at clause density $\alpha$

set $F$ of $m \sim \text{Poisson}(n\alpha)$ clauses

set $V$ of $n$ variables
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set $E$ of random edges, each clause degree $k$ (here $k = 3$)
Random $k$-SAT at clause density $\alpha$

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set $E$ of random edges, each clause degree $k$ (here $k = 3$) randomly divided into affirmative and negative
Random \( k \)-SAT at clause density \( \alpha \)

set \( F \) of \( m \sim \text{Poisson}(n\alpha) \) clauses

set \( V \) of \( n \) variables

set \( E \) of random edges, each clause degree \( k \) (here \( k = 3 \))
randomly divided into affirmative and negative
— altogether forms a random \( k \)-SAT instance \( G \):
an ‘average-case’ version of \( k \)-SAT
Threshold conjecture
SAT threshold conjecture

**SAT threshold conjecture.** For each fixed $k$ (with $k \geq 2$), random $k$-SAT has a sharp satisfiability threshold $\alpha_{sat}(k)$:

\[ \mathbb{P}(\text{SAT}) \]

Selman–Mitchel–Levesque '96, 3-SAT with $n = 50$ variables
**SAT threshold conjecture.** For each fixed $k$ (with $k \geq 2$), random $k$-SAT has a sharp satisfiability threshold $\alpha_{\text{sat}}(k)$:

\[
\mathbb{P}(\text{SAT}) \quad \text{converges to sharp threshold}
\]
\[
in \text{limit } n \to \infty
\]

\[\text{clause-to-variable ratio } \alpha (k \text{ fixed})\]

SAT (with high probability)

UNSAT (with high probability)
**SAT threshold conjecture.** For each fixed $k \ (\text{with } k \geq 2)$, random $k$-SAT has a sharp satisfiability threshold $\alpha_{\text{sat}}(k)$:

- $P(SAT)$ converges to sharp threshold in limit $n \to \infty$

Since early '90s, known for $k = 2$, open for $k \geq 3$.

$(k = 2)$ Goerdt '92, '96, Chvátal–Reed '92, de la Vega '92
Friedgut’s theorem

Friedgut ('99) proved there is a threshold sequence $\alpha_{\text{sat}}(n)$:
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- Sharp threshold $\alpha_{\text{sat}}$ independent of $n$
- Increasing $\alpha$
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Best prior bounds:
- Coja-Oghlan–Panagiotou ’14
- Kirousis–Kranakis–Krizanc–Stamatiou ’96

Algorithmic:
- Frieze–Suen ’96
- Coja-Oghlan ’10

Achlioptas–Peres ’03
- Achlioptas–Moore ’02

This talk: sharp threshold MMZ ’06, DSS ’14
Friedgut’s theorem

Friedgut ('99) proved there is a threshold sequence $\alpha_{\text{sat}}(n)$:

- Increasing $\alpha_{\text{sat}}$ independent of $n$
- (earlier rigorous lower bounds)
  - algorithmic: Frieze–Suen '96, Coja-Oghlan '10
  - Achlioptas–Moore '02
  - Achlioptas–Peres '03
- Best prior bounds: Coja-Oghlan–Panagiotou '14
- Kiousis–Kranakis–Krizanc–Stamatiou '96

This talk: sharp threshold MMZ '06, DSS '14

Threshold conjecture: Friedgut’s theorem (8/28)
First moment

Let $Z(G) \equiv |\text{SOL}(G)| \equiv \#\text{satisfying assignments of } G$. 
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$$\mathbb{E} Z = 2^n (1 - 1/2^k)^{n\alpha} = \exp \left\{ n \left( \ln 2 + \alpha \ln (1 - 1/2^k) \right) \right\}.$$
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Exponent zero at $\alpha_1 = 2^k \ln 2$. Above $\alpha_1$, $\mathbb{E}Z \ll 1$. 

Thus $\mathbb{E}Z$ is dominated by a rare event where $Z$ is extremely large.
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$P(Z \neq 0) \leq E[Z]$, so $Z = 0$ whp. So if $\alpha_{\text{sat}}$ exists, it is $\leq \alpha_1$. 

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The bound isn’t tight: there is a non-trivial interval $(\alpha_{\text{sat}}, \alpha_1)$ where $E Z \gg 1$ even though $Z = 0$ with high probability.
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Statistical physicists made major advances on this front by showing how to adapt heuristics from the study of spin glasses (disordered magnets) to explain the CSP solution space.

Mézard–Parisi '85, '86, '87; Fu–Anderson '86

Some remarkable physics conjectures for spin glasses & CSPs on dense graphs have been rigorously proved: Aldous '00, Guerra '03, Talagrand '06, Panchenko '11, Wästlund '10 (for conjectures of Parisi, Mézard, Krauth in '70s and '80s)

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KMRSZ '07
A ‘universality class’ of sparse random CSPs

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Krząkała–Montanari–Ricci-Tersenghi–Semerjian–Zdeborová '07,
Zdeborová–Krząkała '07, Montanari–Ricci-Tersenghi–Semerjian '08

Such models are believed to exhibit a complex phase diagram: solution space SOL exhibits several distinct behaviors.

Structural phenomena have been linked to algorithmic barriers.

e.g. Achlioptas–Coja-Oghlan '08, Sly '10, Gamarnik–Sudan '13, Rahman–Virag '14
The **1RSB** threshold

Increasing $\alpha$

The **1RSB** models are predicted to exhibit a very specific clustering structure in the regime of $\alpha$ preceding $\alpha_{\text{sat}}$.

(more on this later)
The 1RSB threshold

The 1RSB models are predicted to exhibit a very specific clustering structure in the regime of $\alpha$ preceding $\alpha_{\text{sat}}$. 

On the basis of this structural assumption, one can derive an explicit conjecture $\alpha_{\text{sat}} = \alpha_\star$. This is the 1RSB threshold formula. Similar formulas can be derived in other models.

derivation for random $k$-SAT: Mertens–Mézard–Zecchina ’06
Moment method and 1RSB

In prior literature, best bounds on $\alpha_{\text{sat}}$ are by moment method on $Z$ (number of solutions), with increasingly sophisticated truncation/conditioning to handle the non-concentration of $Z$.

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The physics explains the source of non-concentration (‘RSB’) — strongly suggests moment method on $Z$ cannot detect $\alpha_{\text{sat}}$. The 1RSB hypothesis indicates a better path to the threshold.

The predicted threshold value $\alpha_\star$ is a complicated function — makes it (highly) unlikely that a rigorous determination of $\alpha_{\text{sat}}$ can be made without relying on the physics insight.
Replica symmetry breaking
SAT as graphical model

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$$\nu \equiv \text{uniform probability measure over } \text{SOL}.$$
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Fix the $k$-SAT instance, thereby fixing $\nu$, and consider $X \sim \nu$: a $\{+, -\}$-valued stochastic process indexed by the variables.
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Asking about the geometric structure of $\text{SOL}$ can be recast as asking about the behavior of typical samples $\underline{X} \sim \nu$.

The (random) measure $\nu$ is an example of a graphical model (or factor model/Gibbs measure/Markov random field).
Physicists classify graphical models \( \nu \) as replica symmetric or replica symmetry breaking (\( RS/RSB \)) as follows. For simplicity, assume variables \( X_i \) take values in \( \{+, -\} \) for all \( 1 \leq i \leq n \).
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We say $\nu$ is $\text{RS}$ if faraway variables are ‘nearly independent’ (correlation decay). In particular, if $X^1, X^2 \overset{iid}{\sim} \nu$ (replicas), then

$$\text{overlap}(X^1, X^2) \equiv \frac{1}{n} \sum_{i=1}^{n} X_i^1 X_i^2$$
RS(B) in graphical models

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\[
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\]

Otherwise, \( \nu \) has long-range dependencies and it is RSB. In this case \( \text{overlap}(X^1, X^2) \) has a non-trivial distribution.

failure of correlation decay is a key source of difficulty in the analysis.
Random $k$-SAT exhibits both RS and RSB regimes:

In the $k$-SAT context, the solution space can be visualized as a tree. As the density parameter $\alpha$ increases:

1. The solution space starts as a single large cluster.
2. As $\alpha$ increases further, the cluster splits into smaller clusters, resembling the replica symmetry breaking (RSB) regimes.
3. Finally, when $\alpha$ is sufficiently large, the solution space becomes a set of isolated points, indicating the replica symmetry breaking (RSB) regime.

This diagram illustrates the transition from a single solution to a set of solutions, reflecting the change in the solution landscape as the problem density increases.
RS(B) in SAT context

Random $k$-SAT exhibits both RS and RSB regimes:

In one regime, SOL has exponentially many clusters, each carrying an exponentially small fraction of the total mass.
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Increasing $\alpha$:

In one regime, SOL has exponentially many clusters, each carrying an exponentially small fraction of the total mass.

Replicas $X^1, X^2 \sim \nu$ are in different clusters with high probability, and are nearly orthogonal (clusters are far apart).
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- Increasing \( \alpha \)
  - \( \alpha_{\text{sat}} \)

Nearer to \( \alpha_{\text{sat}} \), almost all mass in \textit{bounded} number of clusters.
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Replicas \( X^1, X^2 \sim iid \nu \) are either in different clusters with overlap \( \neq 0 \), or in the same cluster with overlap \( \neq 1 \).
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Both events occur with non-negligible probability, so overlap distribution is non-trivial.
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Clustering of solutions

Why does the solution space SOL exhibit clustering?

As \( \alpha \) increases, the clusters of solutions become more pronounced.
Clusters of solutions

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Clustering of solutions

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Clusters of solutions

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Clusters of solutions

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The random SAT graph is sparse — each variable participates in bounded number of clauses. Each clause has some freedom. A typical $\bar{x} \in $ SOL thus has $\geq n\pi$ free variables.

By sparsity, extract $n\pi'$ free variables with no shared clauses. Flipping any subset of these variables gives another $\tilde{x} \in $ SOL:

$\bar{x} \in $ cluster $\subseteq $ SOL with $|\text{cluster}| \geq 2^{n\pi'}$.

Such clustering is a generic feature of sparse random CSPs.
Condensation, 1RSB, and cluster encodings
Where are we?

Random $k$-SAT with $n$ variables, $m \sim \text{Poisson}(n\alpha)$ clauses: 

![Diagrams representing various states of $k$-SAT problems with $n$ variables and $m$ clauses.](image-url)
Random $k$-SAT with $n$ variables, $m \sim \text{Poisson}(n\alpha)$ clauses:

So far, we’ve tried to give a tour of the phase diagram — the (conjectural) geometry of $\text{SOL} \subseteq \{+, -\}^n$, as $\alpha$ varies.

geometry $\leftrightarrow$ correlation decay properties
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How did physicists actually come up with such a picture (complete with exact numerical predictions)?

What are the implications for the rigorous approaches to $\alpha_{\text{sat}}$? For example, how does all this relate back to $\mathbb{FZ}$?
Cluster complexity function

Under an additional set of assumptions (1RSB) (more later)
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expected #clusters of size $\exp\{ns + o(n)\}$

$\doteq$ typical #clusters of size $\exp\{ns + o(n)\}$

$\doteq \exp\{n\Sigma(s) + o(n)\}$
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for explicit \(\Sigma(s)\), the so-called ‘cluster complexity function.’
(Implicitly, \(\Sigma(s) = \Sigma(s; \alpha)\).)
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\mathbb{E}Z = \sum_{0 \leq s \leq \ln 2} \exp\{n[s + \Sigma(s)]\}.
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Dominated by \( s = s_\star \) where \( \Sigma'(s_\star) = -1 \). Since we know \( \Sigma \), we can see how \( \max_s [s + \Sigma(s)] \) changes with \( \alpha \).
RS to RSB (condensation/Kauzmann transition)

$$\mathbb{E} Z = \exp\{n[s_\star + \Sigma(s_\star)]\} \text{ where } \Sigma'(s_\star) = -1. \text{ As } \alpha \text{ increases:}$$
The expected partition function is given by
\[ \mathbb{E} Z = \exp\{n[s_\star + \Sigma(s_\star)]\} \] where \( \Sigma'(s_\star) = -1 \). As \( \alpha \) increases:

These results are shown graphically with a plot of \( \Sigma(s) \) against \( s \), illustrating the transition from RS to RSB.
RS to RSB (condensation/Kauzmann transition)

\[ \mathbb{E} Z = \exp\{n[s^* + \Sigma(s^*)]\} \text{ where } \Sigma'(s^*) = -1. \]  

As \( \alpha \) increases:

- The free energy \( \mathbb{E} Z \) changes its behavior.

---

**Diagram:**
- The graph shows the evolution of the free energy as \( \alpha \) increases.
- The red line represents the RS (replica symmetric) regime, while the blue line shows the RSB (replica symmetry breaking) regime.
- The red dot marks the transition point, indicating the onset of RSB.
- The figure illustrates the condensation/Kauzmann transition, where large clusters form.

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1RSB: Condensation (21/28)
RS to RSB (condensation/Kauzmann transition)

\[ \mathbb{E} Z = \exp\{n[s_\star + \Sigma(s_\star)]\} \text{ where } \Sigma'(s_\star) = -1. \] As \( \alpha \) increases:

- RS
- RSB
RS to RSB (condensation/Kauzmann transition)

\[ \mathbb{E}Z \equiv \exp\{n[s_* + \Sigma(s_*])\} \text{ where } \Sigma'(s_*) = -1. \] As \( \alpha \) increases:

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\[ \mathbb{E} Z \equiv \exp\{n[s_* + \Sigma(s_*)]\} \text{ where } \Sigma'(s_*) = -1. \]

As \( \alpha \) increases:

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- RSB
- UNSAT

\[ \Sigma(s) \]

\[ 0 \]

\[ s \]
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typical picture is dominated by \( O(1) \) clusters of this size (the condensation phenomenon)
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Upon onset of RSB (condensation/Kauzmann transition), \( \mathbb{E} Z \) becomes dominated by atypically large clusters. \( Z \ll \mathbb{E} Z \) whp.
Definition of 1RSB

The detailed phase diagram is derived with the assumption

\[ \text{expected \ #clusters \ of \ size \ } \exp\{ns\} \]
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The \textit{one-step replica symmetry breaking} (1RSB) heuristic postulates that solution clusters are replica symmetric even when individual satisfying assignments are RSB.
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The one-step replica symmetry breaking (1RSB) heuristic postulates that solution clusters are replica symmetric even when individual satisfying assignments are RSB.

This assumption underlies the explicit derivation of $\Sigma(s)$, and yields $\alpha_* = \max\{\alpha : \Sigma_{\max}(\alpha) \equiv \max_s \Sigma(s; \alpha) > 0\}.$
Exact formulas
1RSB formulas

The *one-step replica symmetry breaking* (1RSB) heuristic postulates that solution clusters are replica symmetric even when individual satisfying assignments are RSB. No 'clusters within clusters'.

How to get from this to formulas? What does it really mean for clusters to be RS? Need *graphical model of clusters*. Here, graphical model = (weighted) CSP.
**1RSB formulas**

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The random graphs are locally tree-like — few short cycles. Trees are great for formulas (fixed-point equations).

The measure $\nu$ over SOL reflects $k$-SAT on the random graph. To reduce to ‘$k$-SAT on a tree,’ we need to understand the marginal $\nu_U$ over large neighborhoods $U$, say $U = B_t(\nu)$. 
Cavity measure and fixed points

Markov: \( \nu_U(x_U) \equiv 1\{x_U \text{ satisfies all clauses in } U\} \nu_{G \setminus U}(x_{\partial U}). \)
Cavity measure and fixed points

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The central difficulty is to understand the law of \( x_{\partial U} \) in the cavity graph \( G \setminus U \).
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Markov: $\nu_U(x_U) \cong \mathbf{1}\{x_U \text{ satisfies all clauses in } U\} \nu_{G\setminus U}(x_{\partial U})$.

The central difficulty is to understand the law of $x_{\partial U}$ in the cavity graph $G\setminus U$. Ideally, $\exists$ measure $q$ on $\{+, -\}$ so that

$$\nu_{G\setminus U}(x_{\partial U}) \overset{\circ}{=} \prod_{u \in \partial U} q(x_u) \text{ for any } U. \quad (*)$$
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If \( W \subseteq U \), \( \nu_W \) is marginal of \( \nu_U \)
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If $W \subseteq U$, $\nu_W$ is marginal of $\nu_U \rightsquigarrow$ fixed-point eqn. for $q$. 

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Formulas: Tree recursions (24/28)
Cavity measure and fixed points

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If $W \subseteq U$, $\nu_W$ is marginal of $\nu_U \leadsto$ fixed-point eqn. for $q$. Solve for $q_*$. Write $Z_n$ as telescoping product of $Z_i/Z_{i-1}$:

$$(Z_n)^{1/n} \doteq \frac{Z_n}{Z_{n-1}} = \sum_{x_v} \Psi(x_v, x_{\partial v}) \prod_{u \in \partial v} q_*(x_u) \equiv \phi$$
Cavity measure and fixed points

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\]

so \( (\star) \) yields \( Z_n \equiv \phi^n \) for explicit \( \phi \)!
BP fixed points

\[ \nu_{G \setminus U}(x_{\partial U}) = \prod_{u \in \partial U} q(x_u) \quad \text{for any } U \]  

(\star)

breaks down upon onset of RSB.
BP fixed points

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breaks down upon onset of RSB. 1RSB says that a modification of \((\ast)\) holds within each individual cluster \(\gamma\): no ‘clusters within clusters’

\[ \]
BP fixed points

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(\ast)

breaks down upon onset of RSB. 1RSB says that a modification of (\ast) holds \textit{within each individual cluster }\gamma:\n
\[ \nu_{G \setminus U}^\gamma(x_{\partial U}) \doteq \prod_{u \in \partial U} q_u^\gamma(p(u)(x_u)) \quad \text{for any } \gamma, U. \]

no ‘clusters within clusters’
BP fixed points

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breaks down upon onset of RSB. 1RSB says that a modification of (\star) holds within each individual cluster \( \gamma \):

\[ \nu_{\gamma \setminus U}(x_{\partial U}) \doteq \prod_{u \in \partial U} q_{u \rightarrow p(u)}(x_u) \quad \text{for any } \gamma, U. \]

Instead of recursion for single \( q \), have the (vector) BP eqns.: \( q^\gamma = \text{BP}(q^\gamma; G) \). 1RSB correspondence \( \gamma \leftrightarrow \nu^\gamma \leftrightarrow q^\gamma \).
BP fixed points

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no ‘clusters within clusters’

\[ \nu_{G \setminus U}^{\gamma}(x_{\partial U}) = \prod_{u \in \partial U} q_{u \to p(u)}^{\gamma}(x_u) \quad \text{for any } \gamma, U. \]

Instead of recursion for single \( q \), have the (vector) BP eqns.:

\( q^\gamma = \text{BP}(q^\gamma; G) \). 1RSB correspondence \( \gamma \leftrightarrow \nu^\gamma \leftrightarrow q^\gamma \).

Lift \( G \) to new CSP \( G^{\text{BP}} \) whose constraints are the BP eqns.:

\{clusters \( \gamma \} \leftrightarrow \{\text{BP fixed points } q^\gamma\} \leftrightarrow \{\text{solutions of } G^{\text{BP}}\}.\]
Cluster complexity function

\{\text{clusters } \gamma\} \leftrightarrow \{\text{BP fixed points } q^\gamma\} \leftrightarrow \{\text{solutions of } G^{\text{BP}}\}.
Cluster complexity function

\{ \text{clusters } \gamma \} \leftrightarrow \{ \text{BP fixed points } q^{\gamma} \} \leftrightarrow \{ \text{solutions of } G^{\text{BP}} \}.

\( G^{\text{BP}} \) is just another CSP — but \textit{1RSB says that } G^{\text{BP}} \text{ is RS even if } G \text{ is RSB}. The desired condition (\ast) then holds.
Cluster complexity function

\{\text{clusters } \gamma\} \leftrightarrow \{\text{BP fixed points } q^\gamma\} \leftrightarrow \{\text{solutions of } G^{\text{BP}}\}.

$G^{\text{BP}}$ is just another CSP — but 1RSB says that $G^{\text{BP}}$ is RS even if $G$ is RSB. The desired condition $(\ast)$ then holds.

Thus can get to a fixed-point equation for a single $Q$, solve for $Q_*$, and get the partition function $Z(G^{\text{BP}})$. 
Cluster complexity function

\{\text{clusters } \gamma\} \leftrightarrow \{\text{BP fixed points } q^\gamma\} \leftrightarrow \{\text{solutions of } G^{\text{BP}}\}.

\(G^{\text{BP}}\) is just another CSP — but \textit{1RSB says that } G^{\text{BP}} \text{ is RS even if } G \text{ is RSB}. The desired condition \((\star)\) then holds.

Thus can get to a fixed-point equation for a single \(Q\), solve for \(Q_\star\), and get the partition function \(Z(G^{\text{BP}})\).

On \(G\), \(q\) is a measure on \(x \in \{+, -\}\), and \(Z(G)\) is the number of \(k\)-SAT solutions.
Cluster complexity function

\{\text{clusters } \gamma\} \leftrightarrow \{\text{BP fixed points } q^\gamma\} \leftrightarrow \{\text{solutions of } G^{BP}\}.

$G^{BP}$ is just another CSP — but 1RSB says that $G^{BP}$ is RS even if $G$ is RSB. The desired condition (*) then holds.

Thus can get to a fixed-point equation for a single $Q$, solve for $Q_*$, and get the partition function $Z(G^{BP})$.

On $G$, $q$ is a measure on $x \in \{+,-\}$, and $Z(G)$ is the number of $k$-SAT solutions. On $G^{BP}$, $Q$ is a measure on $q \in \mathcal{P}\{+,-\}$, and the partition function $Z(G^{BP})$ is the number of clusters.
Cluster complexity function

\{\text{clusters } \gamma \} \leftrightarrow \{\text{BP fixed points } q^\gamma \} \leftrightarrow \{\text{solutions of } G^{\text{BP}}\}.

\(G^{\text{BP}}\) is just another CSP — but \textit{1RSB says that } G^{\text{BP}} \text{ is RS even if } G \text{ is RSB}. The desired condition (*) then holds.

Thus can get to a fixed-point equation for a single \(Q\), solve for \(Q_\star\), and get the partition function \(Z(G^{\text{BP}})\).

On \(G\), \(q\) is a measure on \(x \in \{+, -\}\), and \(Z(G)\) is the number of \(k\)-SAT solutions. On \(G^{\text{BP}}\), \(Q\) is a measure on \(q \in \mathcal{P}\{+, -\}\), and the partition function \(Z(G^{\text{BP}})\) is the number of clusters.

\[Z(G^{\text{BP}}) \doteq \exp\{n\Sigma_{\text{max}}(\alpha)\} \text{ for explicit } \Sigma_{\text{max}} \rightsquigarrow \text{ explicit } \alpha_\star.\]
Cluster complexity function

\{\text{clusters } \gamma \} \leftrightarrow \{\text{BP fixed points } q^\gamma \} \leftrightarrow \{\text{solutions of } G^{BP}\}.

\(G^{BP}\) is just another CSP — but \textit{1RSB says that }\(G^{BP}\) \textit{is RS even if }\(G\) \textit{is RSB}. The desired condition (⋆) then holds.

Thus can get to a fixed-point equation for a single \(Q\), solve for \(Q_\star\), and get the partition function \(Z(G^{BP})\).

On \(G\), \(q\) is a measure on \(x \in \{+, -\}\), and \(Z(G)\) is the number of \(k\)-SAT solutions. On \(G^{BP}\), \(Q\) is a measure on \(q \in \mathcal{P}\{+, -\}\), and the partition function \(Z(G^{BP})\) is the number of clusters.

\[Z(G^{BP}) = \exp\{n\Sigma_{\max}(\alpha)\}\text{ for explicit }\Sigma_{\max} \rightsquigarrow \text{ explicit }\alpha_\star.\]

With more work, can predict full curve \(\Sigma(s; \alpha)\).
Combinatorial cluster encoding

If only interested in $\max_s \Sigma(s)$, can further reduce BP to WP:
$q_{x \rightarrow y}$ (measure on $\{+, -\}$) projects to $\pi_{x \rightarrow y} \in \{+, -, \text{free}\}$.

see Parisi '02, Braunstein–Mézard–Zecchina '02
Maneva–Mossel–Wainwright '05
Combinatorial cluster encoding

If only interested in $\max_s \sum(s)$, can further reduce BP to WP:
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see Parisi '02, Braunstein–Mézard–Zecchina '02
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\{	ext{clusters } \gamma\} \leftrightarrow \{\text{BP fixed points } q^\gamma\} \leftrightarrow \{\text{WP fixed points } \pi^\gamma\},
Combinatorial cluster encoding

If only interested in max$_s \Sigma(s)$, can further reduce BP to WP:
$q_{x \to y}$ (measure on \{+,-\}) projects to $\pi_{x \to y} \in \{+, -, \text{free}\}$.

see Parisi '02, Braunstein–Mézard–Zecchina '02
Maneva–Mossel–Wainwright '05

\{clusters $\gamma$\} $\leftrightarrow$ \{BP fixed points $q^\gamma$\} $\leftrightarrow$ \{WP fixed points $\pi^\gamma$\},
\[\pi^\gamma \in \{+, -, \text{free}\}^{2E}\] with $\pi^\gamma = \text{WP}(\pi^\gamma; G)$. 
Combinatorial cluster encoding

If only interested in $\max_s \sum(s)$, can further reduce BP to WP: $q_{x \rightarrow y}$ (measure on $\{+,-\}$) projects to $\pi_{x \rightarrow y} \in \{+,-,\text{free}\}$.

see Parisi '02, Braunstein–Mézard–Zecchina '02
Maneva–Mossel–Wainwright '05

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\{\text{clusters } \gamma\} \leftrightarrow \{\text{BP fixed points } q^\gamma\} \leftrightarrow \{\text{WP fixed points } \pi^\gamma\},
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Combinatorial cluster encoding

If only interested in \( \max_s \Sigma(s) \), can further reduce BP to WP: 
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see Parisi '02, Braunstein–Mézard–Zecchina '02
Maneva–Mossel–Wainwright '05

\{clusters \( \gamma \)\} \leftrightarrow \{BP \, \text{fixed points} \, q^{\gamma}\} \leftrightarrow \{WP \, \text{fixed points} \, \pi^{\gamma}\}, 
\pi^{\gamma} \in \{+, -, \text{free}\}^{2E} \, \text{with} \, \pi^{\gamma} = WP(\pi^{\gamma}; G).

WP has nice interpretation: if variable \( v \) neighbors clause \( a \), 
\[ \pi^{\gamma}_{a \rightarrow v} = +/\, - \, \text{iff} \, a \, \text{forces} \, x_v = +/\, - \, \text{in cluster} \, \gamma; \]
Combinatorial cluster encoding

If only interested in $\max_s \Sigma(s)$, can further reduce BP to WP: $q_{x \rightarrow y}$ (measure on $\{+, -\}$) projects to $\pi_{x \rightarrow y} \in \{+, -, \text{free}\}$.

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{$\text{clusters } \gamma \}$ ↔ {$\text{BP fixed points } q^\gamma \}$ ↔ {$\text{WP fixed points } \pi^\gamma \}$,

$\pi^\gamma \in \{+, -, \text{free}\}^{2E}$ with $\pi^\gamma = \text{WP}(\pi^\gamma; G)$.

WP has nice interpretation: if variable $v$ neighbors clause $a$,

$\pi^\gamma_{a \rightarrow v} = +/- \text{ iff } a \text{ forces } x_v = +/- \text{ in cluster } \gamma$;

$\pi^\gamma_{v \rightarrow a} = +/- \text{ iff } \partial v \setminus a \text{ forces } x_v = +/- \text{ in cluster } \gamma$. 
Combinatorial cluster encoding

If only interested in \( \max_s \Sigma(s) \), can further reduce BP to WP: 
\[ q_{x \rightarrow y} \text{ (measure on \( \{+, -\} \)) projects to } \pi_{x \rightarrow y} \in \{+, -, \text{free}\}. \]

see Parisi '02, Braunstein–Mézard–Zecchina '02
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\{clusters \( \gamma \}\} \leftrightarrow \{BP fixed points \( q^\gamma \}\} \leftrightarrow \{WP fixed points \( \pi^\gamma \}\},
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random regular NAE-SAT, random regular IND-SET, random SAT

So far, in models where \( \alpha_{\text{sat}} \) was rigorously determined, lower bounds go through WP configurations \( \pi \).
Combinatorial cluster encoding

If only interested in \( \max_s \sum(s) \), can further reduce BP to WP: 
\[ q_{x \rightarrow y} \text{ (measure on } \{+, -\} \text{) projects to } \pi_{x \rightarrow y} \in \{+, -, \text{free}\}. \]

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\{clusters } \gamma \} \leftrightarrow \{BP fixed points } q^\gamma \} \leftrightarrow \{WP fixed points } \pi^\gamma \} , 
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random regular NAE-SAT, random regular IND-SET, random SAT

So far, in models where \( \alpha_{\text{sat}} \) was rigorously determined, lower bounds go through WP configurations \( \pi \). Informal idea is to show that the \( \pi \)'s ‘do not cluster’ — partially confirms 1RSB.
Some open questions
Open questions

What is the typical value of $Z$?

Other aspects of phase diagram (structural properties of SOL)?

How does the picture change at positive temperature?

Models with higher levels of RSB (MAX-CUT)?
Let $\mathcal{P} \equiv$ space of probability measures on $[0, 1]$. Define the distributional recursion $R_\alpha : \mathcal{P} \rightarrow \mathcal{P}$,

$$(R_\alpha \mu)(B) \equiv \sum_{d=(d^+,d^-)} \pi_\alpha(d) \int \mathbb{1} \left\{ \frac{(1-\Pi^-)\Pi^+}{\Pi^+ + \Pi^- - \Pi^+\Pi^-} \in B \right\} \prod_{i,j} d\mu(\eta_{ij}^\pm)$$

with $\pi_\alpha(d) \equiv \frac{e^{-k\alpha}(k\alpha/2)^{d^+ + d^-}}{(d^+)!(d^-)!}$, $\Pi^\pm \equiv \Pi^\pm(d,\eta) \equiv \prod_{i=1}^{d^\pm} \left( 1 - \prod_{j=1}^{k-1} \eta_{ij}^\pm \right)$.

We show $(R_\alpha)^\ell \mathbb{1}_{1/2} \xrightarrow{\ell \rightarrow \infty} \mu_\alpha$, and use $\mu_\alpha$ to define

$$\Phi(\alpha) = \sum_{d} \pi_\alpha(d) \int \ln \left( \frac{\Pi^+ + \Pi^- - \Pi^+\Pi^-}{(1 - \prod_{j=1}^{k} \eta_{j})^{\alpha(k-1)}} \right) \prod_{i,j} d\mu_\alpha(\eta_{ij}) \prod_{i,j} d\mu_\alpha(\eta_{ij}^\pm).$$

For $k \geq k_0$, the random $k$-SAT threshold $\alpha_{sat} = \alpha_*$ is the unique solution of $\Phi(\alpha) = 0$ in the interval $2^k \ln 2 - 2 \leq \alpha \leq 2^k \ln 2$. 