Constraints, Gadgets, and Invariants

Andrei A. Bulatov
Simon Fraser University

Counting and Phase Transitions Boot Camp, Simons Institute, 2016
Relations and Functions

Let $A$ be a finite set

Relation (k-ary): $R \subseteq A^k$, can be viewed as a function

$R: A^k \rightarrow \{0,1\}$

Function (k-ary): $R: A^k \rightarrow \mathbb{R}$ (for optimization)

$R: A^k \rightarrow \mathbb{R}^+$ (for partition functions)
Constraint Problems
Constraint Problems

Instance: $(V; A; C)$ where $\text{CSP}(\Gamma)$

- $V$ is a finite set of variables
- $A$ is a set of values
- $C$ is a set of constraints $\{R_1(s_1), \ldots, R_q(s_q)\}$

$R_1, \ldots, R_q$ can be relations on $A$, or (nonnegative, real/complex) functions on $A$

Often assumed to be from a fixed set $\Gamma$
Constraint Problems II
Constraint Problems III

Instance: \((V; A; C)\) where \(\text{CSP(}\Gamma\text{)}, \, \$\text{CSP(}\Gamma\text{)}, \, \#\text{CSP(}\Gamma\text{)}\)

- \(V\) is a finite set of variables
- \(C\) is a set of constraints \(\{R_1(s_1), \ldots, R_q(s_q)\}\)

Objective (Decision): whether there is \(h: V \rightarrow A\) such that, for any \(i\), \(R_i(h(s_i))\) is true

Objective (Optimization): find \(h\) that maximizes \(\sum_i R_i(h(s_i))\)

Objective (Counting): find the number of such solutions \(h\)

Objective (Partition function): find the number \(\sum_h \prod_i R_i(h(s_i))\)
Classification

The Classification Problem: Find the complexity of $\text{CSP}(\Gamma)$, $\$\text{CSP}(\Gamma)$, $\#\text{CSP}(\Gamma)$ for every constraint language $\Gamma$
Gadgets and Reductions
Gadgets and Reductions

`express’ R

The hope is $\text{CSP}(R) \leq \text{CSP}(Q)$
No auxiliary variables

Then \( \text{CSP}(R) \leq \text{CSP}(Q) \) (in all possible meanings)

More generally, if for every \( R \in \Gamma \) there is an instance of \( \text{CSP}(\Delta) \) with relations/functions \( Q_1, \ldots, Q_n \in \Delta \) such that

- \( R(\overline{x}) = Q_1(\overline{x}_1) \land \cdots \land Q_n(\overline{x}_n) \) then \( \text{CSP}(\Gamma) \leq \text{CSP}(\Delta) \)
- \( R(\overline{x}) = Q_1(\overline{x}_1) + \cdots + Q_n(\overline{x}_n) \) then \( \$\text{CSP}(\Gamma) \leq \$\text{CSP}(\Delta) \)
- \( R(\overline{x}) = Q_1(\overline{x}_1) \times \cdots \times Q_n(\overline{x}_n) \) then \( \#\text{CSP}(\Gamma) \leq \#\text{CSP}(\Delta) \)
Define relation $R$ on $A$

$R = \emptyset$ if $|A| = 2$

$R$ is AllDifferent otherwise
The set of all functions/relations that can be expressed by an instance of CSP($\Delta$) is called the weak clone generated by $\Delta$, and denoted $\langle \Delta \rangle$. 
Quantification (Decision)

If for every $R \in \Gamma$ there is an instance of CSP($\Delta$) with relations $Q_1, \ldots, Q_n \in \Delta$ such that

$$R(x) = \exists y \ Q_1(x_1, y_1) \land \cdots \land Q_n(x_n, y_n)$$

then $\text{CSP}(\Gamma) \leq \text{CSP}(\Delta)$

(Jeavons, et al., 1997)

The set of all functions/relations that can be expressed by an instance of CSP($\Delta$) + existential quantification is called the clone generated by $\Delta$, and denoted $\langle \Delta \rangle_\exists$
Define relation $R$ on $A = \{0, 1, 2\}$

$R$ is NotAllDifferent
Gadgets & Reductions: Optimization

Optimization (Maximization):

For a constraint language $\Delta$ by $\langle \Delta \rangle_{\text{max}}$ we denote the set of functions

$$R(\vec{x}) = \max_{\vec{y}}(Q_1(\vec{x}_1, \vec{y}_1) + \cdots + Q_n(\vec{x}_n, \vec{y}_n)),$$

the max-clone.

If $\Gamma \subseteq \langle \Delta \rangle_{\text{max}}$, then $\text{CSP}(\Gamma) \leq \text{CSP}(\Delta)$. 
Small Example III

<table>
<thead>
<tr>
<th></th>
<th>0</th>
<th>1</th>
</tr>
</thead>
<tbody>
<tr>
<td>Ferroising</td>
<td>λ</td>
<td>1</td>
</tr>
<tr>
<td></td>
<td>λ</td>
<td>1</td>
</tr>
<tr>
<td>0</td>
<td>λ</td>
<td>1</td>
</tr>
</tbody>
</table>

R: $x \rightarrow FI \rightarrow FI \rightarrow y$

00 $\max\{\lambda + \lambda, 1+1\}$

01 $\max\{1 + \lambda, \lambda + 1\}$

R: $\begin{array}{c|cc}
0 & 0 & 1 \\
0 & 2\lambda & 1+\lambda \\
1 & 1+\lambda & 2\lambda \\
\end{array}$

$1 < \frac{2\lambda}{1+\lambda} < \lambda$
Gadgets & Reductions: Counting

**Counting:**

For a constraint language $\Delta$ by $\langle \Delta \rangle_\Sigma$ we denote the set of functions

$$R(\overline{x}) = \sum_{\overline{y}} Q_1(\overline{x}_1,\overline{y}_1) \times \cdots \times Q_n(\overline{x}_n,\overline{y}_n),$$

the $\Sigma$-clone

If $\Gamma \subseteq \langle \Delta \rangle_\Sigma$, then $\#\text{CSP}(\Gamma) \leq \#\text{CSP}(\Delta)$

For relations: If $\Gamma \subseteq \langle \Delta \rangle_\exists$ then $\#\text{CSP}(\Gamma) \leq \#\text{CSP}(\Delta)$

(B., Dalmau, 2003)
Define relation $R$ on $A = \{0, 1\}$

$R$ is FerroIsing
Polymorphisms
Polymorphisms

Operation $f(x_1, \ldots, x_n)$ is a polymorphism of relation $R$ if for any $\bar{a}_1, \ldots, \bar{a}_n \in R$, it holds $f(\bar{a}_1, \ldots, \bar{a}_n) \in R$.

$\text{Pol}(R)$, $\text{Pol}(\Gamma)$ is the set of all polymorphisms of $R$, $\Gamma$.

$R \in \langle \Gamma \rangle_\exists$ if and only if $\text{Pol}(\Gamma) \subseteq \text{Pol}(R)$

If $\text{Pol}(\Gamma) \subseteq \text{Pol}(\Delta)$ then $\text{CSP}(\Delta) \leq \text{CSP}(\Gamma)$

$\#\text{CSP}(\Delta) \leq \#\text{CSP}(\Gamma)$
Polymorphisms: Examples

Let $R = \{(0,1),(1,2),(2,0)\}$ on $A = \{0,1,2\}$ and $f(x,y,z) = x - y + z$. $f$ is a polymorphisms of $R$

$$
\begin{align*}
  f \begin{pmatrix} 0 & 2 & 1 \\ 1 & 0 & 2 \end{pmatrix} &= \begin{pmatrix} 2 \\ 0 \end{pmatrix}, \\
  f \begin{pmatrix} 0 & 2 & 0 \\ 1 & 0 & 1 \end{pmatrix} &= \begin{pmatrix} 1 \\ 2 \end{pmatrix}, ...
\end{align*}
$$

$f(x,y,z)$ is a majority operation, if $f(x,x,y) = f(x,y,x) = f(y,x,x) = x$. If relation $R$ has a majority polymorphism, then $\bar{a} \in R$ if and only if every its binary projection belongs to the corresponding binary projection of $R$. 

21/36
Polymorphisms: Results

**Dichotomy Conjecture** for decision CSPs (~): CSP(Γ) is poly time if and only if Γ has a nontrivial polymorphism. Otherwise it is NP-complete

**Exact counting**: More complicated, but can be described through polymorphisms
A multimorphism is a collection of operations $m_1, \ldots, m_n$ on $A$. $m_1, \ldots, m_n$ is a multimorphism of function $R$ on $A$ if for any $\bar{a}_1, \ldots, \bar{a}_n$

$$R(\bar{a}_1) + \cdots + R(\bar{a}_n) \geq R(m_1(\bar{a}_1, \ldots, \bar{a}_n)) + R(m_n(\bar{a}_1, \ldots, \bar{a}_n))$$

Submodularity: $m_1 = \land, m_2 = \lor$

$$R(\bar{a}) + R(\bar{b}) \geq R(\bar{a} \land \bar{b}) + R(\bar{a} \lor \bar{b})$$
Optimization: Fractional Polymorphisms

Fix a set $A$ and let $O^k$ denote the set of all $k$-ary operations $m: A^n \to A$. A probability distribution $\mu$ on $O^k$, $\mu: O^k \to [0,1]$ is called a fractional polymorphism of function $R: A^k \to \mathbb{R}$ if for any $\bar{x}_1, \ldots, \bar{x}_n \in A^k$

$$E_{m \sim \mu} [R(m(\bar{x}_1, \ldots, \bar{x}_n))] \leq \text{avg}(R(\bar{x}_1), \ldots, R(\bar{x}_n))$$

Submodularity:

$k = 2$, $\mu(\land) = \mu(\lor) = \frac{1}{2}$, that is,

$$\frac{1}{2} \left( R(\bar{x}_1 \land \bar{x}_2) + R(\bar{x}_1 \lor \bar{x}_2) \right) \leq \frac{1}{2} \left( R(\bar{x}_1) + R(\bar{x}_2) \right)$$
FPol(R), FPol(\Gamma) denote the set of all fractional polymorphisms of function \( R \) or constraint language \( \Gamma \)

\[ R \in \langle \Gamma \rangle_{max} \text{ iff } FPol(\Gamma) \subseteq FPol(R) \quad \text{(Zivny et al. 2009)} \]

\( \text{CSP}(\Gamma) \) is polynomial time iff \( \Gamma \) has a `nontrivial’ fractional polymorphism. Otherwise it is NP-hard.

(Thapper, Zivny, 2013, Kolmogorov et al. 2015)
Approximation: Approximation Polymorphisms

Fix a set $A$ and let $O^k$ denote the set of all $k$-ary operations $m: A^k \rightarrow A$.

A probability distribution $\mu$ on $O^k, \mu: O^k \rightarrow [0,1]$ is called an $\alpha$-approximation polymorphism of function $R: A^k \rightarrow \mathbb{R}$ if for any $\bar{x}_1, ..., \bar{x}_n \in A^k$

$$\alpha \cdot E_{m \sim \mu} \left[ R(m(\bar{x}_1, ..., \bar{x}_n)) \right] \geq avg(R(\bar{x}_1), ..., R(\bar{x}_k))$$

Let $\alpha_{\Gamma}$ be the greatest constant such that there is a `nontrivial' $\alpha_{\Gamma}$-approximation polymorphism of $\Gamma$. Then (assuming the Unique Games Conjecture) $\alpha_{\Gamma}$ is the approximation threshold for $CSP(\Gamma)$. (Raghavendra, 2008)
Approximate Counting
Approximate Counting: Clones

Clones for approximate counting are $\langle \Gamma \rangle_\Sigma + \text{limits} = \langle \Gamma \rangle_\omega$, that is, $R \in \langle \Gamma \rangle_\omega$ iff there are $R_1, R_2, \ldots \in \langle \Gamma \rangle_\Sigma$ such that $\lim R_k = R$

If $\Gamma \subseteq \langle \Delta \rangle_\omega$ then $\#\text{CSP}(\Gamma) \leq_{AP} \#\text{CSP}(\Delta)$

Any `morphisms’ for approximate counting?
Morphisms for Approximate Counting

**Observation:** For any constraint language $\Gamma$ of rational-valued functions there is a constraint language $\Delta$ of relations (possibly on a different set) such that $\#\text{CSP}(\Gamma) \approx \#\text{CSP}(\Delta)$

Partial operation $f(x_1, \ldots, x_n)$ is a partial polymorphism of relation $R$ if for any $\bar{a}_1, \ldots, \bar{a}_n \in R$, it holds $f(\bar{a}_1, \ldots, \bar{a}_n)$ belongs to $R$ or does not exist.

$\text{PPol}(R), \text{PPol}(\Gamma)$ is the set of all partial polymorphisms of $R, \Gamma$.

$R \in \langle \Gamma \rangle$ if and only if $\text{PPol}(\Gamma) \subseteq \text{PPol}(R)$
Can We Do Better?

We need to find some sort of `morphisms’ for $\langle \Gamma \rangle_{\Sigma}$ or/and $\langle \Gamma \rangle_{\omega}$

Nothing known yet, but there are options …
**Option 1.** Does one of the existing types of `morphism`s` work?

Function \( f : \{0,1\}^k \rightarrow \mathbb{R}^+ \) is Log-Super-Modular (LSM) if for any \( \vec{x}_1, \vec{x}_2 \in \{0,1\}^k \)

\[
f(\vec{x}_1)f(\vec{x}_2) \leq f(\vec{x}_1 \land \vec{x}_2)f(\vec{x}_1 \lor \vec{x}_2)
\]

FerroIsing \( \in \) LSM, AntiFerroIsing \( \not\in \) LSM

LSM is closed under \( \langle \cdot \rangle_\Sigma \) and \( \langle \Gamma \rangle_\omega \)

Not clear if it is true for other multimorphisms
Conservative Case

Set of operations (constraint language) $\Gamma$ on $A$ is conservative if it contains all the unary operations on $A$.

Almost complete complexity classification of conservative constraint languages

(many people in different combinations, 2014, 2015)
Option 2. Properties of Fourier coefficients?

Let $f : \{0,1\}^n \rightarrow \mathbb{R}^+$ be a function and $S = \{i_1, \ldots, i_k\} \subseteq \{1, \ldots, n\}$

Fourier coefficient $\hat{f}(S)$ is given by

$$\hat{f}(S) = \frac{1}{2^n} \sum_{x_1, \ldots, x_n \in \{0,1\}^n} f(x_1, \ldots, x_n) (-1)^{x_{i_1} + \cdots + x_{i_k}}$$

Let PF denote the set of functions $f$ such that $\hat{f}(S) \geq 0$ for all $S$. PF is closed under $\langle \cdot \rangle_\Sigma$ and $\langle \cdot \rangle_\omega$

Some interesting constraint languages from PF and LSM
Option 3. Looking for `morphisms’ w.r.t. $\langle \cdot \rangle_\omega$ is wrong.

We may want to relax the closure operator

A probability distribution $\mu$ on $O^k$, $\mu: O^k \to [0,1]$ is called a log-approximation polymorphism of function $R: A^k \to \mathbb{R}^+$ if it is a 1-approximation polymorphism of $\log R$, that is,

$$E_{m \sim \mu} \left[ \log R \left( m(x_1, \ldots, x_k) \right) \right] \geq \text{avg} \left( \log R(x_1), \ldots, \log R(x_k) \right)$$
Log-Approximation Polymorphisms

If $\mu$ is an approximation polymorphism of $\Gamma$, it is a log-approximation polymorphism of any $R \in \langle \Gamma \rangle$

For any $\Gamma$, $\langle \Gamma \rangle \subseteq \langle \Gamma \rangle_{\omega}$

Thus $\#\text{CSP}(\Gamma) \leq_{AP} \#\text{CSP}(\Delta)$ whenever $\langle \Gamma \rangle \subseteq \langle \Delta \rangle$
Thank You!