Lower Bounds for Problems Parameterized by Clique-width

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Satisfiability Lower Bounds and Tight Results for Parameterized and Exponential-Time Algorithms
Berkeley, November 6, 2015

The talk is based on the following papers:

1. F. V. Fomin, P. A. Golovach, D. Lokshtanov, and S. Saurabh, Almost Optimal Lower Bounds for Problems Parameterized by Clique-Width. SIAM J. Comput. 43(5): 1541-1563 (2014)
2. H. Broersma, P. A. Golovach, and V. Patel, Tight complexity bounds for FPT subgraph problems parameterized by the clique-width. Theor. Comput. Sci. 485: 69-84 (2013)

Outline

Clique-width

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We call the t-graph consisting of exactly one vertex v labeled by some integer i from $\{1, 2, \ldots, t\}$ an initial t-graph.

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- join: connecting all vertices labeled by i with all vertices labeled by j by edges (denoted by $\eta_{i,j}$).

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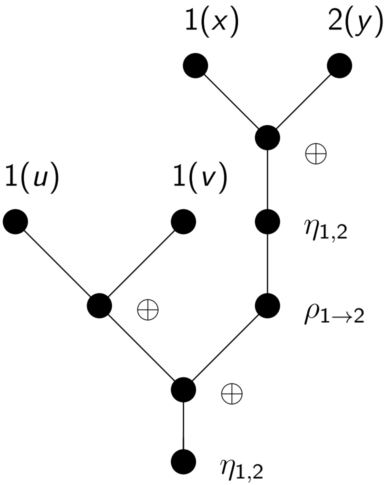
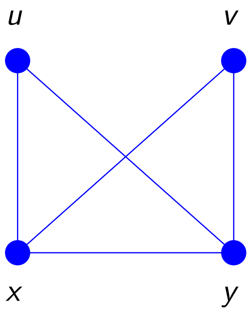
The width of the tree T is the number of different labels appearing in T.

Clique-width





Expression tree

Clique-width

Theorem (Courcelle, Makowsky, and Rotics, 2000)

All problems expressible in MSO_1-logic are fixed parameter tractable (FPT), when parameterized by the clique-width of the input graph. Or in other words, any problem expressible in MSO_1-logic can be solved, for graphs of clique-width at most t, in time $f(t) \cdot |I|^{O(1)}$, where $|I|$ is the size of the input and f is a computable function depending on the parameter t only.

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Similar results can be obtained for some variants of these problems, e.g., for MAXIMUM (MINIMUM) BISECTION.

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We give tight algorithmic lower and upper bounds for some double-parameterized graph problems when the clique-width of the input graph is one of the parameters. We prove the following for n-vertex graphs G of clique-width at most t,

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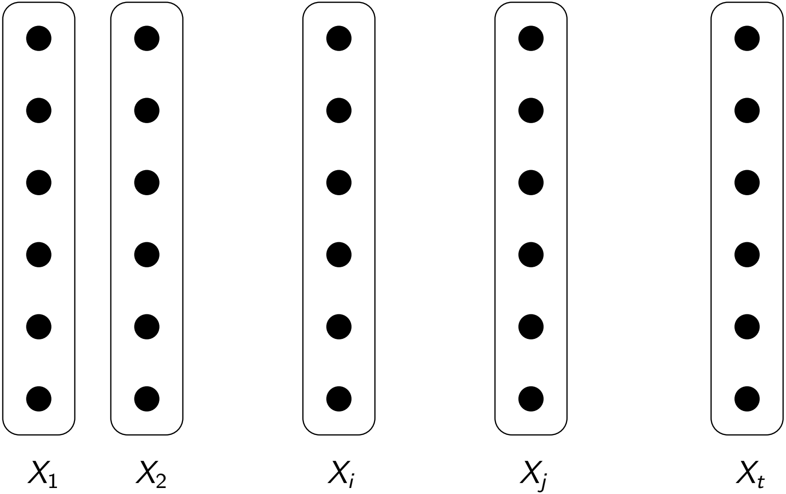
- The d-Regular Induced Subgraph problem, can be solved in time $d^{O(t)} \cdot n$, but it cannot be solved in time $2^{o(t \log d)} \cdot n^{O(1)}$ unless ETH fails.

Algorithmic upper bound for Max-Cut



Partial solution for a node of the expression tree

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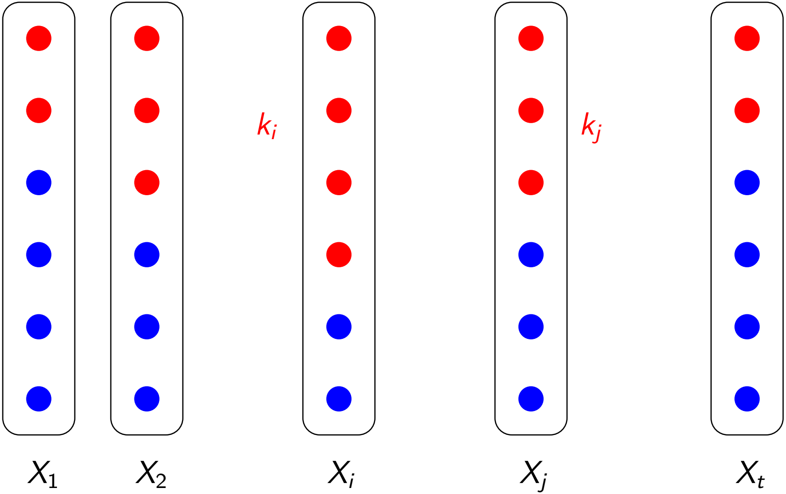

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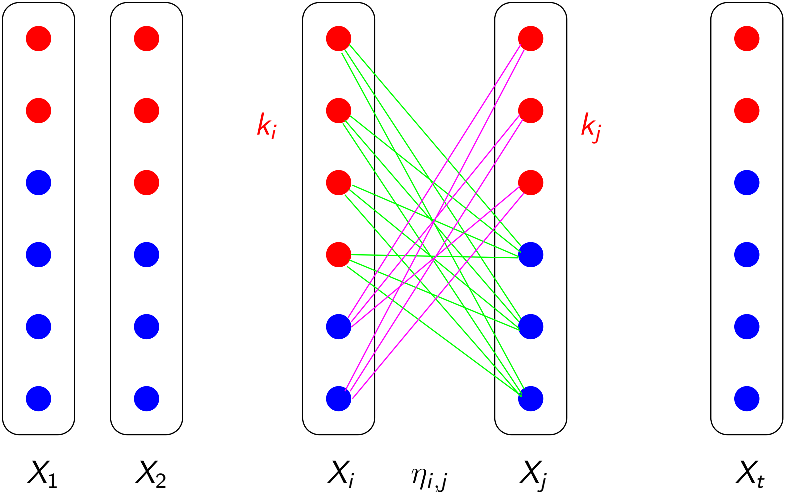

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Algorithmic lower bounds

Theorem (Cai and Juedes 2001, Downey et al. 2003, Chen et al. 2006)

There is no algorithm for k-Clique (finding a clique of size k) running in time $f(k)n^{o(k)}$ unless ETH fails.

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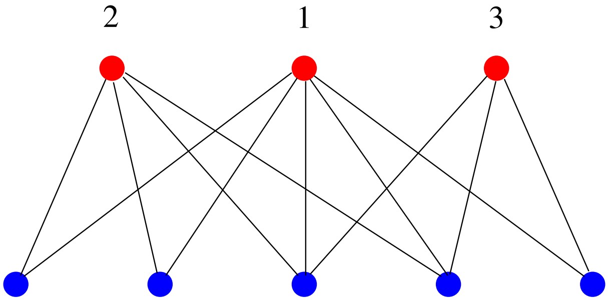
Theorem (Cai and Juedes 2001, Downey et al. 2003, Chen et al. 2006)

There is no algorithm for k-CLIQUE (finding a clique of size k) running in time $f(k)n^{o(k)}$ unless ETH fails.

Corollary

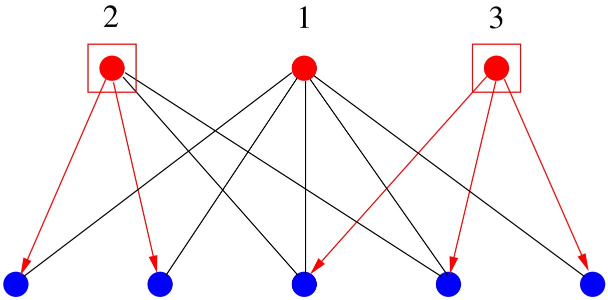
There is no algorithm for MULTICOLORED k-CLIQUE (finding a clique of size k in a k-partite graph) running in time $f(k)n^{o(k)}$ unless ETH fails.

Capacitated Domination



Red-blue Capacitated Dominating Set

Capacitated Domination



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Capacitated Domination

Problem (Red-Blue CDS)

Input: *A graph G with a partition (R, B) of $V(G)$, a capacity function $c\colon R \to \mathbb{N}$ and a positive integer k.*

Question: *Is there a capacitated dominating set $S \subseteq R$ of size at most k?*

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Red-Blue Saturated CDS: each vertex $v \in S$ is assigned exactly $c(v)$ neighbors to dominate.

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RED-BLUE EXACT SATURATED CDS: a variant of Red-Blue Saturated CDS, where $|S| = k$.

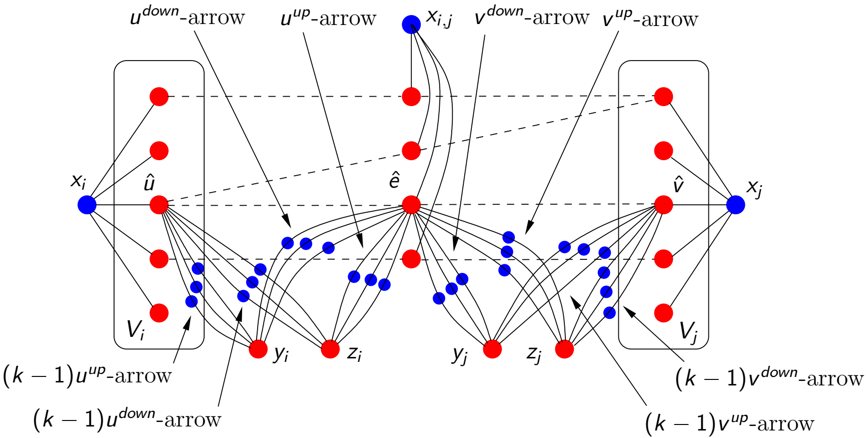
Algorithmic lower bounds

Theorem

There is no algorithm for RED-BLUE CDS (RED-BLUE SATURATED CDS, RED-BLUE EXACT SATURATED CDS) *running in time $f(t)n^{o(t)}$ unless ETH fails, where t is the feedback vertex number of an input graph even if the input restricted to graphs G such that*

- *every minimum feedback vertex set X is independent, and*
- *only leaves of the forest $G - X$ are adjacent to X and each leaf is adjacent to exactly one vertex of X.*

Algorithmic lower bounds



Reduction for RED-BLUE CDS

Algorithmic lower bounds

Corollary

There is no algorithm for Red-blue CDS *(*Red-Blue Saturated CDS, Red-Blue Exact Saturated CDS*) running in time $f(t)n^{o(t)}$ unless ETH fails, where t is the clique-width of an input graph/clique-width of the incidence graph of an input graph, even if an expression tree (clique-decomposition) of width at most t is given.*

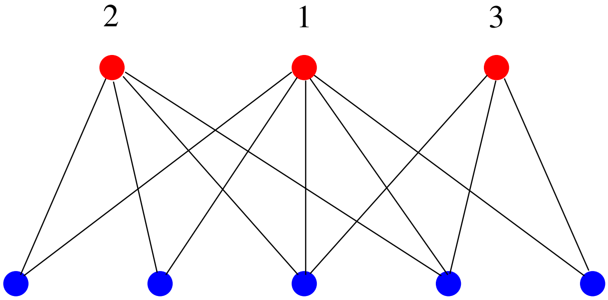
Lower bound for Edge Dominating Set

Theorem

EDGE DOMINATING SET *cannot be solved in time* $f(t) \cdot n^{o(t)}$ *unless the ETH fails even if an expression tree (clique-decomposition) of width at most* t *is given.*

The idea of reduction

Consider an instance of RED-BLUE EXACT SATURATED CDS.



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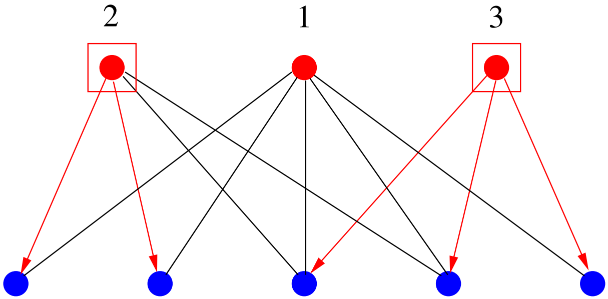

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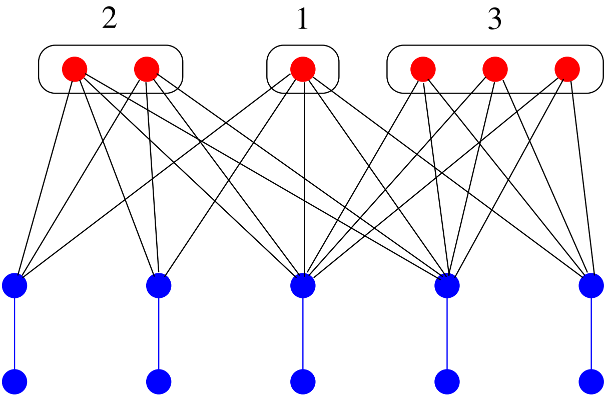

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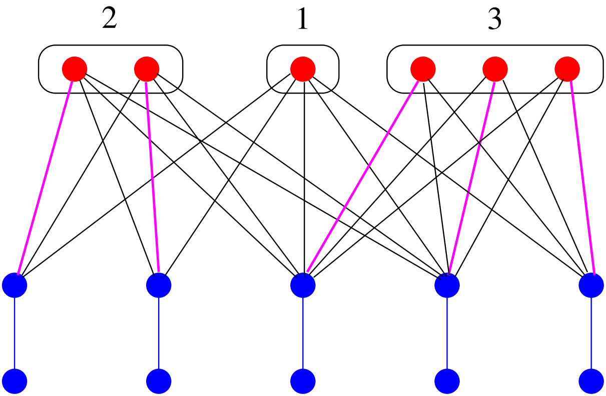

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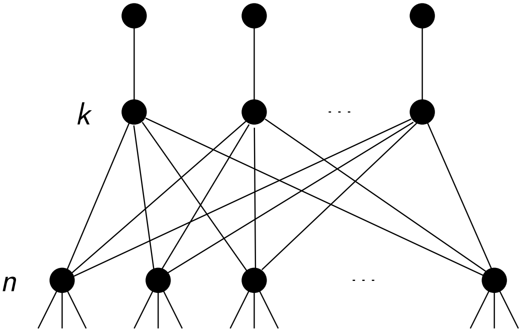
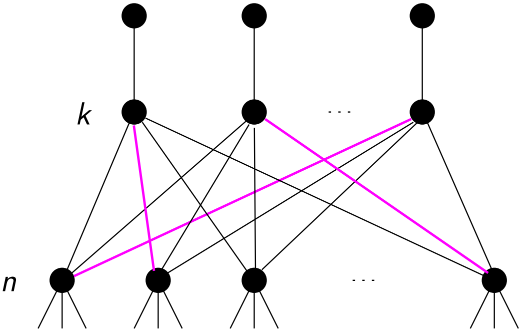


Selection gadget

The idea of reduction



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Double parameterization

Theorem (Lokshtanov, Marx, and Saurabh, 2011)

The $k \times k$-CLIQUE problem (the variant of MULTICOLORED k-CLIQUE where all sets of the k-partition have size k) cannot be solved in time $2^{o(k \log k)} \cdot n^{O(1)}$, where n is the number of vertices of the input graph G, unless ETH fails.

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- Is it possible to give tight algorithmic upper and lower bounds for Hamiltonian Cycle when parameterized by the clique-width of the input graph?
- Is it possible to give tight algorithmic upper and lower bounds for d-Regular Induced Subgraph when parameterized by the tree-width of the input graph?

Thank You!