Lower bounds on the running time for scheduling and packing problems

L. Chen, K. Jansen, F. Land, K. Land, and G. Zhang

University of Kiel and Zhejiang University
Parameterized Complexity I

Assumption: $P \neq NP$. 
Express running time in term of a parameter $k$. 

Parameterized Complexity II

$O(\sqrt{k})$ 

$O(k)$
Natural parameter $k$ for 3-SAT: the number $n$ of variables or $m$ of clauses.
Conjecture

**Exponential Time Hypothesis (ETH) (Impagliazzo, Paturi, Zane 2001)** There is a positive real $\delta$ such that 3-SAT with $n$ variables and $m$ clauses cannot be solved in time $2^{\delta n} (n + m)^{O(1)}$. 
The ETH assumption implies that there is no algorithm for 3-SAT (Impagliazzo, Paturi, Zane 2001) with \( n \) variables and \( m \) clauses that runs in time \( 2^{\delta m} (n + m)^{O(1)} \) for a real \( \delta > 0 \).
Known lower bounds

- \(2^{o(n)} n^{O(1)}\) for independent set, vertex cover, dominating set and Hamiltonian path,

- \(2^{o(k)} n^{O(1)}\) for vertex cover (where \(k = \text{OPT}(I)\)),

- \(f(m) \| I \|^{o(m)}\) for \(P|\text{prec}|C_{max}\) (Chen et al. 2006)

- \(f(\epsilon) \| I \|^{o(\sqrt{1/\epsilon})}\) for 2D vector knapsack (Kulik, Shachnai 2010)

- \(f(m) \| I \|^{o(m/\log m)}\) for unary bin packing (Jansen et al. 2013)
Goal

Find bounds for scheduling and packing problems

- prove lower bounds based on the ETH
- find algorithms to obtain upper bounds

Best results: matching lower and upper bounds
Exact algorithms
Lower bounds

**Theorem:** Subset Sum, Partition, Knapsack, Bin Packing and $Pm||C'_{max}$ for $m \geq 2$ cannot be solved in time $2^{o(n)}||I||^{O(1)}$, unless the ETH fails.

Matching upper bounds

- naive enumeration for Subset Sum, Partition, Knapsack.
- algorithms based on subsets of job solve many scheduling problems.
Variables $x_1, \ldots, x_n$ and clauses $C_1, \ldots, C_m$.

For $x_i$ create items $t_i$ and $f_i$ with

$$s(t_i) = \sum_{j: x_i \in C_j} 10^{n+j-1} + 10^{i-1}$$

$$s(f_i) = \sum_{j: \overline{x}_i \in C_j} 10^{n+j-1} + 10^{i-1}$$
For $C_j$ create items $d_j$ and $d'_j$ with

$$s(d_j) = s(d'_j) = 10^{n+j-1}$$

and use a capacity $B$ with

$$B = \sum_{j=1}^{m} 3 \cdot 10^{n+j-1} + \sum_{i=1}^{n} 10^{i-1}.$$
Reduction for \((\overline{x_1} \lor x_2 \lor x_3) \land (x_1 \lor \overline{x_2})\)

<table>
<thead>
<tr>
<th></th>
<th>(s(t_1))</th>
<th>(s(t_2))</th>
<th>(s(t_3))</th>
<th>(s(f_1))</th>
<th>(s(f_2))</th>
<th>(s(f_3))</th>
<th>(s(d_1))</th>
<th>(s(d_2))</th>
<th>(s(d_1'))</th>
<th>(s(d_2'))</th>
<th>(B)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>1 0</td>
<td>0 0 1</td>
<td>0 1</td>
<td>0 0 1</td>
<td>1 0</td>
<td>0 1 0</td>
<td>0 1</td>
<td>1 0 0</td>
<td>0 0</td>
<td>1 0 0</td>
<td>0 1</td>
</tr>
</tbody>
</table>

**Notice:** there is no carry over.
|       | $s(t_1)$ | 1 0 0 0 1 | $s(t_2)$ | 0 1 0 1 0 | $s(t_3)$ | 0 1 1 0 0 | $s(f_1)$ | 0 1 0 0 1 | $s(f_2)$ | 1 0 0 1 0 | $s(f_3)$ | 0 0 1 0 0 | $s(d_1)$ | 0 1 0 0 0 | $s(d_2)$ | 1 0 0 0 0 | $s(d'_1)$ | 0 1 0 0 0 | $s(d'_2)$ | 1 0 0 0 0 | $B$ | 3 3 1 1 1 |
|-------|---------|----------|---------|----------|---------|----------|---------|----------|---------|----------|---------|----------|---------|----------|---------|----------|---------|----------|---------|

**Assignment:** $\phi(x_1) = \phi(x_3) = true$ and $\phi(x_2) = false$. 
### Truth assignment

<table>
<thead>
<tr>
<th></th>
<th>(s(t_1))</th>
<th>(s(t_2))</th>
<th>(s(t_3))</th>
<th>(s(f_1))</th>
<th>(s(f_2))</th>
<th>(s(f_3))</th>
<th>(s(d_1))</th>
<th>(s(d_2))</th>
<th>(s(d'_1))</th>
<th>(s(d'_2))</th>
<th>(B)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>1 0 0 0 1</td>
<td>0 0 0 1 0</td>
<td>0 1 1 0 0</td>
<td>0 1 0 0 1</td>
<td>1 0 0 1 0</td>
<td>0 0 1 0 0</td>
<td>0 1 0 0 0</td>
<td>1 0 0 0 0</td>
<td>0 1 0 0 0</td>
<td>1 0 0 0 0</td>
<td>3 3 1 1 1</td>
</tr>
</tbody>
</table>

**Subset Sum solution:** \(A = \{t_1, t_3, f_2, d_1, d_2, d'_1\}\).
Properties of reduction

(a) 3-SAT instance is satisfiable, iff the constructed subset sum instance has a solution.

(b) constructed instance has $2n + 2m \leq 8m$ items, using $n \leq 3m$ (i.e. a strong linear reduction),

(c) the existence of an algorithm for Subset Sum in time $2^{o(n)} \|I\|^{O(1)}$ implies that 3-SAT can be decided in time $2^{o(m)}(n + m)^{O(1)}$. 
Size of constructed instance

<table>
<thead>
<tr>
<th></th>
<th>1 0</th>
<th>0 0 1</th>
<th>0 1 0</th>
<th>0 1 0</th>
<th>0 1 0</th>
<th>0 1 0</th>
<th>0 1 0</th>
</tr>
</thead>
<tbody>
<tr>
<td>(s(t_1))</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>(s(t_2))</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>(s(t_3))</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>(s(f_1))</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>(s(f_2))</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>(s(f_3))</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>(s(d_1))</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>(s(d_2))</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>(s(d'_1))</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>(s(d'_2))</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>(B)</td>
<td>3</td>
<td>3</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Notice: \(\|I\| \leq (2n + 2m + 1)(n + m) = O(m^2)\).
Further results I

**Theorem:** Subset Sum, Partition, Knapsack, Bin Packing and $P_{m||C_{max}}$ for $m \geq 2$ cannot be solved in time $2^{o(\sqrt{||I||})}$, unless the ETH fails.

Matching upper bounds

- Subset Sum, Partition (*O’Neil, Kerlin 2010*),
- Knapsack, Bin Packing (*O’Neil 2011*).
**Theorem:** For any $\delta > 0$, there is no $2^{O(m^{1/2-\delta}\sqrt{|I|})}$ time algorithm for $Pm||C_{max}$, unless the ETH fails.

Upper bound: $2^{O(\sqrt{m \log^2(m) ||I||})}$ for $Pm||C_{max}$. 
Approximation schemes
**Lower bounds**

**Theorem:** There is no EPTAS for multiple knapsack (MK) with running time $2^{o(1/\epsilon)} ||I||^{O(1)}$, unless the ETH fails, even for 2 knapsacks of equal capacity and when either

(i) all items have the same profit or

(ii) the profit of each item equals its size.

Upper bound for MK: $2^{O(1/\epsilon \log^4(1/\epsilon))} + ||I||^{O(1)}$ (Jansen 2012).
Proof sketch I

Consider a restricted version $MK_{res}(\alpha, C)$, where

(i) $I$ has $m = 2$ knapsacks of capacity $\frac{1}{2}s(A)$ (where $s(A)$ must be even).

(ii) $||C|| \leq ||A||^{O(1)}$,

(iii) $profit(A) \leq \alpha C n$ where $\alpha = O(1)$,

(iv) $profit(a) \geq C$ for all $a \in A$
Proof Sketch II

**Idea:** reduce an instance of Partition to this restricted version of MK where the sizes remain the same.

**Notice:** If there is a solution for Partition, then there is a packing into 2 knapsacks.

Suppose that there is an approximation scheme $A_\epsilon$ for MK that finds an $(1 + \epsilon)$ solution in time $2^{o(1/\epsilon)} ||I||^{O(1)}$. Set $\epsilon = 1/(\alpha n)$. 
Proof Sketch III

Claim: the approximation scheme packs all items (if Partition has a solution).

\[ \text{profit}(A) \leq \alpha C n \iff \frac{1}{\alpha n} \text{profit}(A) \leq C \]

If all items can be packed, then \( A_\epsilon \) has profit at least

\[ \frac{1}{1+\epsilon} OPT(I) = (1 - \frac{\epsilon}{1+\epsilon}) \text{profit}(A) = \text{profit}(A) - \frac{1}{1+\alpha n} \text{profit}(A) \]
\[ > \text{profit}(A) - \frac{1}{\alpha n} \text{profit}(A) \geq \text{profit}(A) - C. \]

Since \( \text{profit}(a) \geq C \) for all \( a \in A \), there is no unpacked item.
Proof Sketch IV

**Consequence:** We can decide whether a partition instance admits a solution by running a \((1 + \epsilon)\) approximation algorithm.

Since \(\text{profit}(A) \leq \alpha Cn\) and \(\|C\| \leq \|A\|^{O(1)}\), we have \(\|I\| = \|A\|^{O(1)}\). Using \(\epsilon = 1/(\alpha n)\), the approximation scheme \(A_\epsilon\) runs in time \(2^{o(1/\epsilon)}\|I\|^{O(1)} = 2^{o(n)}\|A\|^{O(1)}\). This gives a contradiction.
MK with \( \text{profit}(a) = 1 \) for all \( a \in A \)

By a reduction from Partition with even \( s(A) \) to \( MK \) with \( \text{profit}(a) = 1 \) for all \( a \in A \).

Then, \( \text{profit}(A) = n \) and \( \text{profit}(a) \geq 1 \). This means \( \alpha = 1 \) and \( C = 1 \) works. Therefore, we obtain a instance of \( MK_{res}(1, 1) \).

Notice: If \( s(A) \) is odd then we have a no-instance.
By a reduction from Partition-\(\psi\), where there exists a \(C \in \mathbb{N}\) such that \(C \leq s(a) \leq 3C\) for all \(a \in A\).

The property above implies \(s(A) \leq 3Cn\). Using \(profit(a) = s(a)\), we get \(profit(a) \geq C\) and \(profit(A) \leq 3Cn\). This means \(\alpha = 3\) and the value \(C\) works. We obtain a instance of \(MK_{res}(3, C)\).

Notice: There is also no algorithm that decides Partition-\(\psi\) in time \(2^{o(n)} \|A\|^{O(1)}\).
Theorem: There is no PTAS for 2D vector knapsack with running time \( n^{o(1/\epsilon)} \| I \|^{O(1)} \), unless the ETH fails.

Matching upper bound: \( n^{O(1/\epsilon)} \| I \|^{O(1)} \) (Caprara et al. 2010).
Other results

**Theorem:** For any $\delta > 0$, there is no $2^{O((1/\epsilon)^{1-\delta})} + n^{O(1)}$ EPTAS for $P\|C_{max}$, unless the ETH fails.

Upper bound: $2^{O(1/\epsilon^2 \log^3(1/\epsilon))} + \|I\|^{O(1)}$ for $P\|C_{max}$ and $Q\|C_{max}$ ([Jansen 2010](#)).
Other results

**Theorem:** For any $\delta > 0$, there is no $2^{O((1/\epsilon)^{1-\delta})} + n^{O(1)}$ EPTAS for $P||C_{\text{max}}$, unless the ETH fails.

Improved upper bound: $2^{O(1/\epsilon \log^4(1/\epsilon))} + ||I||^{O(1)}$ for $P||C_{\text{max}}$ and $Q||C_{\text{max}}$ (Jansen, Klein, Verschae 2015).
Other results

**Theorem:** For any $\delta > 0$, there is no $(1/\epsilon)^{O(m^{1-\delta})} + n^{O(1)}$ FPTAS for $Pm\|C_{max}$, unless the ETH fails.

Upper bound: FPTAS for $Rm\|C_{max}$ with running time $(m/\epsilon)^{O(m)} + O(n)$ and $(1/\epsilon)^{O(m)} + O(n)$ for $\epsilon < 1/m$ (Jansen, Mastrolilli 2010).
Summary and Open problems

For further results we refer to:


Open problems:

- Show a lower bound for $d$ dimensional vector knapsack.

- Close the gaps for $MK$ and $P||C_{max}$. 